Preliminary Examination: Complex Analysis

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1. Find all singular points of the following holomorphic functions and classify them: (a) $cot\ z - \frac{1}{z}$. (b) $sin\left(exp\frac{1}{z}\right)$.

2. Show that $\sum_{n=1}^{\infty} \frac{cos\ nz}{n!}$ converges on any compact subset of **C**.

3. Find the Laurent expansion of $\frac{1}{(z-1)^2(z+1)}$ in the annulus $1 < |z| < 2$.

4. Prove that $\Delta(1) - E$, where E is a discret subset, is connected.

5. Let f be a holomorphic function in $\Delta(1)$ with $|f(z)| \le 1$ for any $z \in \Delta(1)$. Prove that $|f^{(n)}(0)| \le n!$.

6. Let $f(z)$ has an essential singularity at $z = a$. Prove that there exists a sequence of points z_n converging to a such that $(z_n - a)^n f(z)$ goes to ∞.

7. Let u be a harmonic fucntion defined on a domain Ω in **C**. Suppose that $u \equiv 0$ on an open subset of Ω. Prove that u equals to zero identically on Ω. Can u vanish on a set with a limit point in Ω and not be identically zero?

8. Let $P(z)$ and $Q(z)$ be two polynomials with $deg(P) = m$ and $deg(Q) = m + 2$. Prove that the sum of all residues of $\frac{P}{Q}$ are zero.

9. Does not there a sequence of polynomails which converges uniformaly to $\frac{1}{z}$ on $\{z \mid |z| = 1\}$? Explain.

10. Give a statement of Schwarz lemma. Prove that any holomorphic function $f : \Delta(1) \to \Delta(1)$, but not one-to-one, must satisfy $|f'(0)| < 1$.