Department of Mathematics, University of Houston
Topology & Geometry
Sample Qualifying Exam

Answer at least 6 questions from the Topology part and at least 3 questions from the Geometry of Manifolds part. Identify the problems that should be graded.

$\mathbf{R}^n$ is Euclidean n-space, $\mathbf{R} = \mathbf{R}^1$.

Topology

1. (a) Which subspaces of a compact Hausdorff space are compact?
   (b) What is a locally compact space?
   (c) Prove that an open subspace of a compact Hausdorff space is locally compact.
   (d) Is every locally compact Hausdorff space homeomorphic to an open subspace of a compact Hausdorff space? Say why or give a counterexample.
2. (a) What is a first countable topological space?
   (b) What is a net, and what does it mean for a net in a topological space to converge?
   (c) Is a convergent net in $\mathbf{R}$ bounded? Prove it or give a counterexample.
   (d) Prove that in a compact first countable space, every sequence has a convergent subsequence.
3. (a) How is a quotient topology defined?
   (b) Show that the quotient topology obtained from $\mathbf{R}$ by identifying two numbers if they differ by a rational number, is the indiscrete topology.
4. State as many characterizations as you know of separable metric spaces.
5. (a) How is the product topology defined?
   (b) Show that the 'projection map' from a product topological space $\prod_{j\in J} X_j$ (with the product topology) to one of the spaces X_j, is an open map.
   (c) Let X be the product of an infinite countable number of copies of the two point set $\{0, 1\}$ with its usual (discrete) topology. Give X the product topology. What topological properties does it have? Is it normal? Metrizable? Compact? Explain. What are its connected components?

6. Let (X, d) be a metric space and $f : X \to X$ a continuos function that has no fixed points (that is, there is no $x \in X$ such that $f(x) = x$).

   (a) If X is compact show that there is an $\varepsilon > 0$ such that $d(x, f(x)) > \varepsilon$ for each $x \in X$.

   (b) Show that the result of (a) is false when compactness is not assumed.

7. (a) Show that if Y is compact then the projection $\pi_1 : X \times Y \to X$ is a closed map.

   (b) Does the result of (a) remain true if Y is not compact?

8. Let X be a completely regular space, $\beta(X)$ its Stone-Čech compactification, and Y any compactification of X (that is, Y is a compact Hausdorff space that contains X as a dense subset). Show that there is a unique continuous map $g : \beta(X) \to Y$ which is the identity on X. Prove that this map is surjective and closed.

Geometry of Manifolds

1. Let

$$\mathbf{RP}^m = \{[x] \mid x = (x_0, \ldots, x_m) \in \mathbf{R}^{m+1} \backslash \{\mathbf{0}\}\},$$

   where $[x]$ is the equivalence class of x, and the equivalence relation "$\sim$" is defined as: $x \sim y$ if and only if $x = \lambda y$ for some λ in $\mathbf{R}$. Prove that $\mathbf{RP}^m$ is an m-dimensional smooth manifold.

2. Let M, N be smooth manifolds and f a smooth map from M to N.

   (a) Let $p \in M$. Give the definition of $f_{*,p}$, the differential of f at p (also called the derivative of f at p in our notes).

   (b) Suppose that M is connected. Show that f is constant if and only if $f_{*p} = 0$ for all $p \in M$.

3. Let $\omega = xydx + zdy - yzdz$, $\eta = xdx - yz^2dy - 2xdz$, and $f : \mathbf{R}^2 \to \mathbf{R}^3$ defined by

$$f(u, v) = (uv, u^2, 3u + v), \qquad (u, v) \in \mathbf{R}^2.$$

   Find: (1) $d\omega$; (2) $d\eta$; (3) $d\omega \wedge \eta - \omega \wedge d\eta$; (4) $f^*\omega$ and $f^*(d\omega)$.

4. Let Γ be the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ in $\mathbf{R}^3$ and $\omega = zdx \wedge dy - ydz \wedge dx$. Calculate $\int_\Gamma \omega$.