# NAIMARK'S PROBLEM *for* GRAPH $C^*$-ALGEBRAS

A Dissertation Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Nishant Suri

May 2017

**NAIMARK'S PROBLEM *for* GRAPH $C^*$-ALGEBRAS**

Nishant Suri

APPROVED:

Dr. Mark Tomforde (Committee Chair)
Department of Mathematics
University of Houston

Dr. Mehrdad Kalantar
Department of Mathematics
University of Houston

Dr. William Ott
Department of Mathematics
University of Houston

Dr. Scott LaLonde
Department of Mathematics
University of Texas at Tyler

Dean, College of Natural Sciences and Mathematics

*To my parents, who gave me their love, and their genes.*

# Acknowledgements

There are many people without whom I could not have accomplished this work. Foremost among them are my parents. To them I owe a debt greater than one I can put into words. Apart from expressing my gratitude for the efforts and sacrifices they have made in order to best meet the moral responsibility of parenthood, and for the love and support and strength and joy they have given me, I can only hope that I am capable of meeting my own moral responsibilities with the grace they demonstrated in meeting theirs. I am also grateful to my sister, whose love and friendship I treasure.

Here in Houston, I was astoundingly fortunate to have found friends who cared about me and whose company I enjoyed. They became my family, and anchored me so I could take on the tempests of everyday life. They helped me keep my strength and my sanity. My deepest thanks to Aarti, Akshay, Anando, Ananya, James, Nandini, Pankaj, Parth, and Rohit. I will forever cherish the times we have spent together.

I am also thankful to the Department of Mathematics. In it, I found a warm and welcoming community of staff, faculty and fellow students who were always friendly, professional and helpful. I am especially indebted to my teachers from whom I learned the fundamentals of the subject. Many of the best educators I have had the good fortune to be able to learn from are here in the Department of Mathematics. Among them is Dr. William Ott. I am hard pressed to recall a teacher I have had who was more dedicated, organized, empathetic and inspirational.

Finally, I would like to thank the person most directly responsible for helping me succeed in research as a graduate student: my advisor, Dr. Mark Tomforde. Without his unwavering support and constant guidance I would have been as lost as a drunken sailor in a raging storm on a starless night. I think it's fair to say I wouldn't have made it. I have benefitted immensely from his expertise and his experience, and I shall forever be grateful for the dedication with which he has endeavored to help me succeed.

# NAIMARK'S PROBLEM *for* GRAPH $C^*$-ALGEBRAS

An Abstract of a Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Nishant Suri

May 2017

# Abstract

*Naimark's problem* asks if a $C^*$-algebra with exactly one irreducible representation up to unitary equivalence is isomorphic to $K(\mathcal{H})$, the algebra of compact operators on some Hilbert space $\mathcal{H}$. A $C^*$-algebra that satisfies the premise of this question but not its conclusion is a *counterexample to Naimark's problem.* It is known that neither separable $C^*$-algebras nor Type I $C^*$-algebras can be counterexamples to Naimark's problem. In 2004, Akemann and Weaver constructed an $\aleph_1$-generated counterexample using Jensen's $\diamondsuit$ axiom (pronounced "diamond axiom"), which is known to be independent of ZFC. In fact, they showed that the existence of an $\aleph_1$-generated counterexample is independent of ZFC. The general problem remains open. In this thesis we focus on Naimark's problem for a subclass of $C^*$-algebras called *graph $C^*$-algebras.* We show that *approximately finite-dimensional* (denoted AF) graph $C^*$-algebras cannot be counterexamples to Naimark's problem. We also show that, as a consequence, $C^*$-algebras of row-countable graphs cannot be counterexamples to Naimark's problem. Since $C^*$-algebras with unique irreducible representations up to unitary equivalence must be simple, and since simple graph $C^*$-algebras are either AF or purely infinite, a complete answer to Naimark's problem for all graph $C^*$-algebras now hinges on an examination of the class of purely infinite graph $C^*$-algebras.

# Contents

# CHAPTER 1

## Introduction and Historical Background

### 1.1 $C^*$-algebras

At the beginning of the 20th century, the mathematician Fredholm began an investigation of integral equations of the type

$$u(y) - \int_0^1 k(y,x)u(x)\mathrm{d}x = f(y),$$

where the problem was to find the function $u$, given the functions $k$ and $f$ [5]. Such equations arose in connection with theoretical physics, in particular the Dirichlet problem, and today find applications in signal processing. Fredholm realized that since the integral can the thought of as a limit of finite sums, such an integral equation could itself be viewed as an infinite-dimensional counterpart to a finite-dimensional system of linear algebraic equations represented by the matrix equation

$$\vec{u} - K\vec{u} = \vec{f},$$

with finite dimensional vectors $\vec{u}$ and $\vec{f}$ being replaced by functions and the matrix $K$ being replaced by the *integral operator*

$$T_k = \int_0^1 k(y,x) \cdot dx \quad ; \quad u \mapsto T_k(u) = \int_0^1 k(y,x)u(x)dx.$$

It was soon recognized that such integral operators are best viewed as continuous (bounded) linear operators on the *normed vector space* of *square integrable functions* on the compact interval $[0,1]$ (i.e., functions $u$ on $[0,1]$ such that $\int_0^1 |u(x)|^2 dx < \infty$), with addition and scalar multiplication of functions given by point-wise addition and point-wise scalar multiplication respectively, and the norm of a square integrable function $u$ given by

$$\|u\| = \left( \int_0^1 |u(x)|^2 dx \right)^{1/2}.$$

This norm is *complete* (in the sense of Cauchy), and is induced by an inner product

$$\langle u, v \rangle = \int_0^1 u(x)\overline{v(x)}dx.$$

The square integrable functions on $[0,1]$ therefore constitute a *Hilbert space,* denoted $L^2([0,1])$.

**Definition 1.1.1** (Hilbert Space)**.** *A* Hilbert space $\mathcal{H}$ *is a real or complex vector space with an inner product, such that* $\mathcal{H}$ *is complete (in the sense of Cauchy) with respect to the norm induced by the inner product.*

This shift in viewpoint marked the birth of the study of operator algebras. Fredholm set forth on a systematic exploration of the relationship between integral equations and matrix equations, and found, for instance, that the kernel and cokernel of an integral operator both have finite, and equal, dimension. His work sparked the interest of Hilbert, who made a detailed study of Fredholm's integral operators in the special case where the function $k$ is symmetric in the variables $x$ and $y$. These symmetric integral operators are infinite-dimensional counterparts of real symmetric matrices, and Hilbert showed that they are diagonalizable, just as real symmetric matrices are [8]. In other words, given a symmetric integral operator $T_k$, there exist infinitely many functions $u_1, u_2, u_3, \ldots$ which satisfy

$$T_k(u_n) = \int_0^1 k(y,x)u_n(x)dx = \lambda_n u_n(y)$$

for some scalars $\lambda_n$, such that any function $f(y)$ in $L^2([0,1])$ may be written as an infinite linear combination

$$f(y) = \sum a_n u_n(y)$$

of them, with the coefficient $a_n$ recoverable from the inner product of $f$ and $u_n$, and the sum converging in the $L^2$-norm. The functions $u_n$ are called *eigenfunctions* of the operator, and they are mutually orthogonal. Hilbert's result may be restated more succinctly as the fact that eigenfunctions of a symmetric integral operator form an *orthogonal basis* of $L^2([0,1])$. This result is known as the *spectral theorem for symmetric integral operators,* so called because the *spectrum* of an operator $T$ on a Hilbert space $\mathcal{H}$ is the collection of all scalars $\lambda$ such that the operator $(T - \lambda I)$ does not have a bounded inverse. This generalizes the notion of the set of eigenvalues of a matrix. The theorem ignited a flurry of activity in the field. For instance, Koopman observed [11] that a measure-preserving automorphism $\phi$ on a measure space $X$ induces a *unitary* (i.e. bounded, surjective and inner-product preserving) operator $U$ on $L^2(X)$ given by

$$U(f)(x) = f(\phi(x)).$$

Building on this observation, von Neumann discovered a spectral theorem for unitary operators that mirrored Hilbert's result for symmetric ones, and used it to show that if no non-constant function in $L^2(X)$ is fixed by $U$ (which is one way of saying that the transformation which induces $U$ is "sufficiently complicated", or *ergodic*), then, for every function $f \in L^2(X)$, the sequence

$$\frac{1}{n}\sum_{k=1}^{n} U^k f$$

of averages converges in the strong operator topology to the constant function whose value everywhere is the average value of $f$ [22]. This result is known as the *mean ergodic theorem*, and its discovery laid the foundation of the subject now known as *ergodic theory.*

It was in the 1920s, however, that operator theory underwent a renaissance triggered by the development of quantum mechanics. The key realization that revitalized the field was that observables of a physical system such as energy and momentum can be viewed as *self-adjoint* operators on a complex Hilbert space corresponding to that system, i.e., as operators that coincide with their *adjoints.* (The adjoint of an operator $T$ on a Hilbert space $\mathcal{H}$ is the unique operator $T^*$ on $\mathcal{H}$ such that

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

for every $x, y$ in $\mathcal{H}$.) The set of all observed values of the observable is then precisely the spectrum of the associated self-adjoint operator. Building on the ideas of Heisenberg, Shrödinger, Dirac, Weyl, Jordan, and others, von Neumann began the axiomatization of quantum theory in order to provide the mathematical setting for modeling the algebra of observables. This led to a deep structural analysis of the operator algebras we now call *von Neumann algebras.* In 1943, Gelfand and Naimark introduced the notion of a $C^*$-*algebra* (pronounced "C star algebra") as an abstraction of those subalgebras of the bounded linear operators on a complex

Hilbert space $\mathcal{H}$ (denoted $B(\mathcal{H})$), which are topologically closed under the operator norm and algebraically closed under taking adjoints [6]. Such subalgebras of $B(\mathcal{H})$ are sometimes called *concrete $C^*$-algebras.*

**Definition 1.1.2** ($C^*$-algebra)**.** *A* $C^*$-algebra *is a complete normed complex algebra closed under a unary operation called an* involution, *denoted* $*$ *and pronounced "star", satisfying the following conditions for all elements* $x$ *and* $y$ *of the algebra and all complex scalars* $\lambda$ :

*(1)* $(x^*)^* = x$

*(2)* $(\lambda x)^* = \overline{\lambda} x^*$

*(3)* $(x + y)^* = x^* + y^*$

*(4)* $(xy)^* = y^* x^*$

*(5)* $\|x^*\| = \|x\|$

*(6)* $\|xx^*\| = \|x\|^2$ .

The involution is precisely the adjoint in a concrete $C^*$-algebra. A *homomorphism of $C^*$-algebras* is a map between $C^*$-algebras that preserves all this structure. It turns out that it is enough for a map to be a $*$-algebra homomorphism, often simply called a $*$-homomorphism, in order for it to be a homomorphism of $C^*$-algebras.

**Definition 1.1.3** ($*$-homomorphism)**.** *A map* $\phi : A \to B$ *between two* $C^*$-*algebras is a* $*$-homomorphism *if it preserves the* $*$-*algebra structure, i.e., if it is linear and satisfies*

$$\phi(xy) = \phi(x)\phi(y) \quad \text{and} \quad \phi(x^*) = \phi(x)^*$$

*for all elements* $x$ *and* $y$ *in* $A$.

Condition (6) in the definition of a $C^*$-algebra, called the *$C^*$-identity,* is a particularly restrictive constraint and has the consequence that $*$-homomorphisms are automatically contractive, and therefore continuous. It also implies that an injective

$*$-homomorphism is automatically isometric. $C^*$-algebras with $*$-homomorphisms constitute a category, with the bijective $*$-homomorphisms, called $*$-isomorphisms, being the isomorphisms in the category.

Gelfand showed that a commutative $C^*$-algebra is isomorphic to $C_0(\Omega)$, the algebra of continuous complex-valued functions on a locally-compact Hausdorff topological space $\Omega$ that vanish at infinity. This correspondence is known as the *Gelfand representation.* In particular, a commutative $C^*$-algebra is *unital* if and only if the corresponding topological space $\Omega$ in its Gelfand representation is compact.

The prototypical example of a $C^*$-algebra is $M_n(\mathbb{C})$, the algebra of $n \times n$ square matrices with complex entries. This is unital, and noncommutative for $n \geq 2$. In the special case when $n = 1$, it yields the algebra $\mathbb{C}$ of complex numbers. Operators on $\mathbb{C}^n$ written as matrices generalize to bounded operators on infinite-dimensional Hilbert spaces. Since $C^*$-algebras are defined as abstractions of closed and self-adjoint subalgebras of $B(\mathcal{H})$, it is no surprise that $B(\mathcal{H})$ is itself a $C^*$-algebra for any Hilbert space $\mathcal{H}$.

The smallest non-zero closed ideal in $B(\mathcal{H})$ is the closure of operators of finite rank, and is denoted $K(\mathcal{H})$. Operators in $K(\mathcal{H})$ are called *compact operators.*

**Definition 1.1.4** (Compact Operator)**.** *A* compact operator *on a Hilbert space $\mathcal{H}$ is one that can be approximated in the norm topology by operators of finite rank. A compact operator takes bounded subsets to precompact ones. The algebra of all compact operators on $\mathcal{H}$ is denoted $K(\mathcal{H})$.*

The algebra $K(\mathcal{H})$ is also noncommutative whenever $dim(\mathcal{H}) \geq 2$, and it is unital if and only if $dim(\mathcal{H}) < \infty$.

## 1.2 Representations of $C^*$-algebras

In the study of abstract groups, it is useful to *represent* group elements as linear transformations of a vector space, and study these concrete objects instead. Indeed, one can learn a lot about an abstract group by studying all of the various ways in which it can be represented on vector spaces. This approach reduces abstract group-theoretic problems to problems in linear algebra, a subject that is well-understood and is more amenable to computations. In a similar manner, one often wishes to study an abstract $C^*$-algebra by representing its elements as bounded linear operators on a Hilbert space, i.e., by representing it as a concrete $C^*$-algebra. Studying all possible such representations illuminates the structure of the $C^*$-algebra.

**Definition 1.2.1** ($C^*$-algebra Representation)**.** *A* representation *of a $C^*$-algebra $A$ is a $*$-homomorphism $\pi : A \to B(\mathcal{H})$ from $A$ into the bounded linear operators on some Hilbert space $\mathcal{H}$. In this case, $A$ is said to be* represented on $\mathcal{H}$*.*

An injective (and consequently isometric) representation is also called *faithful*.

**Definition 1.2.2** (Unitary Equivalence)**.** *Two representations*

$$\pi : A \to B(\mathcal{H}_\pi) \quad \text{and} \quad \rho : A \to B(\mathcal{H}_\rho)$$

*of $A$ are* unitarily equivalent*, denoted $\pi \sim_u \rho$, if there is a unitary operator $U : \mathcal{H}_\pi \to \mathcal{H}_\rho$ such that $\pi(a) = U^*\rho(a)U$ for every $a \in A$.*

It is easy to check that $\sim_u$ is an equivalence relation on representations of $A$. The set of equivalence classes is called the *spectrum of* $A$ and is denoted $\operatorname{Spec} A$. Unitarily equivalent representations are geometrically indistinguishable — the unitary operator simply being a change of coordinates. Unsurprisingly, then, unitarily equivalent representations share the same kernel.

A nonzero vector $\xi$ in $\mathcal{H}$ is *cyclic* for a representation $\pi$ of $A$ on $\mathcal{H}$ if

$$\pi(A)(\xi) := \{\pi(a)(\xi) : a \in A\}$$

is norm-dense in $\mathcal{H}$. A representation which has a cyclic vector is itself called a *cyclic representation.* The *Gelfand-Naimark-Segal construction* (*GNS construction* for short) establishes a correspondence between equivalence classes of cyclic representations of a $C^*$-algebra $A$ on the one hand and positive linear functionals of unit norm on $A$ on the other. These functionals are called *states* on $A$, because in the $C^*$-algebraic formulation of quantum mechanics they correspond to states of a physical system, i.e., mappings which take physical observables, modeled as self-adjoint elements of the $C^*$-algebra, to their expected measurement outcomes.

The set of all states on a $C^*$-algebra $A$, called the *state space of* $A$ and denoted $\mathrm{S}(A)$, is a convex and weak*-compact subset of the Banach dual $A^*$ of $A$. Consequently it has extreme points, and by the *Krein-Milman theorem,* it is the weak*-closure of their convex-hull. These extreme points are called *pure states.* Pure states are "atomic" in the sense that they cannot be approximated (in the weak*-topology) by convex-linear combinations of *other* states. Recall that the Gelfand representation shows that every commutative $C^*$-algebra $A$ is of the form $C_0(\Omega)$, the algebra of complex-valued continuous functions on a locally compact Hausdorff space $\Omega$ that vanish at infinity. In this special case, the states on $A$ correspond to positive Radon measures on $\Omega$, with the pure states on $A$ corresponding to the evaluation functionals

$$\xi_x : C_0(\Omega) \to \mathbb{C} \ ; \ f \mapsto f(x)$$

for every point $x$ in $\Omega$. States may therefore be viewed as non-commutative generalizations of positive Radon measures. Furthermore, the correspondence between states on a $C^*$-algebra $A$ and cyclic representations on $A$ established by the GNS construction restricts to a correspondence between pure states on $A$ and special cyclic representations of $A$ called *irreducible representations.* These representations are themselves "atomic" in an important sense which is made clear in the following discussion.

A *subrepresentation* of $\pi$ is a representation $\pi_W : A \to B(W)$ of $A$ on a closed, $\pi$*-invariant* subspace $W$ of $\mathcal{H}$. (A subspace $W$ is $\pi$-invariant if $\pi(a)(W) \subseteq W$ for every $a$ in $A$.) It is easy to see that if $W$ is a non-trivial, proper, $\pi$-invariant closed subspace of $\mathcal{H}$, then so is its *orthogonal compliment*

$$W^{\perp} := \{\xi \in \mathcal{H} : \text{ there exists } \ w \in W \text{ such that } \langle \xi, w \rangle = 0\}.$$

Consequently, given a proper, non-trivial subrepresentation $\pi_W$ of $\pi$, we may decompose $\pi$ as a *direct sum of subrepresentations* $\pi = \pi_W \oplus \pi_{W^\perp}$ defined by

$$x \mapsto \pi_W(p_W(x)) + \pi_{W^\perp}(p_{W^\perp}(x)),$$

where $p_W$ is the projection onto $W$ and $p_{W^\perp}$ is the projection onto $W^{\perp}$.

**Definition 1.2.3** (Irreducible Representation)**.** *A representation $\pi$ is* irreducible *if it has no proper, non-trivial subrepresentations, and can therefore not be decomposed non-trivially as a direct sum of subrepresentations. In other words, $\pi : A \to B(\mathcal{H})$ is irreducible if the only closed proper subspace of $\mathcal{H}$ that is invariant under the $A$-action $\pi$ is $\{0\}$.*

An ideal $I$ in $A$ is called a *primitive ideal* if it is the kernel of some irreducible representation of $A$. Since unitarily equivalent representations share the same kernel, we have a correspondence between irreducible representations of a $C^*$-algebra $A$ and primitive ideals in $A$. In particular, a $C^*$-algebra with a unique irreducible representation up to unitary equivalence has a unique primitive ideal. The following lemma shows that in this case the primitive ideal is zero, and moreover that any such $C^*$-algebra is simple.

**Lemma 1.3.** *If $A$ is a $C^*$-algebra with a unique irreducible representation up to unitary equivalence, then $A$ is simple.*

*Proof.* It is a standard result that every closed proper ideal in a $C^*$-algebra is the intersection of the primitive ideals containing it. (See [19, Proposition A.17, p.212] for a statement and proof.) Since $A$ has only one irreducible representation up to unitary equivalence, $A$ has a unique primitive ideal $I$. Thus, every closed proper ideal of $A$ must equal $I$. Since $\{0\}$ is a closed proper ideal of $A$, it follows that any closed proper ideal of $A$ is equal to $\{0\}$. In particular, $A$ is simple. □

The set of all primitive ideals of $A$ is called the *primitive spectrum of $A$* and is denoted $\operatorname{Prim} A$.

After proposing their definition of a $C^*$-algebra, Gelfand and Naimark used the GNS construction to show that every $C^*$-algebra can be isometrically embedded in some $B(\mathcal{H})$ as a concrete $C^*$-algebra. (See [19, Theorem A.11, p.207] for a statement and proof.) This parallels Cayley's theorem from group theory and Whitney's embedding theorem from manifold theory, in that it confirms that a "nice" abstraction of a concrete $C^*$-algebra exists, and that Gelfand and Naimark's definition captures it. In particular, this means that the primitive spectrum (and therefore also the spectrum) of a $C^*$-algebra is never empty.

## 1.3 Naimark's Problem

In 1948, Naimark observed that every irreducible representation of $K(\mathcal{H})$ is unitarily equivalent to the inclusion representation $i : K(\mathcal{H}) \hookrightarrow B(\mathcal{H})$ ; $T \mapsto T$ [14]. Said differently, for any Hilbert space $\mathcal{H}$, the $C^*$-algebra $K(\mathcal{H})$ of compact operators on $\mathcal{H}$ has a unique irreducible representation up to unitary equivalence. In 1951, he asked whether this property characterized the $C^*$-algebras $K(\mathcal{H})$ [15]. In other words, if $A$ is a $C^*$-algebra with only one irreducible representation up to unitary equivalence, is $A$ isomorphic to $K(\mathcal{H})$ for some Hilbert space $\mathcal{H}$? This question is known as *Naimark's problem.* A $C^*$-algebra which satisfies the premise of this question but not its conclusion is a *counterexample to Naimark's problem.* In the years following its proposal, various results were obtained which provided partial answers to Naimark's problem.

In 1951, Kaplansky introduced the *GCR $C^*$-algebras* and began developing their representation theory [9]. Today these are more commonly known as *Type I $C^*$-algebras.*

**Definition 1.3.1** (Type I $C^*$-algebras)**.** *A $C^*$-algebra is* Type I *if the image of each of its irreducible representations contains the compact operators.*

It is known that no Type I $C^*$-algebra can be a counterexample to Naimark's problem. Indeed, this is not hard to see.

**Theorem 1.3.1.** *If $A$ is a Type I $C^*$-algebra with $T : A \to B(\mathcal{H})$ its only irreducible representation up to unitary equivalence, then $A \cong K(\mathcal{H})$ via $T$.*

*Proof.* Lemma 1.3 shows that $A$ is simple. Together with the fact that $K(\mathcal{H}) \subseteq \operatorname{im} T$, this forces $\ker T = \{0\}$. In other words, $A \cong \operatorname{im} T$ via $T$. Since $K(\mathcal{H})$ is an ideal in $B(\mathcal{H})$ and therefore an ideal in $\operatorname{im} T$, the simplicity of $A$ in turn forces $\operatorname{im} T = K(\mathcal{H})$. $\square$

Naimark's problem was quickly settled for separable $C^*$-algebras, when in 1953 A. Rosenberg proved the following theorem.

**Theorem 1.3.2** (A. Rosenberg)**.** [20, Theorem 4] *No separable $C^*$-algebra is a counterexample to Naimark's problem. Explicitly, if $A$ is a separable $C^*$-algebra with a unique irreducible representation up to unitary equivalence, then $A$ is isomorphic to $K(\ell^2(\mathbb{N}))$.*

Building on Kaplansky's ideas, Fell showed in 1960 that any two irreducible representations of a Type I $C^*$-algebra that have equal kernels must be unitarily equivalent [4]. That same year, Dixmier proved a partial converse: a *separable* $C^*$-algebra that is not Type I necessarily has unitarily *inequivalent* representations whose kernels are equal [3]. In fact, in 1961 Glimm showed that a separable $C^*$-algebra that is not Type I has *uncountably many* inequivalent irreducible representations [7]. These two results partially recovered Rosenberg's 1953 result of Theorem 1.3.2.

In 2004, however, Akemann and Weaver used Jensen's $\diamondsuit$ axiom (pronounced "diamond axiom"), which is a combinatorial principle known to be independent of ZFC, to construct a counterexample to Naimark's problem that is generated by $\aleph_1$ elements [1]. In fact, they showed that the existence of an $\aleph_1$-generated counterexample is independent of ZFC. Akemann and Weaver's result suggests that there are set-theoretic obstructions to obtaining a general answer to Naimark's problem. In light of this, it is reasonable to consider restrictions of the problem to special subclasses of $C^*$-algebras. In this thesis we focus attention on *graph $C^*$-algebras.*

# CHAPTER 2

## Preliminaries on Graph $C^*$-Algebras

### 2.1 Directed Graphs

**Definition 2.1.1** (Directed Graph)**.** *A* directed graph $(E^0, E^1, r, s)$ *consists of a set $E^0$ of vertices, a set $E^1$ of edges between them, and range and source maps*

$$r : E^1 \to E^0 \quad \textit{and} \quad s : E^1 \to E^0$$

*identifying, respectively, the range and source of each edge.*

From here on, we shall dispense with the qualifier *directed* in an understanding that all graphs under discussion are directed.

A vertex $v$ in $E^0$ is a *sink* if no edges come out of it (i.e., $s^{-1}(v)$ is empty), and it is an *infinite emitter* if infinitely many do (i.e., $|s^{-1}(v)| = \infty$). Sinks and infinite emitters are classified as *singular vertices.* Vertices that are neither sinks nor infinite emitters are called *regular vertices.* A graph is called *row-finite* if every vertex $v$ emits finitely many edges (i.e., $s^{-1}(v)$ is a finite, possibly empty, set), and it is called *row-countable* if every vertex $v$ emits countably many edges (i.e., $s^{-1}(v)$ is a countable, possibly empty, set). A graph $E$ is *finite* if both $E^0$ and $E^1$ are finite sets, and it is *countable* if both $E^0$ and $E^1$ are countable sets.

A *path* $\alpha = e_1 \dots e_n$ in a graph $E$ is a finite succession of edges in which the range vertex of any edge coincides with the source vertex of the next edge in the succession (i.e., $r(e_i) = s(e_{i+1})$ for all $1 \leq i \leq n-1$), and we say that such a path has *length* $|\alpha| = n$. We consider vertices to be paths of length zero (also called empty paths) and edges to be paths of length one. We also let $E^n$ denote the set of all paths of length $n$ in $E$, and we let $E^* := \bigcup_{n=0}^{\infty} E^n$ denote the set of all paths in $E$. We extend the range and source maps to $E^*$ in the obvious way: If $\alpha = e_1 \dots e_n$ then $r(\alpha) := r(e_n)$ and $s(\alpha) := s(e_1)$. An *infinite path* $e_1 e_2 \dots$ is an infinite succession of edges with $r(e_i) = s(e_{i+1})$ for all $i \in \mathbb{N}$. We let $E^\infty$ denote the set of all infinite paths in $E$, and we extend the source map $s$ to $E^\infty$ in the obvious way: If $\alpha = e_1 e_2 \dots$, then $s(\alpha) := s(e_1)$. A *cycle* is a path $\alpha \in E^*$ such that $s(\alpha) = r(\alpha)$. If $\alpha = e_1 \dots e_n$ is a cycle, an *exit* for $\alpha$ is an edge $f \in E^1$ such that $s(f) = s(e_i)$ and $f \neq e_i$ for some $1 \leq i \leq n$. A graph is said to satisfy *Condition (L)* if every cycle in $E$ has an exit.

If $v, w \in E^0$, we say $v$ *can reach* $w$, written $v \geq w$, if there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. A graph is called *cofinal* if whenever $v \in E^0$ and $\alpha := e_1 e_2 \dots \in E^\infty$, then $v \geq s(e_i)$ for some $i \in N$. A subset $H \subseteq E^0$ is called *hereditary* if whenever $e \in E^1$ and $s(e) \in H$, then $r(e) \in H$. A hereditary subset $H$ is called *saturated* if whenever $v$ is a regular vertex and $r(s^{-1}(v)) \subseteq H$, then $v \in H$.

## 2.2 Graph $C^*$-algebras

As a locally compact Hausdorff space $\Omega$ contains enough topological information to produce the commutative $C^*$-algebra $C(\Omega)$, and a discrete group $G$ contains enough algebraic information to produce its group $C^*$-algebra $C^*(G)$, so too a graph $E$ contains enough combinatorial information to produce its *graph $C^*$-algebra* $C^*(E)$. Explicitly, there is a natural recipe for building a $C^*$-algebra from a graph. We may describe this recipe from several viewpoints. One viewpoint that helps make it concrete and transparent is the following.

Given a graph $E = (E^0, E^1, r, s)$, recall that $E^*$ is the set of all (finite, possibly empty) paths in $E$ and $E^\infty$ is the set of all infinite paths in $E$. For this section, we shall denote the set $E^\infty \cup (E^* \setminus E^0)$ of *all* non-empty paths, finite or infinite, by $\Pi$. Each $\delta \in E^*$ (partially) acts on the set $\Pi$ by pre-concatenation with its elements whenever possible:

$$\delta(\pi) = \delta\pi, \ \ \pi \in \Pi.$$

We also define the (partial) inverse of this action as pre-truncation of paths beginning with $\delta$:

$$\delta^*(\delta\pi) = \pi, \ \ \pi \in \Pi.$$

Consequently we have $\delta\delta^*\delta = \delta$ and $\delta^*\delta\delta^* = \delta^*$. This action makes $\delta$ a partial bijection on $\Pi$. Consequently $E^*$ can be viewed as an inverse semigroup. Taking this viewpoint, every (possibly empty) finite path *becomes* a bijection from one subset of $\Pi$ to another. We denote the range of the *partial bijection* $\delta \in E^*$ by $range(\delta)$. (Not to be confused with the *range vertex* of the *path* $\delta$, which is denoted by $r(\delta)$.) If the path $\delta$ is empty (i.e., it is just a vertex, say $v$), then it is, in fact, a partial identity on $\Pi$, and we have $v = v^* = v^2$. Also, different vertices have disjoint ranges in $\Pi$. On the other hand, if $\delta$ is a non-empty finite path $e_1e_2e_3\ldots e_n$, where the $e_i$'s are edges, then every path of the form $\delta(\pi) = \delta\pi$ may be written as $e_1e_2e_3\ldots e_n\pi$, and the partial inverse $\delta^*$ may be written as $e_n^*\ldots e_3^*e_2^*e_1^*$, so that the action of $\delta$ is the composition of the actions of its edges. Again, different edges have disjoint ranges in $\Pi$. Therefore the set $E^0 \cup E^1 = \{v, e : v \in E^0, e \in E^1\}$ of vertices and edges of $E$, subject to the relations

(1) $v = v^* = v^2$

(2) $ee^*e = e$

(3) $e^*e = r(e)$

(4) $range(ee^*) \subseteq range(s(e))$

(5) $range(v) = \coprod_{s(e)=v} range(ee^*)$

generates the inverse semigroup $E^*$.

To move from an inverse semigroup to a $C^*$-algebra, we simply let $E^*$ act in a similar manner, via bounded linear operators, on the Hilbert space $\ell^2(\Pi)$ with orthonormal basis $\{\xi_\pi\}$ indexed by the set of all (possibly infinite) non-empty paths $\pi \in \Pi$. Concretely, for every vertex $v$ in $E$ and every edge $e$ in $E$, define the bounded linear operators $p_v$ and $s_e$ on on $\ell^2(\Pi)$ by

$$p_v(\xi_\pi) := \begin{cases} \xi_\pi & \text{if } \pi \text{ begins at } v \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad s_e(\xi_\pi) := \begin{cases} \xi_{e\pi} & \text{if } e\pi \text{ is a path} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the relations on the vertices and edges that determined the inverse semigroup $E^*$ are carried over appropriately as relations on the $p_v$'s and $s_e$'s in this new setting:

(1) $p_v = p_v^* = p_v^2$

(2) $s_es_e^*s_e = s_e$

(3) $s_e^*s_e = p_{r(e)}$

(4) $s_es_e^* \leq p_{s(e)}$

(5) $p_v = \sum_{s(e)=v} s_es_e^*$ whenever $v \in E^0$ is a regular vertex.

In particular, relation (1) says that the $p_v$'s are projections in $B(\ell^2(\Pi))$ and relation (2) says that the $s_e$'s are partial isometries in $B(\ell^2(\Pi))$. It follows from their definitions that the $p_v$'s are mutually orthogonal, and that the $s_e$'s have mutually orthogonal ranges. Also, for a path $\delta = e_1 e_2 \dots e_n$, we denote the composition $s_{e_1} s_{e_2} \dots s_{e_n}$ of partial isometries (which is itself a partial isometry) by $s_\delta$. The set $\{p_v, s_e : v \in E^0, e \in E^1\}$, together with the relations above, generates a $C^*$-subalgebra of $B(\ell^2(\Pi))$. This $C^*$-algebra encodes much of the combinatorial information contained in the graph, and the recipe described above can serve as a bridge between the study of graphs on the one hand and of their associated $C^*$-algebras on the other. However, we would like to define the graph $C^*$-algebra of a graph $E$ in a manner that is independent of the underlying Hilbert space $\ell^2(\Pi)$. This motivates the following.

**Definition 2.2.1** (Graph $C^*$-algebra)**.** *The* graph $C^*$-algebra $C^*(E)$ *of a graph* $E := (E^0, E^1, r, s)$ *is the universal $C^*$-algebra generated by mutually orthogonal projections* $\{p_v : v \text{ is a vertex in } E\}$ *together with partial isometries with mutually orthogonal ranges* $\{s_e : e \text{ is an edge in } E\}$ *satisfying*

*(CK1)* $s_e^* s_e = p_{r(e)}$ *for all* $e \in E^1$*,*

*(CK2)* $s_e s_e^* \leq p_{s(e)}$ *for all* $e \in E^1$*, and*

*(CK3)* $p_v = \sum_{s(e)=v} s_e s_e^*$ *whenever* $v \in E^0$ *is a regular vertex.*

Such a universal $C^*$-algebra does indeed exist, and its uniqueness is an immediate consequence of universality. The three relations above are called the *Cuntz-Krieger relations.* Given a graph $E$, a family $\{p_v, s_e : v \in E^0, s \in E^1\}$ of mutually orthogonal projections $p_v$ and partial isometries $s_e$ with mutually orthogonal ranges that additionally satisfies the Cuntz-Krieger relations is called a *Cuntz-Krieger E-family.* One can use the Cuntz-Krieger relations to show that $C^*(E) = \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*\}$. Moreover, $C^*(E)$ is separable if and only if $E$ is a countable graph.

The graph $C^*$-algebras are a rich and tractable class. They include, for instance, the finite-dimensional $C^*$-algebras, $K(\mathcal{H})$ for $\mathcal{H}$ not necessarily separable, $n \times n$ matrices over continuous functions on the circle, the Toeplitz algebras, the Cuntz algebras, and the Cuntz-Krieger algebras. Also, all separable AF algebras, as well as all Kirchberg algebras with free $K_1$-group, are Morita equivalent to graph $C^*$-algebras. Furthermore, the class of graph $C^*$-algebras is closed under direct sums, stabilization, and tensoring by $M_n(\mathbb{C})$.

## 2.3 Standard Results for Graph $C^*$-algebras

There are numerous results relating the structure of a graph to the structure of its graph $C^*$-algebra. In this section, we state (with reference) those that we shall need.

*Remark* 2.3.1. Many of these results, as stated in the cited literature, rely on additional assumptions such as row-finiteness of the graph or separability of the graph $C^*$-algebra. Nevertheless, the proofs extend to the more general cases stated.

**Theorem 2.3.1.** [21, Theorem 12] [16, Theorem 4] *Let $E$ be a directed graph. Then the following three conditions are equivalent:*

*(1) $C^*(E)$ is simple.*

*(2) $E$ satisfies Condition (L), $E$ is cofinal, and every vertex of $E$ can reach every singular vertex of $E$.*

*(3) $E$ satisfies Condition (L) and the only saturated hereditary subsets of $E$ are $E^0$ and $\emptyset$.*

In this thesis, we call a $C^*$-algebra an *AF algebra*, or say the $C^*$-algebra is *AF* (short for approximately finite-dimensional), if it is the (generalized) direct limit of finite-dimensional $C^*$-algebras. Note that this differs from the standard usage of the term, which typically requires a *sequential* direct limit, and therefore implies AF algebras are separable. Our notion of AF coincides with the usual definition for separable $C^*$-algebras, but also allows for nonseparable AF algebras, which are direct limits of *directed sets* of finite-dimensional $C^*$-algebras. The following result gives a very nice characterization of AF graph $C^*$-algebras.

**Theorem 2.3.2.** [18, §5.4] *If $E$ is a graph, then $C^*(E)$ is AF if and only if $E$ has no cycles.*

If $E = (E^0, E^1, r_E, s_E)$ is a graph, a *subgraph* of $E$ is a graph $F = (F^0, F^1, r_F, s_F)$ such that $F^0 \subseteq E^0$, $F^1 \subseteq E^*$, and $r_F$ and $s_F$ are restrictions of the range and source maps that $r_E$ and $s_E$ induce on $E^*$ such that $\operatorname{im} r_F \subseteq F^0$ and $\operatorname{im} s_F \subseteq F^0$.

If $H$ is a hereditary subset of the graph $E = (E^0, E^1, r, s)$, the *restriction of $E$ to $H$* is the graph $E_H := (E^0_H, E^1_H, r_H, s_H)$ with vertex set $E^0_H := H$, edge set $E^1_H := s^{-1}(H)$, and range and source maps $r_H := r|_{E^1_H}$ and $s_H := s|_{E^1_H}$. In addition, we let $I_H$ denote the closed two-sided ideal in $C^*(E)$ generated by $\{p_v : v \in H\}$.

*Morita equivalence* is an equivalence relation on the class of all $C^*$-algebras that is weaker than isomorphism but strong enough to preserve the ideal structure and representation theory. Explicitly, if $C^*$-algebras $A$ and $B$ are Morita equivalent then their ideal-lattices (of closed, two-sided ideals) are lattice isomorphic, and this isomorphism takes primitive ideals to primitive ideals. Moreover, there is a correspondence between $\operatorname{Spec} A$ and $\operatorname{Spec} B$ that restricts to a correspondence between $\operatorname{Prim} A$ and $\operatorname{Prim} B$. This is known as the *Rieffel correspondence.* (See [19, §3.3] for a detailed discussion of the Rieffel correspondence.)

**Theorem 2.3.3.** [2, Proposition 3.4] *If $E$ is a graph and $H$ is a hereditary subset of $E$, then $C^*(E_H)$ is Morita equivalent to $I_H$.*

**Definition 2.3.1** (Relative Graph $C^*$-algebra)**.** [13, §3] *If $E := (E^0, E^1, r, s)$ is a graph, let $E^0_{\text{reg}}$ denote the regular vertices of $E$, and let $S \subseteq E^0_{\text{reg}}$. A* Cuntz-Krieger $(E, S)$-family *is a collection of elements $\{s_e, p_v : e \in E^1, v \in E^0\}$ in a $C^*$-algebra such that $\{p_v : v \in E^0\}$ is a collection of mutually orthogonal projections and $\{s_e : e \in E^1\}$ is a collection of partial isometries with mutually orthogonal ranges satisfying the* Cuntz-Krieger relations*:*

*(CK1)* $s_e^* s_e = p_{r(e)}$ *for all* $e \in E^1$*,*

*(CK2)* $s_e s_e^* \leq p_{s(e)}$ *for all* $e \in E^1$*, and*

*(CK3)* $p_v = \sum_{s(e)=v} s_e s_e^*$ *whenever* $v \in S$*.*

*The* relative graph $C^*$-algebra $C^*(E, S)$ *is the $C^*$-algebra generated by a universal Cuntz-Krieger $(E, S)$-family.*

Observe that if $S = E^0_{\text{reg}}$, then $C^*(E, E^0_{\text{reg}})$ is simply the graph $C^*$-algebra $C^*(E)$. If $S = \emptyset$, then $C^*(E, \emptyset)$ is called the Toeplitz algebra of $E$, and often denoted $TC^*(E)$.

If $C^*(E, S)$ is a relative graph $C^*$-algebra and $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a generating Cuntz-Krieger $(E, S)$-family in $C^*(E, S)$, then for any $v \in E^0_{\text{reg}} \setminus S$, we call $q_v := p_v - \sum_{s(e)=v} s_e s_e^*$ the *gap projection* at $v$.

If $I$ is the ideal generated by $\{q_v : v \in E^0_{\text{reg}} \setminus S\}$, then $C^*(E, S)/I \cong C^*(E)$, and hence the graph $C^*$-algebra is a quotient of the relative graph $C^*$-algebra $C^*(E, S)$.

In addition, whenever $E$ is a graph and $S \subseteq E^0_{\text{reg}}$ there exists a graph $E_S$ such that $C^*(E_S)$ is isomorphic to $C^*(E,S)$. Thus every relative graph $C^*$-algebra is isomorphic to a graph $C^*$-algebra (of a possibly different graph).

Furthermore, we have the following Cuntz-Krieger Uniqueness Theorem for relative graph $C^*$-algebras.

**Theorem 2.3.4.** [13, Theorem 3.11] *Let $E$ be a graph, let $S \subseteq E^0_{\mathrm{reg}}$, and let $\phi : C^*(E,S) \to A$ be a homomorphism from $C^*(E,S)$ into a $C^*$-algebra $A$. If $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a generating Cuntz-Krieger $(E,S)$-family in $C^*(E,S)$ and the following three conditions hold*

*(1) $E$ satisfies Condition (L),*

*(2) $\phi(p_v) \neq 0$ for all $v \in E^0$, and*

*(3) $\phi(q_v) \neq 0$ for all $v \in E^0_{\mathrm{reg}} \setminus S$,*

*then $\phi$ is injective.*

# CHAPTER 3

## Structure Results for certain Graph $C^*$-algebras

In this chapter we show that if $C^*(E)$ is AF and has a unique irreducible representation, then the graph $E$ necessarily has one of two specific forms.

*Remark* 3.0.1. All results proved from this point forward are ours.

## 3.1 Preliminary Lemmas

**Lemma 3.2.** *If $F$ is the graph*

$$w_1 \underset{f_1}{\overset{e_1}{\rightrightarrows}} w_2 \underset{f_2}{\overset{e_2}{\rightrightarrows}} w_3 \underset{f_3}{\overset{e_3}{\rightrightarrows}} \cdots \tag{3.2.1}$$

*with distinct vertices $w_1, w_2, \dots$ and distinct edges $e_1, f_1, e_2, f_2, \dots$, then $C^*(F)$ contains a full corner isomorphic to the UHF-algebra $M_{2^\infty}$, and $C^*(F)$ is not a Type I $C^*$-algebra.*

*Proof.* Consider the corner $p_{w_1}C^*(F)p_{w_1}$. Since $F$ has no cycles, the ideals of $C^*(F)$ are in bijective correspondence with the saturated hereditary subsets of $F$. Since $p_{w_1} \in p_{w_1}C^*(F)p_{w_1}$, and any hereditary subset containing $w_1$ must equal $F^0$, we conclude that any ideal containing $p_{w_1}C^*(F)p_{w_1}$ is equal to $C^*(F)$. Thus $p_{w_1}C^*(F)p_{w_1}$ is a full corner of $C^*(F)$.

If $\{s_e, p_v : e \in F^1, v \in F^0\}$ is a generating Cuntz-Krieger $F$-family, then

$$p_{w_1}C^*(F)p_{w_1} = \overline{\text{span}}\{s_\alpha s_\beta^* : s(\alpha) = s(\beta) = w_1 \text{ and } r(\alpha) = r(\beta)\}.$$

For each $n \in \mathbb{N} \cup \{0\}$, define

$$E_n := \{\alpha \in F^* : s(\alpha) = w_1 \text{ and } r(\alpha) = w_n\}$$

and let

$$A_n := \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E_n\}.$$

Then we see that each $A_n$ is a $C^*$-subalgebra of $C^*(F)$, $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$, and

$$C^*(F) = \overline{\bigcup_{n=0}^{\infty} A_n}.$$

For each $n \in \mathbb{N} \cup \{0\}$, we see that $\{s_\alpha s_\beta^* : \alpha, \beta \in E_n\}$ is a set of matrix units, and since $|E_n| = 2^n$ it follows that $A_n \cong M_{2^n}(\mathbb{C})$. Furthermore, for each $s_\alpha s_\beta^*$ with $\alpha, \beta \in E_n$, we see that

$$s_\alpha s_\beta^* = s_\alpha p_{w_n} s_\beta^* = s_\alpha(s_{e_n}s_{e_n}^* + s_{f_n}s_{f_n}^*)s_\beta^* = s_{\alpha e_n}s_{\beta e_n}^* + s_{\alpha f_n}s_{\beta f_n}^*.$$

Hence if for all $n \in \mathbb{N} \cup \{0\}$ we identify $A_n$ with $M_{2^n}(\mathbb{C})$ via an isomorphism, then for each $n$ the inclusion map $A_n \hookrightarrow A_{n+1}$ may be identified with the map $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$. Thus $C^*(F) = \overline{\bigcup_{n=0}^{\infty} A_n}$ is isomorphic to the UHF-algebra $M_{2^\infty}$.

Finally, it follows from [17, Theorem 6.5.7, p.211] that $M_{2^\infty}$ is not Type I. Since any $C^*$-subalgebra of a Type I $C^*$-algebra is Type I [17, Theorem 6.2.9, p.199], we conclude that $C^*(F)$ is not Type I. □

**Proposition 3.3.** *Let $E$ be a row-countable graph such that $C^*(E)$ has a unique irreducible representation up to unitary equivalence. If $v \in E^0$ and we define $H(v) := \{w \in E^0 : v \geq w\}$, then $E_{H(v)}$ is a countable graph, $C^*(E_{H(v)})$ is Morita equivalent to $C^*(E)$, and $C^*(E_{H(v)}) \cong K(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$.*

*Proof.* We see that $H(v) := \{w \in E^0 : v \geq w\}$ is a hereditary subset of $E$. In addition, if we let $H_0 = \{v\}$ and inductively define $H_{n+1} := r(s^{-1}(H_n))$, then one can easily verify that $H(v) = \bigcup_{n=0}^{\infty} H_n$. Since $H_0$ is finite and $E$ is row-countable, an inductive argument shows that $H_n$ is countable for all $n \in \mathbb{N}$. Hence $E^0_{H(v)} := H(v) = \bigcup_{n=0}^{\infty} H_n$ is countable, and $E^1_{H(v)} := s^{-1}(H_v)$ is countable. Thus, $E_{H(v)} := (E^0_{H(v)}, E^1_{H(v)}, r|_{H(v)}, s|_{H(v)})$ is a countable graph.

It follows from Theorem 2.3.3 that $C^*(E_{H(v)})$ is Morita equivalent to $I_{H(v)}$. Because $C^*(E)$ has a unique irreducible representation up to unitary equivalence, it follows from Lemma 1.3 that $C^*(E)$ is simple. Because $I_{H(v)}$ is a nonzero ideal of $C^*(E)$, it follows that $I_{H(v)} = C^*(E)$. Thus $C^*(E_{H(v)})$ is Morita equivalent to $C^*(E)$

Finally, since $C^*(E)$ has a unique irreducible representation up to unitary equivalence and $C^*(E_{H(v)})$ is Morita equivalent to $C^*(E)$, we conclude that $C^*(E_{H(v)})$ has a unique irreducible representation up to unitary equivalence. (This is because the Rieffel correspondence establishes a bijection between representations of Morita equivalent $C^*$-algebras that preserves irreducibility and unitary equivalence.) Moreover, the fact that $E_{H(v)} := (E^0_{H(v)}, E^1_{H(v)}, r|_{H(v)}, s|_{H(v)})$ is a countable graph implies $C^*(E_{H(v)})$ is a separable $C^*$-algebra. Hence by Rosenberg's result (Theorem 1.3.2) we have $C^*(E_{H(v)}) \cong K(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$. $\square$

## 3.2 A Structure Result for AF Graph $C^*$-algebras

**Proposition 3.3.** *Let $E$ be a directed graph such that $C^*(E)$ has a unique irreducible representation up to unitary equivalence. Then the following are equivalent:*

*(1) $E$ is row-countable.*

*(2) $E$ is row-finite.*

*(3) $C^*(E)$ is AF.*

*Proof.* (2) $\implies$ (1): This is immediate from the definitions.

(1) $\implies$ (3): Let $v \in E^0$, and set $H(v) := \{w \in E^0 : v \geq w\}$. Then $H(v)$ is a hereditary subset of $E$, and by Proposition 3.3 $C^*(E_{H(v)}) \cong K(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$. Consequently $C^*(E_{H(v)})$ is AF, and Theorem 2.3.2 implies the graph $E_{H(v)}$ has no cycles. Hence $E$ have no cycles with vertices in $H(v)$. Furthermore, $C^*(E)$ is simple by Lemma 1.3, and thus Theorem 2.3.1 implies that $E$ is cofinal. Since vertices in the hereditary set $H(v)$ cannot reach cycles containing vertices in $E^0 \setminus H(v)$, we may conclude that $E$ has no cycles. Thus Theorem 2.3.2 implies $C^*(E)$ is AF.

(3) $\implies$ (2): Since $C^*(E)$ is AF, Theorem 2.3.2 implies the graph $E$ has no cycles. In addition, since $C^*(E)$ has a unique irreducible representation up to unitary equivalence, Lemma 1.3 implies that $C^*(E)$ is simple. It then follows from Theorem 2.3.1 that every vertex of $E$ can reach every singular vertex of $E$. Let $v \in E^0$, and suppose $v$ is not a sink. Then there exists $e \in s^{-1}(v)$. Since $E$ has no cycles, it follows that $r(e)$ cannot reach $v$. But this implies that $v$ is not a singular vertex, and hence $v$ emits a finite number of edges. Since every vertex of $E$ that is not a sink emits a finite number of edges, $E$ is row-finite. □

## 3.3 A Forbidden Subgraph

**Proposition 3.4.** *Let $E$ be a directed graph such that $C^*(E)$ has a unique irreducible representation up to unitary equivalence. If $C^*(E)$ is AF, then $E$ does not contain a subgraph of the form*

$$v_1 \underset{\alpha_1}{\overset{\beta_1}{\rightrightarrows}} v_2 \underset{\alpha_2}{\overset{\beta_2}{\rightrightarrows}} v_3 \underset{\alpha_3}{\overset{\beta_3}{\rightrightarrows}} \cdots \qquad (3.4.1)$$

*where $v_1, v_2, \ldots$ are distinct vertices and $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ are distinct paths.*

*Proof.* For the sake of contradiction, suppose that $E$ has a subgraph of the form in (3.4.1), and use the labeling of vertices and paths listed in (3.4.1) Since $C^*(E)$ is AF, Proposition 3.3 implies $E$ is row-finite. If $H(v_1) := \{w \in E^0 : v_1 \geq w\}$, then $H(v_1)$ is a hereditary subset of $E$ and Proposition 3.3 implies $C^*(E_{H(v)}) \cong K(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$. Thus $C^*(E_{H(v)})$ is a Type I $C^*$-algebra. Furthermore, since $v_1$ can reach every vertex on each path $\alpha_i$ and each path $\beta_i$ for all $i \in \mathbb{N}$, we conclude that the graph $E_{H(v)}$ has a subgraph of the form in (3.4.1).

Let $\{s_e, p_v : e \in E^1_{H(v)}, v \in E^0_{H(v)}\}$ be a generating Cuntz-Krieger $E_{H(v)}$-family in $C^*(E_{H(v)})$, and consider the set $\{p_{v_i}, s_{\alpha_i}, s_{\beta_i}\}_{i=1}^{\infty}$. Also let $F$ be the graph

$$w_1 \underset{f_1}{\overset{e_1}{\rightrightarrows}} w_2 \underset{f_2}{\overset{e_2}{\rightrightarrows}} w_3 \underset{f_3}{\overset{e_3}{\rightrightarrows}} \cdots \qquad (3.4.2)$$

and let

$$S := \{w_i : i \in \mathbb{N} \text{ and } p_{v_i} = s_{\alpha_i}s^*_{\alpha_i} + s_{\beta_i}s^*_{\beta_i}\}.$$

Then $\{p_{v_i}, s_{\alpha_i}, s_{\beta_i}\}_{i=1}^{\infty}$ is an $(F, S)$-family in $C^*(E_{H(v)})$, and there exists a homomorphism $\phi : C^*(F, S) \to C^*(E_{H(v)})$ (where $C^*(F, S)$ denotes the relative graph $C^*$-algebra of $F$ with the (CK3) relation imposed at the vertices in $S$). We observe that $F$ has no cycles, and whenever $i \in \mathbb{N}$ with $w_i \notin S$, then the gap projection

$p_{v_i} - s_{\alpha_i} s_{\alpha_i}^* - s_{\beta_i} s_{\beta_i}^* \neq 0$ whenever $v_i \notin S$, it follows from the Cuntz-Krieger Uniqueness Theorem for relative graph $C^*$-algebras that $\phi$ is injective. Hence $\operatorname{im}\phi$ is a $C^*$-subalgebra of $C^*(E_{H(v)})$ isomorphic to $C^*(F, S)$.

It follows from Proposition 3.2 that $C^*(F)$ is not a Type I $C^*$-algebra. Because $C^*(F)$ is a quotient of $C^*(F, S)$, and all quotients of Type I $C^*$-algebras are Type I [17, Theorem 6.2.9, p.199], it follows that $C^*(F, S)$ is not a Type I $C^*$-algebra. Thus $\operatorname{im}\phi \cong C^*(F, S)$ is a $C^*$-subalgebra of $C^*(E_{H(v)})$ that is not Type I, and since all $C^*$-subalgebras of Type I $C^*$-algebras are Type I [17, Theorem 6.2.9, p.199], it follows that $C^*(E_{H(v)})$ is not a Type I $C^*$-algebra. But this contradicts the fact that $C^*(E_{H(v)}) \cong K(\mathcal{H})$. □

## 3.4 The Graph of a Simple AF Algebra

We are now in a position to prove the result promised at the beginning of this chapter.

**Theorem 3.4.1.** *Let $E$ be a directed graph such that $C^*(E)$ is AF and has a unique irreducible representation up to unitary equivalence. Then one of two distinct possibilities must occur: Either*

*(1) $E$ has exactly one sink and no infinite paths; or*

*(2) $E$ has no sinks and $E$ contains an infinite path $\alpha := e_1 e_2 e_3 \ldots$ in which every vertex emits exactly one edge, i.e.,*

$$s^{-1}(s(e_i)) = \{e_i\}$$

*for all $i \in \mathbb{N}$.*

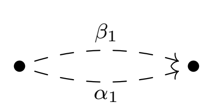
*Proof.* Because $C^*(E)$ has a unique irreducible representation up to unitary equivalence, it follows from Lemma 1.3 that $C^*(E)$ is simple, and it follows from Theorem 2.3.1 that $E$ is cofinal, satisfies Condition (L), and every vertex of $E$ can reach every singular vertex of $E$.

The fact that every vertex of $E$ can reach every singular vertex of $E$ implies that $E$ has at most one sink. If $E$ has one sink, then the cofinality of $E$ implies that $E$ has no infinite paths (because a sink cannot reach a vertex on the infinite path), and hence we are in situation (1) of the proposition.

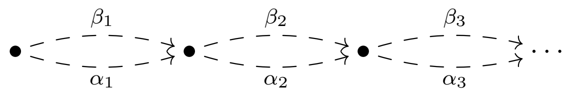
If $E$ has no sinks, then $E$ must contain an infinite path $f_1 f_2 \ldots$. To show that we are in situation (2) it suffices to show that there exists $N \in \mathbb{N}$ such that $s^{-1}(s(f_i)) = \{f_i\}$ for all $i \geq N$. (For then we can take $e_i := f_{N+i}$, and $e_1 e_2 \ldots$ is the desired path.)

Suppose to the contrary that the infinite path $f_1 f_2 \ldots$ does not have our desired property. Then for each $k \in \mathbb{N}$ there exists $n \geq k+1$ such that $s^{-1}(s(f_n))$ contains an element different from $e = f_n$. Choose a natural number $n_0$ such that $s^{-1}(s(f_{n_0}))$ contains an element $g$ different from $f_{n_0}$, and define $w_0 := s(f_{n_0})$. By cofinality there exists a path $\mu$ with $s(\mu) = r(g)$ and $r(\mu) = s(f_{m_0})$ for some $m_0 \in \mathbb{N}$.

Since $C^*(E)$ is AF, it follows from Theorem 2.3.2 that $E$ has no cycles, and hence $m_0 > n_0$. Using our hypothesis, we may choose $n_1 > m_0$ such that $s^{-1}(s(f_{n_1}))$ contains an element distinct from $f_{n_1}$. If we let $\alpha_1 := f_{n_0} f_{n_0+1} \ldots f_{m_0} \ldots f_{n_1-1}$ and $\beta_1 := g\mu f_{m_0} f_{m_0+1} \ldots f_{n_1-1}$, then $\alpha_1$ and $\beta_1$ are distinct paths in $E$ with $s(\alpha_1) = s(\beta_1)$ and $r(\alpha_1) = r(\beta_1)$ and such that $r(\alpha_1) = r(\beta_1) = s(f_{n_1})$ with $s^{-1}(s(f_{n_1}))$ containing an element distinct from $f_{n_1}$.



Proceeding inductively, we are able to construct a subgraph of $E$ of the following form.




Since $C^*(E)$ has a unique irreducible representation up to unitary equivalence, Proposition 3.4 implies that $E$ does not have such a subgraph. Hence, we have a contradiction. $\square$

# CHAPTER 4

# Naimark's Problem for certain Graph $C^*$-Algebras

## 4.1 Main Results

In this chapter we prove our two main results: (1) Naimark's Problem has an affirmative answer for the class of AF graph $C^*$-algebras, and (2) Naimark's Problem has an affirmative answer for the class of $C^*$-algebras of row-countable graphs.

If $\mathcal{H}$ is a Hilbert space, then for any $x, y \in \mathcal{H}$, we let $\Theta_{x,y} : \mathcal{H} \to \mathcal{H}$ denote the rank-one operator given by

$$\Theta_{x,y}(z) := \langle y, z \rangle x.$$

Since $K(\mathcal{H})$ is the closure of the finite-rank operators, we see that if $\beta$ is a basis for $\mathcal{H}$, then $K(\mathcal{H}) = \overline{\text{span}}\{\Theta_{x,y} : x, y \in \beta\}$.

If $V : \mathcal{H}_1 \to \mathcal{H}_2$ is an isometry between Hilbert spaces, we let $\text{Ad}_V : K(\mathcal{H}_1) \to K(\mathcal{H}_2)$ denote the homomorphism given by $\text{Ad}_V(T) := VTV^*$. It is straightforward to verify that $\mathrm{Ad}_V$ is injective and for any $x, y \in \mathcal{H}_1$ we have $\mathrm{Ad}_V(\Theta_{x,y}) = \Theta_{Vx,Vy}$.

**Theorem 4.1.1.** *Let $E$ be a graph such that $C^*(E)$ is AF. If $C^*(E)$ has a unique irreducible representation up to unitary equivalence, then $C^*(E) \cong K(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

*Proof.* Throughout, let $\{s_e, p_v : e \in E^1, v \in E^0\}$ be a generating Cuntz-Krieger $E$-family. By Proposition 3.4.1 there are two cases to consider.

Case I: $E$ has exactly one sink and no infinite paths.

Let $v_0$ denote the sink of $E$, and let $E^*(v_0) := \{\alpha \in E^* : r(\alpha) = v_0\}$. Define $I_{v_0} := \overline{\mathrm{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*(v_0)\}$. Since no path ending at the sink $v_0$ can be extended, for any $\alpha, \beta, \gamma, \delta \in E^*(v_0)$ we have

$$(s_\alpha s_\beta^*)(s_\gamma s_\delta^*) := \begin{cases} s_\alpha s_\delta^* & \text{if } \beta = \gamma \\ 0 & \text{if } \beta \neq \gamma, \end{cases}$$

and $\{s_\alpha s_\beta^* : \alpha, \beta \in E^*(v_0)\}$ is a set of matrix units indexed by $E^*(v_0)$. Hence $I_{v_0} \cong K(\mathcal{H})$, where $\mathcal{H} := \ell^2(E^*(v_0))$. Furthermore, since $v_0$ is a sink, $I_{v_0}$ is a nonzero ideal in $C^*(E)$. By Lemma 1.3 $C^*(E)$ is simple, and hence $I_{v_0} = C^*(E)$. Thus the result holds in this case.

Case II: $E$ contains an infinite path $\alpha := e_1 e_2 \ldots$ with $s^{-1}(s(e_i)) = \{e_i\}$ for all $i \in \mathbb{N}$.

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \xrightarrow{e_3} \cdots$$

For convenience of notation, let $v_i := s(e_i)$, and for each $n \in \mathbb{N}$ define $E^*(v_n) := \{\alpha \in E^* : r(\alpha) = v_n\}$. Let $\mathcal{H}_n := \ell^2(E^*(v_n))$, and for each $\alpha \in E^*(v_n)$ let $\delta_\alpha \in \mathcal{H}_n$ denote the point mass function at $\alpha$, so that $\{\delta_\alpha : \alpha \in E^*(v_n)\}$ forms an orthonormal basis for $\mathcal{H}_n$.

For each $n \in \mathbb{N}$ define $V_n : \mathcal{H}_n \to \mathcal{H}_{n+1}$ to be the isometry with

$$V_n(\delta_\alpha) := \delta_{\alpha e_n}$$

for each $\alpha \in \ell^2(E^*(v_n))$. Also define $\mathrm{Ad}_{V_n} : K(\mathcal{H}_n) \to K(\mathcal{H}_{n+1})$ by $\mathrm{Ad}_{V_n}(T) := V_n T V_n^*$.

For each $n \in \mathbb{N}$ define $A_n := \overline{\mathrm{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^*(v_n)\}$. If we consider the generating set $\{s_\alpha s_\beta^* : \alpha, \beta \in E^*(v_n)\}$, then for any $\beta, \gamma \in E^*(v_n)$, we have $r(\beta) = r(\gamma) = v_n$, and since $E$ has no cycles the only way for one of $\beta$ and $\gamma$ to extend the other is if $\beta = \gamma$. Hence for any $\alpha, \beta, \gamma, \delta \in E^*(v_n)$, we have

$$(s_\alpha s_\beta^*)(s_\gamma s_\delta^*) := \begin{cases} s_\alpha s_\delta^* & \text{if } \beta = \gamma \\ 0 & \text{if } \beta \neq \gamma \end{cases}$$

and $\{s_\alpha s_\beta^* : \alpha, \beta \in E^*(v_n)$ is a set of matrix units indexed by $E^*(v_n)$. Hence there exists an isomorphism $\phi_n : K(\mathcal{H}_n) \to A_n$ satisfying $\phi_n(\Theta_{\alpha,\beta}) = s_\alpha s_\beta^*$.

Let $\iota_n : A_n \hookrightarrow A_{n+1}$ denote the embedding $s_\alpha s_\beta^* \mapsto s_{\alpha e_n} s_{\beta e_n}^*$. For each $n \in \mathbb{N}$ and for all $\alpha, \beta \in E^*(v_n)$ we have

$$\begin{aligned} \phi_{n+1} \circ \mathrm{Ad}_{V_n}(\Theta_{\delta_\alpha,\delta_\beta}) &= \phi_{n+1}(V_n \Theta_{\delta_\alpha,\delta_\beta} V_n^*) = \phi_{n+1}(\Theta_{V_n\delta_\alpha, V_n\delta_\beta}) \\ &= \phi_n(\Theta_{\delta_{\alpha e_n},\delta_{\beta e_n}}) = s_{\alpha e_n} s_{\beta e_n}^* = s_\alpha s_{e_n} s_{e_n}^* s_\beta^* = s_\alpha p_{s(e_n)} s_\beta^* = s_\alpha p_{s(e_n)} s_\beta^* \\ &= s_\alpha s_\beta^* = \iota_n \circ \phi_n(\Theta_{\delta_\alpha,\delta_\beta}). \end{aligned}$$

Thus for each $n \in \mathbb{N}$ we have $\phi_{n+1} \circ \mathrm{Ad}_{V_n} = \iota_n \circ \phi_n$ and the diagram

$$\begin{array}{ccc} K(\mathcal{H}_n) & \xrightarrow{\mathrm{Ad}_{V_n}} & K(\mathcal{H}_{n+1}) \\ \phi_n \downarrow & & \downarrow \phi_{n+1} \\ A_n & \xrightarrow{\iota_n} & A_{n+1} \end{array} \tag{4.1.1}$$

commutes. Since the direct limit of the sequence

$$A_1 \xrightarrow{\iota_1} A_2 \xrightarrow{\iota_2} A_3 \xrightarrow{\iota_3} \dots$$

is equal to $\overline{\bigcup_{n=1}^{\infty} A_n}$, and since for all $n \in \mathbb{N}$ the map $\phi_n : A_n \to A_{n+1}$ is an isomorphism and the diagram in (4.1.1) commutes, we may conclude that

$$\varinjlim K(\mathcal{H}_n) \cong \overline{\bigcup_{n=1}^{\infty} A_n}, \tag{4.1.2}$$

where $\varinjlim K(\mathcal{H}_n)$ is the direct limit of the sequence

$$K(\mathcal{H}_1) \xrightarrow{\mathrm{Ad}_{V_1}} K(\mathcal{H}_2) \xrightarrow{\mathrm{Ad}_{V_2}} K(\mathcal{H}_3) \xrightarrow{\mathrm{Ad}_{V_3}} \dots .$$

Next we consider the set of infinite paths $E^\infty$. For any infinite path $\mu \in E^\infty$, we must have $\mu = \alpha e_i e_{i+1} e_{i+1} \dots$ for some $\alpha \in E^*$ and some $i \in N$, for otherwise the vertex $v_1$ could not reach a vertex on $\mu$, contradicting the cofinality of $E$.

Define $\mathcal{H}_\infty := \ell^2(E^\infty)$ and for $\mu \in E^\infty$ let $\delta_\mu$ denote the point mass function at $\mu$. Then $\{\delta_\mu : \mu \in E^\infty\}$ is an orthonormal basis for $\mathcal{H}_\infty$. For each $n \in \mathbb{N}$ define an isometry $W_n : \mathcal{H}_n \to \mathcal{H}_\infty$ by

$$W_n(\delta_\alpha) = \delta_{\alpha e_n e_{n+1} \dots}.$$

For each $n \in \mathbb{N}$ and for any $\alpha \in E^*(v_n)$ we have

$$W_{n+1}(V_n(\delta_\alpha)) = W_{n+1}(\delta_{\alpha e_n}) = \delta_{\alpha e_n e_{n+1} e_{n+2} \dots} = W_n(\delta_\alpha)$$

and hence $W_{n+1} \circ V_n = W_n$ for all $n \in \mathbb{N}$.

In addition, for any $n \in \mathbb{N}$ we define $\mathrm{Ad}_{W_n} : K(\mathcal{H}_n) \to K(\mathcal{H}_\infty)$ by $\mathrm{Ad}_{W_n}(T) := W_n T W_n^*$. For any $T \in K(\mathcal{H}_n)$ we have

$$\begin{aligned} \mathrm{Ad}_{W_{n+1}} \circ \mathrm{Ad}_{V_n}(T) &= \mathrm{Ad}_{W_{n+1}}(V_n T V_n^*) = W_{n+1} V_n T V_n^* W_{n+1}^* \\ &= (W_{n+1} V_n) T (W_{n+1} V_n)^* = W_n T W_n^* = \mathrm{Ad}_{W_n}(T) \end{aligned}$$

so that

$$\mathrm{Ad}_{W_{n+1}} \circ \mathrm{Ad}_{V_n} = \mathrm{Ad}_{W_n}$$

for all $n \in \mathbb{N}$. By the universal property of the direct limit there exists a homomorphism $\psi : \varinjlim K(\mathcal{H}_n) \to K(\mathcal{H}_\infty)$ with $\operatorname{im} \operatorname{Ad}_{W_n} \subseteq \operatorname{im} \psi$ for all $n \in \mathbb{N}$, and furthermore, since each $\operatorname{Ad}_{W_n}$ is injective for all $n \in \mathbb{N}$, we may conclude that $\psi : \varinjlim K(\mathcal{H}_n) \to K(\mathcal{H}_\infty)$ is injective.

Moreover, for any $\mu, \nu \in E^\infty$, we may write $\mu = \alpha e_n e_{n+1} \ldots$ and $\nu = \beta e_n e_{n+1} \ldots$ for some $n \in \mathbb{N}$ and some $\alpha, \beta \in E^*(v_n)$, from which it follows that

$$\begin{aligned}\Theta_{\delta_\mu, \delta_\nu} &= \Theta_{\delta_{\alpha e_n e_{n+1}\ldots}, \delta_{\beta e_n e_{n+1}\ldots}} = \Theta_{W_n(\delta_\alpha), W_n(\delta_\beta)} = W_n \Theta_{\alpha,\beta} W_n^* \\ &= \operatorname{Ad}_{W_n}(\Theta_{\delta_\alpha, \delta_\beta}) \in \operatorname{im} \operatorname{Ad}_{W_n} \subseteq \operatorname{im} \psi.\end{aligned}$$

Hence $\{\Theta_{\delta_\mu, \delta_\nu} : \mu, \nu \in E^\infty\} \subseteq \operatorname{im} \psi$, so that $\operatorname{im} \psi = K(\mathcal{H}_\infty)$, and $\psi$ is surjective. Hence $\psi : \varinjlim K(\mathcal{H}_n) \to K(\mathcal{H}_\infty)$ is an isomorphism, and

$$\varinjlim K(\mathcal{H}_n) \cong K(\mathcal{H}_\infty). \tag{4.1.3}$$

Next we let

$$\begin{aligned}H := \{v \in E^0 : p_v = \sum_{i=1}^{k} s_{\alpha_i} s_{\beta_i}^* \text{ for some } \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \bigcup_{n=1}^{\infty} E^*(v_n) \\ \text{satisfying } s(\alpha_i) = s(\beta_i) = v \text{ for all } 1 \leq i \leq k\}\end{aligned}$$

We shall show that $H$ is a saturated and hereditary subset of $E$.

To show that $H$ is hereditary, we first observe that for each $i \in \mathbb{N}$ we have $p_{v_i} = s_{e_i} s_{e_i}^*$ and that $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$, implying that $v_i \in H$. Thus $\{v_1, v_2, \ldots\} \subseteq H$. Next, suppose that $e \in E^1$ and $s(e) \in H$. If $s(e) = v_i$ for some $i \in N$, then from the previous sentence we have that $r(e) = v_{e+i} \in H$. If $s(e) \neq v_i$ for all $i \in \mathbb{N}$, we use the fact that $s(e) \in H$ to write

$$p_{s(e)} = \sum_{i=1}^{k} s_{\alpha_i} s_{\beta_i}^*$$

for some $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \bigcup_{n=1}^{\infty} E^*(v_n)$ with $s(\alpha_i) = s(\beta_i) = s(e)$ for all $1 \leq i \leq k$, and moreover, the fact that $s(e) \neq v_i$ for all $i \in \mathbb{N}$ implies that $\alpha_i$ and $\beta_i$ are paths of length at least 1 for each $1 \leq i \leq k$. Consequently,

$$p_{r(e)} = s_e^* s_e = s_e^* p_{s(e)} s_e = s_e^* \left( \sum_{i=1}^{k} s_{\alpha_i} s_{\beta_i}^* \right) s_e = \sum_{i=1}^{k} s_e^* s_{\alpha_i} s_{\beta_i}^* s_e. \tag{4.1.4}$$

For each $1 \leq i \leq k$, we may use the fact that $\alpha_i$ and $\beta_i$ have lengths at least 1 to write $\alpha_i = f_1 \ldots f_l$ and $\beta_i = g_1 \ldots g_m$ for edges $f_1, \ldots, f_l, g_1, \ldots, g_m \in E^1$, and then we have

$$s_e^* s_{\alpha_i} s_{\beta_i}^* s_e = \begin{cases} s_{f_2 \ldots f_l} s_{g_2 \ldots g_m}^* & \text{if } f_1 = e \text{ and } g_1 = e \\ 0 & \text{otherwise.} \end{cases}$$

For the nonzero case above, we see that $s(f_2) = r(e)$ and $s(g_2) = r(e)$, and also $r(f_l) = r(g_m) = r(\alpha_i) = r(\beta_i)$ so that $s_e^* s_{\alpha_i} s_{\beta_i}^* s_e = s_{f_2 \ldots f_l} s_{g_2 \ldots g_m}^*$ has the properties given in defining the set $H$. Consequently, (4.1.4) shows that $r(e) \in H$. Hence $H$ is hereditary.

To see that $H$ is saturated, suppose that $v \in E^0$ is a regular vertex with $r(s^{-1}(v)) \subseteq H$. For each $e \in s^{-1}(v)$, the fact that $r(e) \in H$ allows us to write

$$p_{r(e)} = \sum_{i=1}^{k_e} s_{\alpha_i^e} s_{\beta_i^e}^*$$

for some $\alpha_1^e, \ldots, \alpha_k^e, \beta_1^e, \ldots, \beta_k^e \in \bigcup_{n=1}^{\infty} E^*(v_n)$ with $s(\alpha_i^e) = s(\beta_i^e) = v$ for all $1 \leq i \leq k_e$. Hence

$$\begin{aligned} p_v = \sum_{s(e)=v} s_e s_e^* &= \sum_{s(e)=v} s_e p_{r(e)} s_e^* = \sum_{s(e)=v} s_e \left( \sum_{i=1}^{k_e} s_{\alpha_i^e} s_{\beta_i^e}^* \right) s_e^* \\ &= \sum_{s(e)=v} \sum_{i=1}^{k_e} s_e s_{\alpha_i^e} s_{\beta_i^e}^* s_e^* = \sum_{s(e)=v} \sum_{i=1}^{k_e} s_{e\alpha_i^e} s_{e\beta_i^e}^* \end{aligned}$$

and since $s(e\alpha_i^e) = s(e\beta_i^e) = v$ and $r(e\alpha_i^e) = r(e\beta_i^e) \in \bigcup_{n=1}^{\infty} E^*(v_n)$, it follows that $v \in H$. Thus, $H$ is saturated.

Because $H$ is a nonempty saturated hereditary subset, and since $C^*(E)$ is simple by Lemma 1.3, it follows from Theorem 2.3.1 that $H = E^0$. Consequently, for any $v \in E^0$ we have that $p_v = \sum_{i=1}^k s_{\alpha_i} s_{\beta_i}^*$ for some paths $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \bigcup_{n=1}^\infty E^*(v_n)$ satisfying $s(\alpha_i) = s(\beta_i) = v$ for all $1 \leq i \leq k$. Thus $p_v \in \overline{\bigcup_{n=1}^\infty A_n}$.

Likewise, for any $e \in E^1$, we have $r(e) \in H$ and $p_{r(e)} = \sum_{i=1}^k s_{\alpha_i} s_{\beta_i}^*$ for some $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \bigcup_{n=1}^\infty E^*(v_n)$ satisfying $s(\alpha_i) = s(\beta_i) = r(e)$ for all $1 \leq i \leq k$. Thus $s_e = s_e p_{r(e)} = \sum_{i=1}^k s_e s_{\alpha_i} s_{\beta_i}^* = \sum_{i=1}^k s_{e\alpha_i} s_{\beta_i}^* \in \overline{\bigcup_{n=1}^\infty A_n}$.

Hence $\{p_v, s_e : v \in E^0, e \in E^1\} \subseteq \overline{\bigcup_{n=1}^\infty A_n}$, and it follows that

$$C^*(E) = \overline{\bigcup_{n=1}^\infty A_n}. \tag{4.1.5}$$

Combining (4.1.2), (4.1.3), and (4.1.5) gives the desired result. □

**Theorem 4.1.2.** *If $E$ is a row-countable directed graph such that $C^*(E)$ has a unique irreducible representation up to unitary equivalence, then $C^*(E) \cong K(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

*Proof.* It follows from Lemma 1.3 that $C^*(E)$ is simple, and since $E$ is row-countable, Proposition 3.3 then implies that $C^*(E)$ is AF. The result then follows from Theorem 4.1.1. □

## 4.2 Conclusion and Future Directions

We know that a graph $C^*$-algebra with a unique irreducible representation is simple. We have shown that if such a graph $C^*$-algebra is AF, then it is isomorphic to $K(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Furthermore, it is known that simple graph $C^*$-algebras exhibit the following dichotomy.

**Theorem 4.2.1.** [12, Corollary 3.10] *Let $E$ be a graph. If $C^*(E)$ is simple, then either*

*(1) $C^*(E)$ is an AF algebra if $E$ contains no cycle, or*

*(2) $C^*(E)$ is purely infinite if $E$ contains a cycle.*

*Remark* 4.2.1. The result, as stated in the cited article, relies on additional assumptions. However, the proof extends to the general case stated above.

**Definition 4.2.1** (Purely Infinite $C^*$-algebra)**.** [10, Definition 4.1] *A $C^*$-algebra $A$ is* purely infinite *if there are no characters on $A$ and if for every pair of positive elements $a, b$ in $A$, $a \leq b$ if and only if $a$ belongs to the closed two-sided ideal $\overline{AbA}$ generated by $b$.*

Consequently, obtaining a complete answer to Naimark's problem for all graph $C^*$-algebras now requires an examination of purely infinite graph $C^*$-algebras.

# Bibliography


[1] C. Akemann and N. Weaver, *Consistency of a counterexample to Naimark's problem*, PNAS **101** (2004), 7522–7525.

[2] T. Bates, J.H. Hong, I. Raeburn, and W. Szymański, *The ideal structure of the $C^*$-algebras of infinite graphs*, Illinois J. Math. **46** (2002), no. 4, 1159–1176.

[3] J. Dixmier, *Sur les $C^*$-algbres*, Illinois J. Math. **4** (1960), 221–230.

[4] J. M. G. Fell, *$C^*$-algebras with smooth dual*, Illinois J. Math. **4** (1960), 221–230.

[5] I. Fredholm, *Sur une classe d'équations fonctionnelles.*, Acta Math. **27** (1903), 365–390.

[6] I. Gelfand and M. A. Naimark, *On the imbedding of normed rings into the ring of operators in Hilbert space*, Rec. Math. [Mat. Sbornik] N.S. **12(54)** (1943), 197–217.

[7] J. Glimm, *Type I $C^*$-algebras*, Annals of Mathematics **73(3)** (1961), 572–612.

[8] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Fortschritte der mathematischen Wissenschaften in Monographien (Bd. 3), 1912.

[9] I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. **70** (1951), 219–255.

[10] E Kirchberg and M. Rørdam, *Non-simple purely infinite $C^*$-algebras*, American Journal of Mathematics **122** (2000), 637–666.

[11] B. O. Koopman, *Hamiltonian systems and transformations in Hilbert space*, PNAS **17(5)** (1931), 315–318.

[12] A. Kumjian, D. Pask, and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific Journal of Mathematics **184(1)** (1998), 161–174.

[13] P. Muhly and M. Tomforde, *Adding tails to $C^*$-correspondences*, Doc. Math. 9 (2004), 79–106.

[14] M. A. Naimark, *Rings with involutions*, Uspekhi Mat. Nauk **3** (1948), 52–145.

[15] ______, *On a problem of the theory of rings with involution*, Uspekhi Mat. Nauk **6** (1951), 160–164.

[16] A.L.T. Paterson, *Graph inverse semigroups, groupoids and their $C^*$-algebras*, J. Operator Theory **48** (2002), no. 3, suppl., 645–662.

[17] G. Pedersen, *$C^*$-algebras and their Automorphism Groups*, Academic Press Inc., New York, 1979.

[18] I. Raeburn and W. Szymański, *Cuntz-Krieger algebras of infinite graphs and matrices*, Trans. Amer. Math. Soc. **356** (2004), no. 1, 39–59.

[19] I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-Trace $C^*$-algebras*, Math. Surveys & Monographs, vol. 60, Amer. Math. Soc., Providence, 1998.

[20] A. Rosenberg, *The number of irreducible representations of simple rings with no minimal ideals*, Amer. J. Math. **75** (1953), 523–530.

[21] W. Szymański, *Simplicity of Cuntz-Krieger algebras of infinite matrices*, Pacific J. Math. **199** (2001), no. 1, 249–256.

[22] J. von Neumann, *Proof of the quasi-ergodic hypothesis*, PNAS **18** (1932), 70–82.