1. Consider the matrices
$$A = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$
and the corresponding ODEs $\dot{x} = Ax$ and $\dot{y} = By$ in $\mathbb{R}^3$. Let $x(t)$ and $y(t)$ be solutions of these ODEs with the same initial condition $x(0) = y(0) = (z_1, z_2, z_3)^T$, where $z_3 \neq 0$.
   (a) What is the behaviour of $|x(t)|$ and $|y(t)|$ as $t \to \infty$? (That is, do they decay to 0? Diverge to ∞? Remain bounded away from both?)
   (b) Which of $|x(t)|$ and $|y(t)|$ dominates the other as $t \to \infty$? That is, does $\lim_{t\to\infty} \frac{|x(t)|}{|y(t)|}$ equal 0 (in which case $|y(t)|$ dominates), or does it equal ∞ (in which case $|x(t)|$ dominates)?

2. (a) Suppose A is an $n \times n$ matrix and $B(t)$ is a continuous map ($B : \mathbb{R} \to \mathbb{R}^n$). Prove that all solutions of
$$\dot{x} = Ax + B(t)$$
are of the form
$$x(t) = e^{At}[\int_0^t e^{-As} B(s)\, ds + C]$$
where C is a constant vector in $\mathbb{R}^n$.
   (b) Using part (a) or otherwise show that if $n = 2$ then all solutions to
$$\dot{x} = Ax + B(t), \qquad x(0) = x_0$$
tend to the origin if
$$A = \begin{pmatrix} -1 & 1 \\ 1 & -4 \end{pmatrix}$$
and the continuous map $B(t)$ is bounded.

3. (a) Find all equilibria for the system $\dot{x} = f(x, \mu)$ where
$$f(x, \mu) = \mu x - x^3$$
and determine bifurcation points in the (μ, x) plane. Plot a bifurcation diagram in the (μ, x) plane indicating stable branches and unstable branches of equilibria.
   (b) Consider a simple population growth model with harvesting,
$$\dot{x} = x - x^2 - p$$
where x represents the size of the population and p is positive.
      i. Find the equilibria of this model.
      ii. Show that if the harvesting rate p satisfies $p > \frac{1}{4}$ then the population dies out.
      iii. Show that if $0 < p < \frac{1}{4}$ then the longterm fate of the population depends on the initial population size.

4. (a) Show that the equilibrium of the system
$$\dot{x} = -2x + y^2 x$$
$$\dot{y} = -3yx^2 + y^2 x^3$$
at the origin is stable by using a Lyapunov function (or otherwise).

(b) Find all equilibria for the system $\dot{x} = f(x, \mu)$ where

$$f(x, \mu) = \mu x - x^2$$

and determine bifurcation points in the (μ, x) plane. Plot a bifurcation diagram in the (μ, x) plane indicating stable branches and unstable branches of equilibria.

5. (a) Show that the Poincaré-Bendixson theorem for planar flows does not extend to smooth flows on $\mathbb{R}^4$ by giving an example of a system

$$\dot{p} = f(p),$$

where $f : \mathbb{R}^4 \to \mathbb{R}^4$ is smooth, which has a bounded flow invariant region without a fixed point yet no periodic solution exists in the region. *Hint: it is possible to find an example of the form* $f(p) = Ap$ *where* A *is a* 4×4 *matrix.*

(b) Consider the planar system of differential equations

$$\begin{aligned} \dot{x} &= \lambda + x + y - xy \\ \dot{y} &= \lambda x(2 - x) \end{aligned}$$

depending on the parameter $\lambda \neq 0$.

i. Find all equilibria for this system.

ii. Determine the type of each equilibrium (source, sink, saddle, etc.) as a function of λ.

6. (a) State the Picard-Lindelöf theorem on existence and uniqueness of solutions to the ODE $\dot{x} = f(x)$ for a vector field $f : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is an open domain.

(b) State the Banach Fixed Point Theorem. Define Picard iteration and explain its role in the proof of the Picard-Lindelöf Theorem. (*You do not need to give a complete proof of the theorem, just describe how Picard iteration is used.*)

(c) Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the ODE $\dot{x} = f(x)$ has multiple solutions for some initial condition x_0. Write down at least two of these solutions.

(d) Give an example of a C^1 function $f : \mathbb{R} \to \mathbb{R}$ such that the ODE $\dot{x} = f(x)$ with initial condition $x(0) = x_0$ does **not** have a solution that is defined on all of $[0, \infty)$. Find the maximal value of T such that your ODE has a solution on $[0, T)$, and explain what happens to $x(t)$ as $t \to T^-$.

7. (a) Consider ODEs of the form

$$x^{(n)} + a_{n-1}x^{(n-1)} + a_{n-2}x^{(n-2)} + \cdots + a_1\dot{x} + a_0 x = 0, \qquad (\star)$$

where $a_j \in \mathbb{R}$. What is the smallest value of n for which the coefficients a_j can be chosen so that each of $x(t) = t\sin(t)$ and $x(t) = t^2 e^t$ is a solution of $(\star)$?

(b) Let $A(t)$ be a time-dependent $n \times n$ matrix with real entries, and $\Phi(0)$ an $n \times n$ matrix. Is $\Phi(t) = e^{\int_0^t A(s)\,ds}\Phi(0)$ always a solution of $\dot{\Phi}(t) = A(t)\Phi(t)$? If so, prove it; if not, state why this may fail. (*A heuristic explanation is enough.*)

8. Determine the stability of the equilibrium at the origin for the system as a function of α.

$$\begin{aligned} \dot{x} &= x^2 - xy \\ \dot{y} &= -y + \alpha x^2 \end{aligned}$$

Hint: construct a center manifold and analyze the flow on it.

9. (a) Show that if A is a 2×2 matrix with eigenvalues $a + ib$, $a - ib$ with $a > 0$ and $b > 0$, then there is a basis for $\mathbb{R}^2$ such that A has the form

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

(b) Give an example of a real 3×3 matrix A such that $\dot{v} = Av$ has a nontrivial periodic solution of period 2π for the initial condition $v_0 = (0, 1, 1)^T \in \mathbb{R}^3$.

10. Show that the system

$$\ddot{x} + (x^2 + 2\dot{x}^2 - 1)\dot{x} + x = 0$$

has a non-constant periodic solution. *Hint: Find a flow-invariant region.*

11. Consider two conjugate flows φ_t and ψ_t on $\mathbb{R}^n$; that is, $\psi_t = h^{-1} \circ \varphi_t \circ h$ where $h : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism (both h and h^{-1} are continuous). Show that

$$h(\omega_\psi(x)) = \omega_\varphi(h(x))$$

where $\omega_\psi(x)$ is the omega limit set of the flow ψ starting at x.

12. (a) Suppose A is a real 3×3 matrix and $(A - 2I)^3 v = 0$ for all vectors $v \in \mathbb{R}^3$. Write down the possible Jordan canonical forms for A.

(b) What is the smallest degree $n > 0$ for which there is a differential equation of the form

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_n x = 0$$

which has all the functions $\cos(2t)$, $t\cos(2t)$, $\sin(2t)$ as solutions? Find such a differential equation explicitly.

13. (a) Suppose that $H(x, y)$ is a C^2 function on the plane and

$$\dot{x} = \frac{\partial H}{\partial y}$$
$$\dot{y} = -\frac{\partial H}{\partial x}.$$

Show that H is constant on the flow lines generated by the vector field, i.e. $H(x(t), y(t))$ is constant in $t > 0$.

(b) Using (a) or otherwise show that if $(x(0), y(0)) = (1, 0)$ is the initial condition for the flow generated by

$$\dot{x} = y$$
$$\dot{y} = -x - x^3,$$

then the solution $(x(t), y(t))$ satisfies $y(t)^2 \leq \frac{3}{2}$ for all t.

14. (a) Suppose A is a real $n \times n$ matrix and $B(t)$ is a continuous map ($B : \mathbb{R} \to \mathbb{R}^n$). Prove that all solutions of

$$\dot{x} = Ax + B(t)$$

are of the form

$$x(t) = e^{At}[\int_0^t e^{-As} B(s)\, ds + C],$$

where C is a constant vector in $\mathbb{R}^n$.

(b) Suppose A is a real $n \times n$ matrix and there exists a constant $C > 0$ such that $|e^{At} v| \leq C|v|$ for all $t > 0$ and $v \in \mathbb{R}^n$, where $|v|$ denotes the usual Euclidean length of a vector $v \in \mathbb{R}^n$. What does this imply about the Jordan canonical form of A? State your reasons carefully.

(c) Suppose that A is a real $n \times n$ matrix and there exists a constant $C > 0$ such that $|e^{At}v| \le C|v|$ for all $t > 0$ and $v \in \mathbb{R}^n$, and furthermore that $B(t)$ is a continuous $\mathbb{R}^n$-valued function which satisfies $\lim_{t\to\infty} B(t) = 0$. Show that for all initial conditions x_0 with $|x_0| \le 1$, there is a constant K (uniform over all $|x_0| < 1$) such that the initial value problem

$$\dot{x} = Ax + B(t), \quad x(0) = x_0$$

satisfies $|x(t)| < K$ for all $t > 0$.

15. (a) Briefly explain (one paragraph or less) the technique of Picard iteration. Compute three Picard iteration terms $x_0(t), x_1(t), x_2(t)$ for the initial value problem:

$$\dot{x} = \sin(x); \qquad x(0) = \pi/2.$$

(b) Consider the system of differential equations

$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= x - y - xz \\ \dot{z} &= xy - \alpha z \end{aligned}$$

where $\alpha \in \mathbb{R}$. Note that the origin is an equilibrium for the system. Determine the stability of the origin as a function of the parameter α. *Hint: it may be helpful to consider a Lyapunov function.*

16. (a) Suppose A is a 4×4 real-valued matrix and the solutions to $\det(A - \lambda I) = 0$, where I is the 4×4 identity matrix are, counting multiplicity, $\lambda \in \{-1, -1, -i, i\}$. Write down all possible upper real Jordan canonical forms B for A. Give the general solution to $\dot{x} = Bx$ for all such upper real Jordan canonical forms B.

(b) Show that if A is a 2×2 real-valued matrix with eigenvalues $a + ib$, $a - ib$, with $a > 0$, $b > 0$, then there is a basis for $\mathbb{R}^2$ such that A has the form

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

17. Fix a parameter $\beta \in \mathbb{R}$ and consider the ODE

$$\begin{aligned} \dot{x} &= \beta x + y - x^3, \\ \dot{y} &= -x - y^3. \end{aligned} \tag{$\dagger$}$$

(a) Show that when $\beta \le 0$, the fixed point at the origin is stable, and its basin of attraction is all of $\mathbb{R}^2$.

(b) Show that when $\beta > 0$, the fixed point at the origin is unstable. Determine for which values of β it is

i. a saddle (both attracting and repelling directions);
ii. a proper unstable node (repelling, real eigenvalues, semisimple);
iii. an improper unstable node (repelling, real eigenvalues, a non-trivial Jordan block);
iv. an unstable focus (repelling, complex eigenvalues).

For each of the above cases that occur, fix a value of β that realizes that case and draw the corresponding phase portrait for ($\dagger$) in a small neighborhood of the origin.

(c) It can be shown that for $\beta \le 1$, the origin is the only equilibrium point of ($\dagger$). Use this fact to prove that for $0 < \beta \le 1$, the system ($\dagger$) has a periodic orbit Γ.

(d) **Extra credit**: Find the smallest value of β for which ($\dagger$) has an equilibrium point besides the origin.

18. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field and $\gamma : \mathbb{R} \to \mathbb{R}^3$ a periodic solution of the ODE $\dot{x} = f(x)$, with period T. Let $A(t) := Df(\gamma(t))$ and let $\Phi(t)$ be a fundamental matrix solution of $\dot{\Phi} = A(t)\Phi$.

    Let $\Gamma = \{\gamma(t) \mid t \in [0, T]\} \subset \mathbb{R}^3$ be the periodic orbit parameterized by γ. Suppose that $\Phi(T)$ has an eigenvalue inside the unit circle, and prove that there exists $x \in \mathbb{R}^3 \setminus \Gamma$ such that $\omega(x) = \Gamma$.

    You may use any theorems proved during the lectures, but you must cite them by name and use the proper terminology.

19. Consider the two-dimensional system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y(4 - 2x^2 - y^2). \end{aligned}$$

    Using the Poincaré-Bendixson theorem or otherwise, show that this system has a non-trivial periodic solution.

20. Using Liapunov's method or otherwise, determine the stability of the origin $(0, 0)$ for the following system. If you use Liapunov's method, state carefully the assumptions and conclusion of the Liapunov stability criterion you use.

$$\begin{aligned} \dot{x} &= -3x^3 + 2xy^2 \\ \dot{y} &= -y^3 \end{aligned}$$

21. (a) Let A be an $n \times n$ matrix. Let λ be a real eigenvalue of A and $E(\lambda, k) = \{v \in \mathbb{R}^n : (A - \lambda I)^k v = 0\}$. Show that $E(\lambda, k)$ is invariant under the flow generated by

$$\dot{x} = Ax, \qquad x(0) = x_0,$$

    i.e. if $x_0 \in E(\lambda, k)$, then $x(t) \in E(\lambda, k)$.

    (b) Give an example of a 3×3 real matrix A such that the solution to

$$\dot{x} = A, \qquad x(0) = x_0$$

    is

    i. periodic with period 2π if x_0 lies in the plane spanned by the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

    and

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

    ii. approaches the origin at an exponential rate if

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

    (Editor's note: There is possibly a typo in the first displayed line in part (b); perhaps it should read $\dot{x} = Ax,\ x(0) = x_0$.)

22. (a) Determine whether or not the equilibrium of the system

$$\dot{x} = -2x + y^2 x$$
$$\dot{y} = -3yx^2 + y^3 x^4$$

at the origin is stable or asymptotically stable or neither by using a Lyapunov function or otherwise.

(b) Consider the two-dimensional system

$$\dot{x} = y$$
$$\dot{y} = -x + y(4 - 2x^2 - y^2).$$

Show that this system has a non-trivial periodic solution.
Hint: Consider $\frac{dr}{dt}$ *where* $r = x^2 + y^2$ *and use Poincaré-Bendixson.*

23. Find an approximate center manifold W_ϵ^c near the origin and the approximation to the flow on W_ϵ^c for the system

$$\dot{x} = \alpha x^2 + y^2$$
$$\dot{y} = -y + x^2.$$

Carry the expansion to high enough order to determine stability in terms of the real parameter α (at least to third order).

24. Consider a simple population growth model with harvesting,

$$\dot{x} = \alpha x - \beta x^2 - \gamma,$$

where x represents the size of the population and α, β, and γ are positive. We will consider α and β as fixed and γ as a parameter to be varied.

(a) Show that if the harvesting rate γ satisfies $\gamma > \frac{\alpha^2}{4\beta}$, then the population dies out.

(b) Show that if $0 < \gamma < \frac{\alpha^2}{4\beta}$, then there are two equilibria, and describe how the longterm fate of the population depends on the initial population size.

25. Suppose $g(t)$ is a real-valued continuous function and $g(t) \geq 0$ for all $t \geq 0$. Suppose in addition that for all $t \geq 0$,

$$g(t) \leq C + K \int_0^t g(s)\, ds,$$

where $C > 0$ and $K > 0$ are positive constants. Show that

$$g(t) \leq Ce^{Kt}$$

(this is a version of Gronwall's inequality). *Hint: Define* $F(t) = C + K \int_0^t g(s)\, ds$*, bound* $\frac{F'(t)}{F(t)}$*, and integrate.*

26. (a) Consider the n^{th} order differential equation

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_0 x = 0, \qquad (*)$$

where $x \in \mathbb{R}$. Find the smallest n for which the following functions are all solutions to equation $(*)$:

$$2t\sin(2t),\ 3e^{-3t},\ 3e^{3t}.$$

(b) Give an explicit example of an equation $(*)$ for which $2t\sin(2t)$, $3e^{-3t}$, $3e^{3t}$ are solutions, i.e. determine constants $a_0, a_1, \ldots, a_{n-1}$ such that equation $(*)$ has such solutions.

27. (a) Give an example of a one-dimensional ordinary differential equation with initial condition of the form
$$\dot{x} = f(x), \qquad x(0) = x_0,$$
where $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable, but the solution $x(t)$ is not defined for all $t > 0$. (Make sure to prove your assertions.)

(b) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 and $f(x, y) = -f(x, -y)$. Is it possible for a solution to
$$\dot{p} = f(p), \qquad p(0) = (1, -1)$$
(where $p \in \mathbb{R}^2$) to satisfy $p(t) = (1, 1)$ for some $t > 0$? Give reasons for your answer.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be C^1 and assume $f(0) = 0$ and
$$\begin{aligned} f(x) < 0 \quad &\text{for} \quad x > 0 \\ f(x) > 0 \quad &\text{for} \quad x < 0. \end{aligned}$$
Prove that the system of differential equations
$$\begin{aligned} \dot{x} &= y^2 + f(x) \\ \dot{y} &= -xy + f(y) \\ \dot{z} &= -z^3 \end{aligned}$$
has an asymptotically stable equilibrium at the origin.

28. (a) Determine the stability of the equilibrium at the origin for the system
$$\begin{aligned} \dot{x} &= \alpha x - y^{3/2} + xy \\ \dot{y} &= -y + x^2 \end{aligned}$$
as a function of $\alpha \neq 0$, stating briefly any theorem that you use.

(b) Determine the stability of the equilibrium at the origin for the system
$$\begin{aligned} \dot{x} &= -xy \\ \dot{y} &= -y + x^2. \end{aligned}$$
Hint: construct a center manifold and analyze the flow on it.

29. (a) Consider ODEs of the form
$$x^{(n)} + a_{n-1}x^{(n-1)} + a_{n-2}x^{(n-2)} + \cdots + a_1\dot{x} + a_0 x = 0, \qquad (\star)$$
where $a_j \in \mathbb{R}$. What is the smallest value of n for which the coefficients a_j can be chosen so that each of $x(t) = \cos(t)$, $x(t) = te^{-t}$, and $x(t) = e^{2t}$ is a solution of $(\star)$?

(b) Fix $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ and let $B : \mathbb{R} \to \mathbb{R}^n$ be continuous. Solve the non-homogeneous ODE
$$\dot{x} = Ax + B(t), \qquad x : \mathbb{R} \to \mathbb{R}^n.$$

30. Consider the matrices
$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$
and the corresponding ODEs $\dot{x} = Ax$ and $\dot{y} = By$ in $\mathbb{R}^3$. Let $x(t)$ and $y(t)$ be solutions of these ODEs with the same initial conditions $x(0) = y(0) = (z_1, z_2, z_3)^T$, where $z_3 \neq 0$. Which of $|x(t)|, |y(t)|$ grows more quickly? That is, does $\lim_{t\to\infty} \frac{|x(t)|}{|y(t)|}$ equal 0 (in which case $|y(t)|$ grows more quickly), or does it equal ∞ (in which case $|x(t)|$ grows more quickly)?

31. Let $\phi, \psi : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 functions and consider the ODE

$$\begin{aligned} \dot{x} &= \phi(x,y)x + \psi(x,y)y, \\ \dot{y} &= -\psi(x,y)x + \phi(x,y)y. \end{aligned} \tag{$\dagger$}$$

Give conditions on ϕ, ψ which guarantee that the annulus $A = \{(x,y) \mid 1 \leq x^2 + y^2 \leq 2\}$ is forward invariant and contains a periodic orbit for ($\dagger$). Ensure that your conditions are non-vacuous—that is, that there do exist ϕ, ψ satisfying your conditions.

32. (a) Given a parameter $\alpha \in \mathbb{R}$, consider the following system, which is ($\dagger$) from Question 31 with $\psi(x,y) = -1$ and $\phi(x,y) = \alpha - x^2 - y^2$.

$$\begin{aligned} \dot{x} &= -y + x(\alpha - x^2 - y^2), \\ \dot{y} &= x + y(\alpha - x^2 - y^2). \end{aligned} \tag{$\ddagger$}$$

Find the stability of the equilibrium point at 0 for each value of $\alpha \in \mathbb{R}$.

(b) Consider the system ($\ddagger$) with $0 < \alpha < 1$, and fix $z_0 = (x_0, y_0)$ with $x_0^2 + y_0^2 > \alpha$. **Using the definition of ω-limit set**, show directly that $\omega(z_0) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = \alpha\}$. *Hint: It may help to use polar coordinates.*