HILBERT C^*-MODULES OVER Σ^*-ALGEBRAS

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Doctor of Philosophy

By

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Abstract

A Σ^*-*algebra* is a concrete C^*-algebra that is sequentially closed in the weak operator topology. In this thesis, we study an appropriate class of C^*-modules over Σ^*-algebras analogous to the class of W^*-modules (selfdual C^*-modules over W^*-algebras).

In Chapter 3, we define our main object of study, the "Σ^*-module", and we present Σ^*-versions of virtually all the first results in the theory of C^*- and W^*-modules. We then study Σ^*-modules possessing a weak sequential form of the condition of being countably generated.

In Chapter 4, we apply the results and techniques of Chapter 3 to develop the appropriate Σ^*-algebraic analogue of the notion of strong Morita equivalence for C^*-algebras. We define strong Σ^*-Morita equivalence, prove a few characterizations, look at the relationship with equivalence of categories of a certain type of Hilbert space representation, study Σ^*-versions of the interior and exterior tensor products, and prove a Σ^*-version of the Brown-Green-Rieffel stable isomorphism theorem.

In Chapter 5, we prove abstract characterizations of weak* sequentially closed subspaces of dual Banach spaces and dual operator spaces, and we make some connections between these and our theory of Σ^*-modules.

Contents

Chapter 1

Introduction

Hilbert C^*-modules (also called Hilbert modules, and which we simply call C^*-modules) are simultaneous generalizations of C^*-algebras, Hilbert spaces, and certain types of vector bundles. They are an amazingly versatile tool used in a broad range of subfields of operator algebra theory—for example, the theory of Morita equivalence, Kasparov's KK-theory and its applications in noncommutative geometry, quantum group theory, and operator space theory.

An important subclass of C^*-modules is the class of selfdual C^*-modules (see Definition 2.1.3) over W^*-algebras, i.e. the *W^*-modules*. Historically, W^*-modules were among the first C^*-modules to appear (they were introduced in 1973 by Paschke in [31]), but today they seem less well known and perhaps under-exploited. Compared to the general theory of C^*-modules, the theory of W^*-modules is much more elegant and similar to that of Hilbert spaces, in large part due to powerful "orthogonality" properties automatically present in W^*-modules.

Between the classes of C^*-algebras and W^*-algebras is the class of Σ^*-*algebras*. First defined and studied by E. B. Davies in [16], a (concrete) Σ^*-*algebra* is a C^*-subalgebra of $B(\mathcal{H})$ that is closed under limits of sequences converging in the weak operator topology (WOT) (equivalently, in the weak*-topology on $B(\mathcal{H})$). Evidently, every von Neumann algebra is a Σ^*-algebra, but the converse does not hold. Conceptually, the theory of Σ^*-algebras may be considered as a halfway world between general C^*-algebra theory and von Neumann algebra theory—indeed, many (but certainly not all) von Neumann algebraic concepts and techniques have some sort of analogue for Σ^*-algebras. It is the general purpose of this thesis to explore the "appropriate" class of C^*-modules over Σ^*-algebras in analogy with the way that W^*-modules are the "appropriate" class of C^*-modules over W^*-algebras.

A major motivation for exploring the Σ^*-world comes from the commutative case. The two most powerful slogans in operator algebra theory are: "C^*-algebra theory is noncommutative topology" and "von Neumann algebra theory is noncommutative measure theory." These are in fact extremely deep statements out of which researchers have gotten a tremendous amount of mileage, but the beginning of the meaning of these can be understood from the following two simple facts: (1) the commutative C^*-algebras are precisely the spaces $C_0(X)$ of continuous functions vanishing at infinity on a locally compact Hausdorff space X; (2) commutative von Neumann algebras are precisely the spaces $L^\infty(X, \mu)$ of essentially bounded measurable functions on a locally finite measure space (X, μ). Turning to the Σ^*-world, the prototypical example of a commutative Σ^*-algebra is the space $\mathrm{Bor}_b(X)$ of bounded Borel-measurable functions on a second countable locally compact Hausdorff space

X. Since spaces of Borel functions play a large role in classical measure theory, one may hope that Σ^*-algebras should play an analogously large role in von Neumann algebra theory. (Indeed, many basic theorems and facts about von Neumann algebras are "really" facts about Σ^*-algebras; e.g., the existence of a bounded Borel functional calculus.)

This work was also inspired in large part by Hamana's paper [23], in which he studies selfdual C^*-modules over monotone complete C^*-algebras. Theorems 1.2, 2.2, and 3.3 of that paper indicate that selfdual C^*-modules are the "appropriate" class of C^*-modules over monotone complete C^*-algebras (and his interesting "conversely" statement in Theorem 2.2 seems to indicate that monotone complete C^*-algebras are the "appropriate" class of coefficient C^*-algebras over which to consider selfdual C^*-modules). We do not have in the case of Σ^*-modules the existence of an "orthonormal basis," which is Hamana's main technical tool in [23], so most of his proof techniques do not work for us, but the overarching philosophy of what we have tried to accomplish is very much in line with that of Hamana's work. For more work on the subject of C^*-modules over monotone complete C^*-algebras, see the paper [21] by M. Frank.

Also somewhat related to the present work is the noncommutative semicontinuity theory initiated by Akemann and Pedersen in [1] and developed further by Brown in [12] (see also [13, 14]). Though the present work has seemingly little to do with this theory (we do not deal with monotone limits, and in this work, the universal representation is mentioned only as an example setting), we were first drawn into this investigation by Brown's mentioning in [13] that the monotone sequentially closed

C^*-algebra generated by the set of semicontinuous elements in the second dual of a C^*-algebra is seemingly the most natural noncommutative analogue of the space of bounded Borel functions on a locally compact Hausdorff space. See Note 2.3.4 for a short discussion of some related interesting open problems.

The organization of this thesis is as follows. In Chapter 2, we give a quick rundown of the necessary elements of C^*-module and W^*-module theory and a slightly more detailed discussion of Σ^*-algebra theory.

The first main chapter in the body is Chapter 3, which contains the definition of our main object of study, the "Σ^*-modules," and an investigation into their basic properties. We also look at the somewhat more manageable class of Σ^*-modules that are "countably generated" in an appropriate way. This chapter contains precisely the material of the paper [3].

Chapter 4 applies the methods and results of Chapter 3 to study an appropriate analogue of Morita equivalence for Σ^*-algebras. We define strong Σ^*-Morita equivalence, prove a few characterizations, look at the relationship with equivalence of categories of a certain type of Hilbert space representation, study Σ^*-versions of the interior and exterior tensor products, and prove a Σ^*-version of the Brown-Green-Rieffel stable isomorphism theorem. This chapter coincides with the preprint [2].

In the final chapter, Chapter 5, we first look abstractly at weak* sequentially closed subspaces of dual Banach spaces, as well as the operator space analogue of these. We then prove several results connecting these with Σ^*-modules.

Chapter 2

Background

In this chapter we fix our notation and review the necessary definitions and results for C^*-modules, W^*-modules, and Σ^*-algebras.

2.1 Hilbert C^*-modules and W^*-modules

Since the basic theory of C^*-modules is well known and covered in many texts, we will be brief here. We generally refer to [9, Chapter 8] for notation and results; other references include [25], [36], [43, Chapter 15], and [5, Section II.7].

Loosely speaking, a (right or left) module X over a C^*-algebra A is called a *(right or left) C^*-module over A* if X is equipped with an "A-valued inner product" $\langle\cdot|\cdot\rangle : X \times X \to A$ (see any of the references above for the complete list of axioms, including, for example, $\langle x|x\rangle \geq 0$ for all $x \in X$) such that X is complete in the canonical norm induced by this inner product (i.e. the norm $\|x\| := \sqrt{\|\langle x|x\rangle\|}$). If X is a right C^*-module, the inner product is taken to be linear and A-linear in the second variable and conjugate-linear in the first variable, and vice versa for left modules. When unspecified, "C^*-module" should be taken to mean "right C^*-module."

For clarity, we sometimes write the inner product with a subscript denoting its range (e.g., $\langle\cdot|\cdot\rangle_A$ for a right module and ${}_A\langle\cdot|\cdot\rangle$ for a left module). For a C^*-module X, we will denote by $\overline{X}$ the *adjoint module* of X (often called the *conjugate module*); see [9], last paragraph of 8.1.1. If X is a right (resp. left) C^*-module over A, then $\overline{X}$ is a left (resp. right) C^*-module over A.

If X and Y are two right C^*-modules over A, $B_A(X,Y)$ denotes the Banach space of bounded A-module maps from X to Y with operator norm; $\mathbb{B}_A(X,Y)$ denotes the closed subspace of adjointable operators; and $\mathbb{K}_A(X,Y)$ denotes the closed subspace generated by operators of the form $|y\rangle\langle x| := y\langle x|\cdot\rangle$ for $y \in Y$, $x \in X$. If $X = Y$, the latter two of these spaces are C^*-algebras, and in this case, X is a left C^*-module over $\mathbb{K}_A(X)$ with inner product $|\cdot\rangle\langle\cdot|$. If X and Y are left modules over A, we write the subscript A on the left; e.g., ${}_AB(X,Y)$.

In this thesis, we will be concerned with modules over Σ^*-algebras, which are a class of C^*-algebras with an extra bit of structure that may be viewed abstractly, but is often most easily captured by fixing a faithful representation of a certain type on a Hilbert space. Reflecting this view, C^*-modules over these algebras are also most easily studied when viewed under a representation induced by a fixed representation of the coefficient C^*-algebra. There is a well-known general procedure for taking a representation of the coefficient C^*-algebra of a C^*-module and inducing representations of the C^*-module and many of the associated mapping spaces mentioned in the previous paragraph. The following paragraph and proposition describe this construction and its relevant features.

Suppose $A \subseteq B(\mathcal{H})$ is a C^*-algebra (assumed to be nondegenerately acting, although this is not strictly necessary for everything that follows) and that X is a right C^*-module over A. We may consider $\mathcal{H}$ as a left module over A and take the algebraic module tensor product $X \odot_A \mathcal{H}$. This vector space admits an inner product determined by the formula $\langle x \otimes \zeta, y \otimes \eta \rangle = \langle \zeta, \langle x|y \rangle \eta \rangle$ for simple tensors (see [25, Proposition 4.5] for details), and we may complete $X \odot_A \mathcal{H}$ in the induced norm to yield a Hilbert space $X \otimes_A \mathcal{H}$. Considering A as a C^*-module over itself and taking the C^*-module direct sum $X \oplus A$, there is a canonical corner-preserving embedding of $B_A(X \oplus A)$ into $B((X \otimes_A \mathcal{H}) \oplus^2 \mathcal{H})$ which allows us to concretely identify many of the associated spaces of operators between X and A with spaces of Hilbert space operators between $\mathcal{H}$ and $X \otimes_A \mathcal{H}$—this is the content of the following proposition. All of the pieces of this proposition can be found in the textbooks mentioned above.

Recall that for a nondegenerate C^*-algebra $A \subseteq B(\mathcal{H})$, the multiplier algebra $M(A)$ may be identified with the space $\{T \in B(\mathcal{H}) : TA \subseteq A \text{ and } AT \subseteq A\}$, and the left multiplier algebra $LM(A)$ with $\{T \in B(\mathcal{H}) : TA \subseteq A\}$ (see [34, 3.12.3]).

Proposition 2.1.1. *If X is a C^*-module over a nondegenerate C^*-algebra $A \subseteq B(\mathcal{H})$, then*

$$\begin{bmatrix} \mathbb{K}_A(X) & \mathbb{K}_A(A, X) \\ \mathbb{K}_A(X, A) & \mathbb{K}_A(A) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbb{B}_A(X) & \mathbb{B}_A(A, X) \\ \mathbb{B}_A(X, A) & \mathbb{B}_A(A) \end{bmatrix}$$

are canonically C^*-algebras,

$$\begin{bmatrix} B_A(X) & B_A(A,X) \\ B_A(X,A) & B_A(A) \end{bmatrix}$$

is canonically a Banach algebra, and there are canonical corner-preserving maps making the following diagram commute:

$$\begin{tikzcd}
\begin{bmatrix} \mathbb{K}_A(X) & \mathbb{K}_A(A,X) \\ \mathbb{K}_A(X,A) & \mathbb{K}_A(A) \end{bmatrix} \arrow[r] \arrow[d] & \begin{bmatrix} \mathbb{K}_A(X) & X \\ \overline{X} & A \end{bmatrix} \arrow[d] \\
\begin{bmatrix} \mathbb{B}_A(X) & \mathbb{B}_A(A,X) \\ \mathbb{B}_A(X,A) & \mathbb{B}_A(A) \end{bmatrix} \arrow[r] \arrow[d] & \begin{bmatrix} M(\mathbb{K}_A(X)) & * \\ * & M(A) \end{bmatrix} \arrow[d] \\
\begin{bmatrix} B_A(X) & B_A(A,X) \\ B_A(X,A) & B_A(A) \end{bmatrix} \arrow[r] \arrow[d] & \begin{bmatrix} LM(\mathbb{K}_A(X)) & * \\ * & LM(A) \end{bmatrix} \arrow[dl] \\
\begin{bmatrix} B(X \otimes_A \mathcal{H}) & B(\mathcal{H}, X \otimes_A \mathcal{H}) \\ B(X \otimes_A \mathcal{H}, \mathcal{H}) & B(\mathcal{H}) \end{bmatrix} &
\end{tikzcd}$$

The top two horizontal maps are $*$-isomorphisms, and the horizontal map in the third row is a Banach algebra isomorphism. All vertical maps are isometric homomorphisms, and in the diagram with the third row deleted, all vertical maps are isometric $*$-homomorphisms.

Proof. We provide a sketch of the proof that $B_A(X)$ embeds into $B(X \otimes_A \mathcal{H})$ and coincides with the canonical copy of $LM(\mathbb{K}_A(X))$ there. (The assertion that $B_A(X) \cong LM(\mathbb{K}_A(X))$ was first proved by Lin in [26]; we generally follows the proof given in [9, 8.1.16].) The other pieces of this proposition, if not standard fare covered in all the introductory texts mentioned above, may be deduced similarly.

For $T \in B_A(X)$, define a map $\tilde{T}$ on linear combinations of simple tensors in $X \otimes_A \mathcal{H}$ by

$$\tilde{T}(\sum_i x_i \otimes \zeta_i) = \sum_i T(x_i) \otimes \zeta_i.$$

It follows from the matrix inequality $[\langle T(x_i)|T(x_j)\rangle] \leq \|T\|^2[\langle x_i|x_j\rangle]$ (see the proof of [9, Proposition 8.2.2]) that $\tilde{T}$ may be extended to a well-defined operator in $B(X \otimes_A \mathcal{H})$ with $\|\tilde{T}\| \leq \|T\|$. To see the converse inequality, note that for $x \in X$, $\|T(x)\| = \sup\{\|T(x) \otimes \zeta\| : \zeta \in \mathrm{Ball}(\mathcal{H})\} \leq \|\tilde{T}\|\|x\|$. Hence $T \mapsto \tilde{T}$ is isometric, and it is easy to see that this map is linear and multiplicative.

Let $\mathfrak{K} = \{\tilde{K} : K \in \mathbb{K}_A(X)\}$ be the image of $\mathbb{K}_A(X)$ under the embedding just defined. By Cohen's factorization theorem (or a C^*-module factorization lemma like [36, Proposition 2.31]), each $x \in X$ can be written as $K(x')$ for some $K \in \mathbb{K}_A(X)$ and $x' \in X$. It follows that $\mathfrak{K}$ acts nondegenerately on $X \otimes_A \mathcal{H}$. We claim that

$$\{\tilde{T} : T \in B_A(X)\} = \{S \in B(X \otimes_A \mathcal{H}) : S\mathfrak{K} \subseteq \mathfrak{K}\}.$$

The inclusion of the former into the latter follows from the equation $T|x\rangle\langle y| = |Tx\rangle\langle y|$. For the converse, let S be in the latter, and define a map $T : X \to X$ by $T(x) = L(x')$ where we factor $x = K(x')$ as a few lines above, and where L is the unique operator in $\mathbb{K}_A(X)$ such that $\tilde{L} = S\tilde{K}$. To see that this is well-defined, let

(e_t) be a cai for $\mathbb{K}_A(X)$. For each t, there is an $S_t \in \mathbb{K}_A(X)$ such that $\tilde{S}_t = S\tilde{e}_t$. Since $\tilde{e}_t \xrightarrow{SOT} I_{X\otimes_A\mathcal{H}}$, we have $S_t(x)\otimes\zeta \to S(x\otimes\zeta) = L(x')\otimes\zeta$ for all $\zeta \in \mathcal{H}$, and it follows that $L(x') = \lim_t S_t(x)$. Hence the definition of $T(x)$ does not depend on the particular factorization $x = K(x')$. It follows quickly from these descriptions of T that $T \in B_A(X)$ and $\tilde{T} = S$. $\square$

Note 2.1.2. We will often use the proposition above many times in the sequel, often without mention and often without distinguishing between a C^*-module operator and its image as a Hilbert space operator. That said, we will sometimes have two C^*-algebras $\mathfrak{A} \subseteq B(\mathcal{K})$ and $\mathfrak{B} \subseteq B(\mathcal{H})$ and a bimodule X that is a left C^*-module over $\mathfrak{A}$ and a right C^*-module over $\mathfrak{B}$; in this case, it is important to distinguish whether we are viewing X as embedded in $B(\mathcal{H}, X\otimes_{\mathfrak{B}}\mathcal{H})$ or in $B(\overline{X}\otimes_{\mathfrak{A}}\mathcal{K}, \mathcal{K})$ (see Note 3.1.3).

Definition 2.1.3. A right C^*-module X over a A is called *selfdual* if every bounded A-module map $X \to A$ is of the form $\langle x|\cdot\rangle$ for some $x \in X$. A *W^*-module* is a selfdual C^*-module over a W^*-algebra.

There are many beautiful characterizations of W^*-modules among C^*-modules. Most elegantly, a C^*-module over a W^*-algebra is a W^*-module if and only if it has a Banach space predual (this was originally proved in [44] and [18], or see [10, Corollary 3.5] for another proof). For the purposes of this paper, the following characterization may be taken as motivation:

Proposition 2.1.4. *A C^*-module Y over a von Neumann algebra $M \subseteq B(\mathcal{H})$ is a W^*-module if and only if the canonical image of Y in $B(\mathcal{H}, Y\otimes_M\mathcal{H})$ is weak*-closed.*

Proof. ($\Longrightarrow$) Assume Y is a W^*-module. By the Krein-Smulian theorem, it suffices to prove that if (y_λ) is a bounded net in Y such that $y_\lambda \xrightarrow{w^*} T$ in $B(\mathcal{H}, X \otimes_M \mathcal{H})$, then $T \in Y$. If we have such a net (y_λ) and operator T, then for any $x \in Y$, $\langle y_\lambda \otimes \zeta, x \otimes \eta\rangle = \langle \zeta, \langle y_\lambda | x\rangle \eta\rangle$ is convergent for all $\zeta, \eta \in \mathcal{H}$, hence $\langle y_\lambda | x\rangle$ converges WOT to some $a_x \in M$, and since the WOT and weak* topology on $B(\mathcal{H})$ coincide on bounded sets, we have $\langle y_\lambda | x\rangle \xrightarrow{w^*} a_x$. Since Y is a W^*-module, the map $x \mapsto a_x$ has the form $\langle y | \cdot\rangle$ for some $y \in Y$, and it follows easily that $T = y$ in $B(\mathcal{H}, Y \otimes_M \mathcal{H})$.

($\Longleftarrow$) Assume the latter condition, let $\varphi \in B_M(Y, M)$, and let (e_t) be a cai (contractive approximate identity) for $\mathbb{K}_M(Y)$. For each t, $\varphi e_t \in \mathbb{K}_M(Y, M)$, and so there is a $y_t \in Y$ such that $\varphi e_t = \langle y_t | \cdot\rangle$ (by the top right corner of the top isomorphism in Proposition 2.1.1). By assumption, Y is a dual Banach space, and so (y_t) has a weak*-convergent subnet $y_{t_s} \xrightarrow{w^*} y$.

Using Cohen's factorization theorem ([9, A.6.2]) to write any $x \in Y$ as $x = Kx'$ for $K \in \mathbb{K}_M(Y)$ and $x' \in Y$, we have $e_t(x) = e_t(Kx') = (e_t K)(x') \to Kx' = x$ in norm in Y, so $(\varphi e_t)(x) = \varphi(e_t x) \to \varphi(x)$ in norm in M. Hence $(\varphi e_t)(x \otimes \zeta) = (\varphi e_t)(x)(\zeta) \to \varphi(x)(\zeta) = \varphi(x \otimes \zeta)$ in $\mathcal{H}$ for all $x \in Y$ and $\zeta \in \mathcal{H}$. Since (φe_t) is bounded and the simple tensors are total in $Y \otimes_M \mathcal{H}$, a triangle inequality argument shows that (φe_t) converges in the SOT (strong operator topology), hence weak*, to φ in $B(Y \otimes_M \mathcal{H}, \mathcal{H})$.

Since $y_{t_s} \xrightarrow{w^*} y$ in $B(\mathcal{H}, Y \otimes_M \mathcal{H})$, we have

$$\langle \langle y_{t_s} | \cdot\rangle (x \otimes \zeta), \eta\rangle = \langle x \otimes \zeta, y_{t_s}(\eta)\rangle \to \langle x \otimes \zeta, y(\eta)\rangle = \langle \langle y | \cdot\rangle (x \otimes \zeta), \eta\rangle.$$

Since $(\langle y_{t_s} | \cdot\rangle)$ is bounded, another triangle inequality argument as in the previous paragraph gives $\langle y_{t_s}|\cdot\rangle \xrightarrow{WOT} \langle y|\cdot\rangle$, so that $\langle y_{t_s}|\cdot\rangle \xrightarrow{w^*} \langle y|\cdot\rangle$ by boundedness again.

Since $\varphi e_t = \langle y_t|\cdot\rangle$, we may combine the previous two paragraphs to conclude that $\varphi = \langle y|\cdot\rangle$. $\square$

2.2 Σ^*-algebras

Recall that the weak operator topology (WOT) on $B(\mathcal{H})$ may be described as follows in terms of convergence of nets: $T_\lambda \xrightarrow{WOT} T$ if and only if $\langle T_\lambda\zeta, \eta\rangle \to \langle T\zeta, \eta\rangle$ for all $\zeta, \eta \in \mathcal{H}$.

Definition 2.2.1 ([16])**.** A *concrete Σ^*-algebra* is a nondegenerate C^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$ that is closed under limits of WOT-convergent sequences, i.e. whenever (b_n) is a sequence in $\mathfrak{B}$ that converges in the weak operator topology of $B(\mathcal{H})$ to an operator T, then $T \in \mathfrak{B}$.

For a Banach space X, a *σ-convergence system* is a set $\mathscr{S}$ whose elements are pairs $((x_n), x)$ consisting of a sequence $(x_n) \subset X$ and an element $x \in X$. (Elements in $\mathscr{S}$ should be thought of as convergent sequences with specified limits.)

An *abstract Σ^*-algebra* $(\mathfrak{B}, \mathscr{S})$ is a C^*-algebra $\mathfrak{B}$ with a σ-convergence system $\mathscr{S}$ such that there exists a faithful representation $\pi : \mathfrak{B} \to B(\mathcal{H})$ in which $\pi(\mathfrak{B})$ is a concrete Σ^*-algebra and $((b_n), b) \in \mathscr{S}$ if and only if $\pi(b_n) \xrightarrow{WOT} \pi(b)$ in $B(\mathcal{H})$. The sequences in $\mathscr{S}$ are called the *σ-convergent* sequences of $\mathfrak{B}$, and we write $b_n \xrightarrow{\sigma} b$ (or $b_n \xrightarrow{\mathscr{S}} b$ if there is a chance of ambiguity with the σ-convergence system) to mean $((b_n), b) \in \mathscr{S}$.

A Σ^*-*representation* of an abstract Σ^*-algebra $(\mathfrak{B}, \mathscr{S})$ is a nondegenerate representation $\pi : \mathfrak{B} \to B(\mathcal{H})$ such that $b_n \xrightarrow{\sigma} b$ implies $\pi(b_n) \xrightarrow{WOT} \pi(b)$. A *faithful* Σ^*-*representation* is an isometric Σ^*-representation π such that $\pi(\mathfrak{B})$ is a concrete Σ^*-algebra and $b_n \xrightarrow{\sigma} b$ if and only if $\pi(b_n) \xrightarrow{WOT} \pi(b)$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be abstract Σ^*-algebras. A map $\varphi : \mathfrak{A} \to \mathfrak{B}$ is σ-*continuous* if $\varphi(a_n) \xrightarrow{\sigma} \varphi(a)$ whenever $a_n \xrightarrow{\sigma} a$ in $\mathfrak{A}$. A Σ^*-*isomorphism* is a $*$-isomorphism $\psi : \mathfrak{A} \to \mathfrak{B}$ such that ψ and ψ^{-1} are σ-continuous. A Σ^*-*subalgebra* of $\mathfrak{B}$ is a C^*-subalgebra $\mathfrak{C}$ closed under limits of σ-convergent sequences in $\mathfrak{C}$. A Σ^*-*embedding* of $\mathfrak{A}$ into $\mathfrak{B}$ is an isometric $*$-homomorphism $\rho : \mathfrak{A} \hookrightarrow \mathfrak{B}$ such that $\rho(\mathfrak{A})$ is a Σ^*-subalgebra of $\mathfrak{B}$ and ρ is a Σ^*-isomorphism onto $\rho(\mathfrak{A})$.

The following lemma will make some proofs later a little shorter:

Lemma 2.2.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ^*-algebras. An injective $*$-homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a Σ^*-embedding if and only if the following two conditions hold:*

(1) *φ is σ-continuous;*

(2) *if (a_n) is a sequence in $\mathfrak{A}$ such that $(\varphi(a_n))$ is σ-convergent in $\mathfrak{B}$, then there is an $a \in \mathfrak{A}$ such that $a_n \xrightarrow{\sigma} a$.*

Proof. ($\Longrightarrow$) Suppose φ is a Σ^*-embedding. Condition (1) holds by definition. For (2), suppose (a_n) is a sequence in $\mathfrak{A}$ such that $(\varphi(a_n))$ is σ-convergent in $\mathfrak{B}$. Since $\varphi(\mathfrak{A})$ is σ-closed, there is an $a \in \mathfrak{A}$ such that $\varphi(a_n) \xrightarrow{\sigma} \varphi(a)$. Since φ^{-1} is σ-continuous, $a_n \xrightarrow{\sigma} a$.

($\Longleftarrow$) Suppose an injective $*$-homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ satisfies conditions (1) and (2). If (a_n) is a sequence in $\mathfrak{A}$ and $b \in \mathfrak{B}$ such that $\varphi(a_n) \xrightarrow{\sigma} b$, then by (2), there is an $a \in \mathfrak{A}$ such that $a_n \xrightarrow{\sigma} a$. By (1), $\varphi(a_n) \xrightarrow{\sigma} \varphi(a)$, so that $b = \varphi(a)$ by the uniqueness of σ-limits. Thus $\varphi(\mathfrak{A})$ is a σ-closed $*$-subalgebra of $\mathfrak{B}$, so it is a Σ^*-subalgebra of $\mathfrak{B}$. To see that $\varphi^{-1} : \varphi(\mathfrak{A}) \to \mathfrak{A}$ is σ-continuous, suppose $\varphi(a_n) \xrightarrow{\sigma} \varphi(a)$. By the argument we just employed, there is an $a' \in \mathfrak{A}$ such that $a_n \xrightarrow{\sigma} a'$ and $\varphi(a) = \varphi(a')$. Since φ is injective, $a = a'$, so $a_n \xrightarrow{\sigma} a$. □

Note 2.2.3. The phrase "$\mathfrak{B}$ is a Σ^*-algebra" should be taken to mean that $\mathfrak{B}$ is an abstract Σ^*-algebra with an implicit collection of σ-convergent sequences; the phrase "$\mathfrak{B} \subseteq B(\mathcal{H})$ is a Σ^*-algebra" should be taken to mean that $\mathfrak{B}$ is a concrete Σ^*-algebra in $B(\mathcal{H})$.

E. B. Davies in [16] proved that a C^*-algebra $\mathfrak{B}$ with a σ-convergence system $\mathscr{S}$ is an abstract Σ^*-algebra if and only if the following four conditions hold:

(1) if $((x_n), x) \in \mathscr{S}$, then (x_n) is a bounded sequence;

(2) if $((x_n), x) \in \mathscr{S}$, then $((x_n y), xy) \in \mathscr{S}$ for all $y \in \mathfrak{B}$;

(3) if (x_n) is a sequence in $\mathfrak{B}$ such that $\phi(x_n)$ converges for every $\mathscr{S}$-continuous functional $\phi \in \mathfrak{B}^*$, then there is an $x \in \mathfrak{B}$ such that $((x_n), x) \in \mathscr{S}$;

(4) if $0 \neq x \in \mathfrak{B}$, then there is an $\mathscr{S}$-continuous functional $\phi \in \mathfrak{B}^*$ such that $\phi(x) \neq 0$.

(We include this mainly as motivation for our definitions of Σ-Banach spaces and Σ-operator spaces in Chapter 5.)

For any subset $S \subseteq B(\mathcal{H})$, denote by $\mathscr{B}(S)$ the smallest WOT sequentially closed subset of $B(\mathcal{H})$ containing S. Such a set exists since the intersection of any family of WOT sequentially closed subsets is also WOT sequentially closed. If there is ambiguity (for example if we represent a C^*-algebra on two different Hilbert spaces), we add a subscript; e.g., $\mathscr{B}_{\mathcal{H}}(S)$. By the following proposition, these closures provide many examples of Σ^*-algebras. The proof of this proposition exhibits a common technique for proving results about Σ^*-algebras.

Proposition 2.2.4. *If $A \subseteq B(\mathcal{H})$ is a nondegenerate C^*-algebra, then $\mathscr{B}(A)$ is a Σ^*-algebra.*

Proof. ([16, Lemma 2.1]) Fix $a \in A$, and let $S = \{b \in \mathscr{B}(A) : ab \in \mathscr{B}(A)\}$. Clearly S is WOT sequentially closed and contains A, so $S = \mathscr{B}(A)$. Hence $ab \in \mathscr{B}(A)$ for all $a \in A$ and $b \in \mathscr{B}(A)$. Similar tricks show that $bc \in \mathscr{B}(A)$ for all $b, c \in \mathscr{B}(A)$ and that $\mathscr{B}(A)$ is a $*$-invariant subspace of $B(\mathcal{H})$. Since $\mathscr{B}(A)$ is also evidently norm-closed, the result follows. $\square$

Example 2.2.5. (1) Every von Neumann algebra is clearly a Σ^*-algebra. Conversely, if $\mathcal{H}$ is separable, then every Σ^*-algebra is a von Neumann algebra. Kadison first proved this fact for Σ^*-algebras in an appendix to [16] using the Kaplansky density theorem with the fact that the unit ball of a von Neumann algebra acting on a separable Hilbert space is WOT-metrizable. Another proof for this goes through Pedersen's up-down theorem ([34, 2.4.3]).

(2) If $\mathcal{H}$ is a Hilbert space, then the ideal $\mathscr{I}$ of operators in $B(\mathcal{H})$ with separable range is the Σ^*-algebra $\mathscr{B}(\mathbb{K}(\mathcal{H}))$, which is of course not unital if $\mathcal{H}$

is not separable. Indeed, every operator with separable range is a limit in the strong operator topology (SOT) of a sequence of finite rank operators (if $T \in B(\mathcal{H})$ has separable range, and $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\overline{T(\mathcal{H})}$, then $\sum_{k=1}^{n} |e_k\rangle\langle T^* e_k| \xrightarrow{SOT} T$ as $n \to \infty$). Conversely, by basic operator theory, every compact operator has separable range, so $\mathbb{K}(\mathcal{H}) \subseteq \mathscr{I}$. To see that $\mathscr{I}$ is WOT sequentially closed, suppose (T_n) is a sequence in $\mathscr{I}$ converging in the WOT to $T \in B(\mathcal{H})$. Then $P := \vee_n r(T_n)$ (where $r(T_n)$ denotes the projection onto $\overline{\mathrm{Ran}(T_n)}$) is a projection with separable range, and $T_n = PT_n \xrightarrow{WOT} PT$. Hence $PT = T$, so T has separable range.

(3) Let A be a C^*-algebra considered as a concrete C^*-algebra in its universal representation $A \subseteq B(\mathcal{H}_u)$. The Σ^*-algebra $\Sigma^*(A) := \mathscr{B}(A)$ obtained here is called the *Davies-Baire envelope of A* (following the terminology of [42]). It was proved by Davies in [16, Theorem 3.2] that $\Sigma^*(A)$ is Σ^*-isomorphic to $\mathscr{B}_{\mathcal{H}_a}(A)$, where $A \hookrightarrow B(\mathcal{H}_a)$ is the atomic representation of A.

(4) (cf. [16, Corollary 3.3], [34, 4.5.14]) Let X be a locally compact Hausdorff space. By basic C^*-algebra theory, the atomic representation of the commutative C^*-algebra $C_0(X)$ is the embedding of $C_0(X)$ into $B(\ell^2(X))$ as multiplication operators. By the last statement in the previous example, $\Sigma^*(C_0(X))$ may be identified with the WOT sequential closure of $C_0(X)$ in $B(\ell^2(X))$. This closure is easily checked to be contained in the copy of the space of all bounded functions on X, $\ell^\infty(X)$, in $B(\ell^2(X))$. Since WOT-convergence of sequences in $\ell^\infty(X) \subseteq B(\ell^2(X))$ coincides with pointwise convergence of bounded sequences of functions, we may identify $\Sigma^*(C_0(X))$ with the bounded pointwise sequential

closure of $C_0(X)$ in $\ell^\infty(X)$—this is the space Baire(X) of bounded Baire functions on X (in the sense of [33, 6.2.10]). Recall two well-known classical facts about the bounded Baire functions: (a) if X is second countable, the bounded Baire functions and the bounded Borel-measurable functions on X coincide; and (b) X is σ-compact if and only if the constant functions are Baire, and if and only if the algebra Baire(X) is unital.

Indeed, for (a), [33, 6.2.9] shows that the monotone sequential closure of $C_0(X)$ in $\ell^\infty(X)$ is the space of bounded Borel functions on X, and by [33, 6.2.2], the latter space is bounded pointwise sequentially closed.

To prove (b), suppose first that X is σ-compact, and write $X = \cup_{n=1}^\infty K_n$ for an increasing sequence of compact sets K_n. For each $n \in \mathbb{N}$, let f_n be a function in $C_0(X)$ such that $f_n(x) = 1$ for all $x \in K_n$. Then $f_n \to 1$ pointwise, so the constant functions are Baire functions, and this evidently implies that Baire(X) is unital. For the final implication, suppose that Baire(X) is unital, with unit e. Then e is the characteristic function of some subset of X, and since $e\chi_S = \chi_S$ for any singleton S, $e = \chi_X$. So $\chi_X \in$ Baire(X). Now let $\mathscr{F}$ be the set of $f \in$ Baire(X) such that $\{x \in X : f(x) \neq 0\} \subseteq \cup_{n=1}^\infty K_n$ for some sequence of compact sets K_n. It is easy to check that $C_0(X) \subseteq \mathscr{F}$ and that $\mathscr{F}$ is closed under limits of pointwise convergent bounded sequences. Thus $\mathscr{F} =$ Baire(X), so the characteristic function of X is in $\mathscr{F}$, and it follows that X is σ-compact.

Thus $\Sigma^*(C_0(X))$ for non-σ-compact X provides another example of a nonunital Σ^*-algebra.

(5) If A is a separable C^*-algebra and ϕ is a faithful state on A, then the GNS

construction gives a faithful representation of A as operators on a separable Hilbert space $\mathcal{H}_\phi$. By (1) above, $\mathscr{B}_{\mathcal{H}_\phi}(A)$ is the weak*-closure of A in $B(\mathcal{H}_\phi)$. In particular, if $A = C(X)$ for a second countable compact Hausdorff space X, and μ is a finite positive Borel measure on X such that $\int f d\mu > 0$ for all nonzero positive $f \in C(X)$ (e.g., take μ to be Lebesgue measure on $X = [0,1]$), then by basic measure theory (see, e.g., [30, Example 4.1.2]), $\mathscr{B}_{L^2(X,\mu)}(C(X)) = L^\infty(X, \mu)$.

We now briefly record a few basic facts about Σ^*-algebras that we will use later, sometimes without mention. For the proof of Proposition 2.2.6, see [17, Lemma 2.1] or [34, 4.5.16]; for the proof of Proposition 2.2.7, see [34, 4.5.7].

Proposition 2.2.6. *Let T be an operator in a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$. If $T = U|T|$ is the polar decomposition of T, then $U \in \mathfrak{B}$.*

Proposition 2.2.7. *If $\mathfrak{B} \subseteq B(\mathcal{H})$ is a unital Σ^*-algebra and x is a selfadjoint element in $\mathfrak{B}$, then $f(x) \in \mathfrak{B}$ for all bounded Borel functions $f : \mathbb{R} \to \mathbb{C}$.*

The simple, well-known principles in the following lemma are crucial:

Lemma 2.2.8. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces.*

(1) *A WOT-convergent sequence in $B(\mathcal{H}_1, \mathcal{H}_2)$ is bounded.*

(2) *Let $\mathscr{T}_1, \mathscr{T}_2$ be sets of total vectors in $\mathcal{H}_1, \mathcal{H}_2$ respectively (i.e. the span of $\mathscr{T}_i$ is dense in $\mathcal{H}_i$). A sequence (T_n) in $B(\mathcal{H}_1, \mathcal{H}_2)$ is WOT-convergent if and only if (T_n) is bounded and $\langle T_n\zeta, \eta\rangle$ converges for all $\zeta \in \mathscr{T}_1$ and $\eta \in \mathscr{T}_2$. Also,*

$T_n \xrightarrow{WOT} T$ *if and only if* (T_n) *is bounded and* $\langle T_n\zeta, \eta\rangle \to \langle T\zeta, \eta\rangle$ *for all* $\zeta \in \mathscr{T}_1$ *and* $\eta \in \mathscr{T}_2$.

Proof. (1) Suppose (T_n) is a WOT-convergent sequence in $B(\mathcal{H}_1, \mathcal{H}_2)$. Then $(\langle\zeta, T_n\eta\rangle)$ is a convergent, hence bounded sequence for each $\eta \in \mathcal{H}_1$ and $\zeta \in \mathcal{H}_2$. By the principle of uniform boundedness applied to the functionals $\langle\cdot, T_n\eta\rangle$ in $\mathcal{H}_2^*$, it follows that $(T_n\eta)$ is a bounded sequence for each $\eta \in \mathcal{H}_1$. By the principle of uniform boundedness applied to the operators T_n, we have that (T_n) is a bounded sequence.

(2) The forward direction of the first statement is obvious using (1). For the converse, a simple triangle inequality argument shows that if (T_n) is a bounded sequence in $B(\mathcal{H}_1, \mathcal{H}_2)$ such that $(\langle T_n\zeta', \eta'\rangle)$ converges for all $\zeta' \in \mathscr{T}_1$ and $\eta \in \mathscr{T}_2$, then $(\langle T_n\zeta, \eta\rangle)$ converges for all $\zeta \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$. Define a sesquilinear form $\mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$, $(\zeta, \eta) \mapsto \lim_n\langle T_n\zeta, \eta\rangle$. Since (T_n) is bounded, this is a bounded sesquilinear form, hence by the correspondence between such forms and operators in $B(\mathcal{H}_1, \mathcal{H}_2)$, there is an operator $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $\langle T\zeta, \eta\rangle = \lim_n\langle T_n\zeta, \eta\rangle$ for all $\zeta \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$. Thus $T_n \xrightarrow{WOT} T$.

The forward direction of the second statement is also obvious using (1). For the converse, employ the same argument as above to get an operator $T' \in B(\mathcal{H})$ such that $\langle T_n\zeta, \eta\rangle \to \langle T'\zeta, \eta\rangle$ for all $\zeta \in \mathcal{H}_1$, $\eta \in \mathcal{H}_2$. Then $\langle T'\zeta', \eta'\rangle = \langle T\zeta', \eta'\rangle$ for all $\zeta' \in \mathscr{T}_1$, $\eta' \in \mathscr{T}_2$. Thus $T = T'$ and $T_n \xrightarrow{WOT} T$. □

Lemma 2.2.9. *If* $\mathfrak{B} \subseteq B(\mathcal{H})$ *is a nonunital* Σ^**-algebra, then its unitization* $\mathfrak{B}^1$ *is a* Σ^**-algebra in* $B(\mathcal{H})$*, and for* $(b_n), b \in \mathfrak{B}$ *and* $(\lambda_n), \lambda \in \mathbb{C}$*, we have* $b_n + \lambda_n I_{\mathcal{H}} \xrightarrow{WOT} b + \lambda I_{\mathcal{H}}$ *if and only if* $b_n \xrightarrow{WOT} b$ *and* $\lambda_n \to \lambda$.

Proof. If $(b_n + \lambda_n I_{\mathcal{H}})$ is a sequence in $\mathfrak{B}^1$ converging WOT to T in $B(\mathcal{H})$, then it follows from Lemma 2.2.8 that (λ_n) is bounded, hence has a subsequence (λ_{n_k}) converging to some $\lambda \in \mathbb{C}$. So $b_{n_k} \xrightarrow{WOT} T - \lambda I_{\mathcal{H}}$, and thus $T \in \mathfrak{B}^1$.

The backward direction of the last claim is obvious. For the forward direction, suppose $b_n + \lambda_n I_{\mathcal{H}} \xrightarrow{WOT} b + \lambda I_{\mathcal{H}}$. Just as above, (λ_n) is bounded. Let λ_{n_k} be a convergent subsequence of (λ_n) with $\lambda_{n_k} \to \lambda'$. Then

$$b_{n_k} = (b_{n_k} + \lambda_{n_k} I_{\mathcal{H}}) - \lambda_{n_k} I_{\mathcal{H}} \xrightarrow{WOT} b + \lambda I_{\mathcal{H}} - \lambda' I_{\mathcal{H}} = b + (\lambda - \lambda') I_{\mathcal{H}}.$$

Since $\mathfrak{B}$ is a Σ^*-algebra, $b + (\lambda - \lambda') I_{\mathcal{H}} \in \mathfrak{B}$. Since $\mathfrak{B}$ is nonunital, $\lambda = \lambda'$. Thus every convergent subsequence of (λ_n) has the same limit λ. Hence $\lambda_n \to \lambda$, and

$$b_n = (b_n + \lambda_n I_{\mathcal{H}}) - \lambda_n I_{\mathcal{H}} \xrightarrow{WOT} b + \lambda I_{\mathcal{H}} - \lambda I_{\mathcal{H}} = b.$$

$\square$

The following simple observation is well known and explains why the weak* sequentially closed operator spaces we will study in Chapter 5 are generalizations of Σ^*-algebras.

Lemma 2.2.10. *A sequence in $B(\mathcal{H})$ is weak*-convergent if and only if it is WOT-convergent. Hence a C^*-algebra $A \subseteq B(\mathcal{H})$ is a Σ^*-algebra if and only if it is sequentially closed in the weak* topology of $B(\mathcal{H})$.*

Proof. One direction is obvious. The other follows by Lemma 2.2.8 together with the well-known fact that a bounded WOT-convergent net is automatically weak*-convergent. $\square$

2.3 Notes

We record in this final section some non-crucial facts for extra context on Σ^*-algebras that some may find interesting or useful.

2.3.1 (Other ways to characterize Σ^*-algebras.)**.** There is an abstract characterization of Σ^*-algebras corresponding to Davies' result mentioned in Note 2.2.3 in which the σ-convergence system $\mathscr{S}$ is replaced by a subspace of the dual space A^* meeting certain axioms, and another characterization involving a closed convex subset of the state space $S(A)$. The latter perspective was mentioned by Davies in the original paper and is the underlying point of view in N. N. Dang's paper [15]. More explicitly, Dang defines a Σ^*-algebra to be a pair (A, S), where A is a C^*-algebra and S is a subset of $S(A)$ such that:

(1) if $\varphi \in S$ and $a \in A$ with $\varphi(a^*a) = 1$, then $\varphi(a^* \cdot a) \in S$;

(2) if ψ is a state on A such that $\psi(a_n)$ converges for all sequences (a_n) in ${}^{\sigma}S = \{(a_n) \in \ell^\infty(A) : \varphi(a_n) \text{ converges for all } \varphi \in S\}$, then $\psi \in S$;

(3) if $a \in A$ is nonzero, then $\varphi(a) \neq 0$ for some $\varphi \in S$;

(4) if $(a_n) \in {}^{\sigma}S$, then there is an $a \in A$ such that $\varphi(a_n) \to \varphi(a)$ for all $\varphi \in S$.

Elementary operator theoretic arguments show that if A is WOT sequentially closed, then the collection of WOT sequentially continuous states meets these requirements. Conversely, as Dang points out, one may use a slight modification of the polarization identity ($b^*xa = \frac{1}{4}\sum_{k=0}^{3} i^k(a + i^k b)^* x (a + i^k b)$ for $a, x, b \in A$), to check that if (A, S)

is a Σ^*-algebra in Dang's sense, then $(A, {}^\sigma S)$ is a Σ^*-algebra in Davies' sense, so that by Davies' result, A admits a representation as a Σ^*-algebra in our sense.

2.3.2 (Variants on Σ^*-algebras.)**.** A class of C^*-algebras similar to the Σ^*-algebras was studied by Pedersen in several papers (see [34, Section 4.5] for the main part of the theory and more references). He studied "Borel $*$-algebras," which are concrete C^*-algebras closed under limits of bounded monotone sequences of selfadjoint elements. In some ways, Borel $*$-algebras are more technically forbidding (e.g., compare Proposition 2.2.4 to [34, Theorem 4.5.4]), but in other ways they seem nicer—for example, it seems to be an open question whether or not a $*$-isomorphism between Σ^*-algebras is always a Σ^*-isomorphism, but it is easy to see that the analogous statement for Borel $*$-algebras is true.

Similarly, one may consider other variants; e.g., SOT-sequentially closed C^*-algebras. We admit that Borel $*$-algebras and the SOT variant both meet the motivations for this project discussed in the introduction just as well as Σ^*-algebras. Besides the fact mentioned above that Σ^*-algebras are sometimes technically easier to deal with than Borel $*$-algebras, one other reason we have for working with Σ^*-algebras rather than the other variants is that because of Lemma 2.2.10, the definition naturally extends to work in other categories; e.g., Banach spaces, as we will cover in Chapter 5.

2.3.3 (Relationship to other classes of C^*-algebras)**.** A *monotone σ-complete C^*-algebra* is a C^*-algebra A such that whenever (x_n) is a norm-bounded increasing sequence in A_{sa}, (x_n) has a supremum in A_{sa}. Evidently, every Σ^*-algebra is monotone σ-complete.

A *weakly Rickart C^*-algebra* is a C^*-algebra in which every maximal abelian $*$-subalgebra is monotone σ-complete (see [41]). Every Σ^*-algebra is a weakly Rickart C^*-algebra. Indeed, suppose M is a maximal abelian $*$-subalgebra of a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, and let (x_n) be a norm-bounded increasing sequence in M_{sa}. Then (x_n) has a supremum x in $\mathfrak{B}_{sa}$, and $x_n \xrightarrow{WOT} x$. So for every $y \in M$, $yx = \text{WOT-}\lim_n yx_n = \text{WOT-}\lim_n x_n y = xy$. It follows from maximality of M that $x \in M$.

A *Rickart C^*-algebra* is a unital weakly Rickart C^*-algebra (see [41]). It follows from the previous paragraph that every unital Σ^*-algebra is a Rickart C^*-algebra.

2.3.4 (Open Questions)**.** (Cf. [34] 4.5.14, [42, Section 5.3.1]) Though we do not address these in the present work, there are some interesting and natural open questions about Σ^*-algebras and similar classes of C^*-algebras.

As mentioned in 2.3.2, it is unknown whether or not every $*$-isomorphism between Σ^*-algebras is a Σ^*-isomorphism. In fact, it appears to be unknown whether or not $*$-isomorphic Σ^*-algebras are always automatically Σ^*-isomorphic.

Related to (3) in Example 2.2.5, if $A \subseteq B(\mathcal{H}_u)$ is a C^*-algebra in its universal representation, it is unknown whether or not one must have $\mathscr{B}(A) = \mathscr{B}^m(A)$, where the latter refers to the monotone sequential closure of A (that is, $\mathscr{B}^m(A) = \mathscr{B}^m(A_{sa}) + i\mathscr{B}^m(A_{sa})$, where $\mathscr{B}^m(A_{sa})$ is the smallest subset of $B(\mathcal{H}_u)_{sa}$ containing A_{sa} and closed under limits of bounded increasing sequences). Clearly the inclusion $\mathscr{B}^m(A) \subseteq \mathscr{B}(A)$ always holds. Pedersen proved that $\mathscr{B}^m(A)$ is always a C^*-algebra ([34, 4.5.4]) and that the equation $\mathscr{B}(A) = \mathscr{B}^m(A)$ does hold if A is type I ([34, Section 6.3]).

One may also replace the monotone sequential closure $\mathscr{B}^m(A)$ in the paragraph above with a number of variants—for example, the SOT sequential closure of A, $\mathscr{B}^s(A)$. Clearly $\mathscr{B}^s(A)$ lies between $\mathscr{B}^m(A)$ and $\mathscr{B}(A)$, but as far as we know, the questions of whether or not $\mathscr{B}^s(A)$ always equals $\mathscr{B}(A)$ or $\mathscr{B}^m(A)$ are still open. (Note that by Lemma 2.2.10 the weak* sequential closure of A coincides with $\mathscr{B}(A)$.)

In fact, as far as we can tell, there is no known example of *any* Borel $*$-algebra that is not a Σ^*-algebra (or a Σ^*-algebra that is not SOT sequentially closed).

Somewhat similar in spirit is the interesting open question of whether or not A^m_{sa} (the set of limits in A^{**}_{sa} of bounded increasing nets in A_{sa}) is always norm-closed. Brown proved in [12] (Corollary 3.25) that this does hold if A is separable. See [14] for an insightful discussion on this problem.

Chapter 3

Hilbert C^*-modules over Σ^*-algebras

This chapter is broken up into two sections. In the first, we define our class of "Σ^*-modules" and prove general results analogous to many of the basic results in C^*- and W^*-module theory. In particular, we show that Σ^*-modules correspond with WOT sequentially closed ternary rings of operator (TROs) and with corners of Σ^*-algebras in the same way that C^*-modules (resp. W^*-modules) correspond with norm-closed (resp. weak*-closed) TROs and with corners of C^*-algebras (resp. W^*-algebras). The other main highlight of this section is the "Σ^*-module completion" of a C^*-module over a Σ^*-algebra, in analogy with the selfdual completion of a C^*-module over a W^*-algebra.

In the second section, we study the subclass of "$\Sigma^*_{\mathfrak{B}}$-countably generated" Σ^*-modules, and are able to prove many satisfying results about these—for example, that all $\Sigma^*_{\mathfrak{B}}$-countably generated Σ^*-modules are selfdual. As expected, there is also a WOT sequential version of Kasparov's stabilization theorem that holds in this case.

3.1 Σ^*-modules

Definition 3.1.1. A right (resp. left) C^*-module $\mathfrak{X}$ over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$ is called a *right (resp. left) Σ^*-module* if the canonical image of $\mathfrak{X}$ in $B(\mathcal{H}, \mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$ (resp. in $B(\overline{\mathfrak{X}} \otimes_{\mathfrak{B}} \mathcal{H}, \mathcal{H})$) is WOT sequentially closed. As with C^*-modules, "$\mathfrak{X}$ is a Σ^*-module" means "$\mathfrak{X}$ is a right Σ^*-module." We usually only explicitly prove results for right Σ^*-modules, but in these cases there is always an easily translated "left version."

Note the evident facts that every Σ^*-algebra is a Σ^*-module over itself (this will be generalized in Theorem 3.1.10) and that a Σ^*-module $\mathfrak{X}$ over a non-unital Σ^*-algebra $\mathfrak{B}$ is canonically a Σ^*-module over $\mathfrak{B}^1$ (indeed, the algebraic module tensor products $\mathfrak{X} \odot_{\mathfrak{B}} \mathcal{H}$ and $\mathfrak{X} \odot_{\mathfrak{B}^1} \mathcal{H}$ coincide, so we have equality of the Hilbert spaces $\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H} = \mathfrak{X} \otimes_{\mathfrak{B}^1} \mathcal{H}$).

We will show shortly (Proposition 3.1.5) that every selfdual C^*-module over a Σ^*-algebra is a Σ^*-module, but the converse is not true. Indeed, if $\mathfrak{B}$ is a nonunital Σ^*-algebra (e.g., the bounded Baire functions on a non-σ-compact locally compact Hausdorff space X, or $\mathscr{B}(\mathbb{K}(\mathcal{H}))$ for nonseparable $\mathcal{H}$) viewed as a Σ^*-module over itself, then $\mathfrak{B}$ is not selfdual since the identity map on $\mathfrak{B}$ is not of the form $x \mapsto y^*x$ for some $y \in \mathfrak{B}$. However, we will show in Theorem 3.2.10 that these notions do coincide in the case of $\Sigma^*_{\mathfrak{B}}$-countably generated C^*-modules over Σ^*-algebras.

Lemma 3.1.2. *Let X be a C^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$. For a sequence $(x_n) \in X$ and $x \in X$, we have $\langle x_n | y \rangle \xrightarrow{WOT} \langle x | y \rangle$ for all $y \in X$ if and only if $x_n \xrightarrow{WOT} x$ in $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$.*

Proof. ($\implies$) Suppose that $\langle x_n | y \rangle \xrightarrow{WOT} \langle x | y \rangle$ for all $y \in X$. For each n, let $\varphi_n : X \to \mathfrak{B}$ be the bounded linear map defined by $\varphi_n(y) = \langle x_n | y \rangle$. Then for any $y \in X$, $\sup_n \|\varphi_n(y)\| = \sup_n \|\langle x_n | y \rangle\| < \infty$ since the sequence $(\langle x_n | y \rangle)$ is WOT-convergent, hence bounded. By the uniform boundedness principle, $\sup_n \|x_n\| = \sup_n \|\varphi_n\| < \infty$. Since

$$\begin{aligned} \langle x_n(\zeta), y \otimes \eta \rangle = \langle x_n \otimes \zeta, y \otimes \eta \rangle = \langle \zeta, \langle x_n | y \rangle \eta \rangle \\ \longrightarrow \langle \zeta, \langle x | y \rangle \eta \rangle = \langle x_n \otimes \zeta, y \otimes \eta \rangle = \langle x(\zeta), y \otimes \eta \rangle \end{aligned}$$

for all $\zeta, \eta \in \mathcal{H}$ and $y \in X$, and since elements of the form $y \otimes \eta$ are total in $X \otimes_{\mathfrak{B}} \mathcal{H}$, it follows from a triangle inequality argument that $x_n \xrightarrow{WOT} x$ in $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$.

($\impliedby$) Assume $x_n \xrightarrow{WOT} x$ in $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$, and take $y \in X$. Then for any $\zeta, \eta \in \mathcal{H}$, we have

$$\langle \zeta, \langle x_n | y \rangle \eta \rangle = \langle x_n(\zeta), y \otimes \eta \rangle \to \langle x(\zeta), y \otimes \eta \rangle = \langle \zeta, \langle x | y \rangle \eta \rangle,$$

so that $\langle x_n | y \rangle \xrightarrow{WOT} \langle x | y \rangle$. $\square$

Note 3.1.3. A similar result holds for left C^*-modules—namely, if X is a left C^*-module over a Σ^*-algebra $\mathfrak{A} \subseteq B(\mathcal{K})$ with $\mathfrak{A}$-valued inner product $\langle \cdot | \cdot \rangle_{\mathfrak{A}}$, then $\langle x_n | y \rangle_{\mathfrak{A}} \xrightarrow{WOT} \langle x | y \rangle_{\mathfrak{A}}$ for all $y \in X$ if and only if $x_n \xrightarrow{WOT} x$ in $B(\overline{X} \otimes_{\mathfrak{A}} \mathcal{K}, \mathcal{K})$. If

X is both a left Σ^*-module over $\mathfrak{A}$ and a right Σ^*-module over $\mathfrak{B}$, there is thus the potential for confusion in an expression like "$x_n \xrightarrow{WOT} x$." To distinguish, we write:

$$x_n \xrightarrow{\mathfrak{A}WOT} x \text{ iff } \langle x_n|y\rangle_{\mathfrak{A}} \xrightarrow{WOT} \langle x|y\rangle_{\mathfrak{A}} \text{ for all } y \in X$$

and

$$x_n \xrightarrow{WOT_{\mathfrak{B}}} x \text{ iff } \langle x_n|y\rangle_{\mathfrak{B}} \xrightarrow{WOT} \langle x|y\rangle_{\mathfrak{B}} \text{ for all } y \in X$$

where $\langle\cdot|\cdot\rangle_{\mathfrak{A}}$ denotes the $\mathfrak{A}$-valued inner product and $\langle\cdot|\cdot\rangle_{\mathfrak{B}}$ denotes the $\mathfrak{B}$-valued inner product on X. Note that these notations make good sense even if $\mathfrak{A}$ and $\mathfrak{B}$ are concrete C^*-algebras that are not necessarily WOT sequentially closed.

The following proposition is often helpful when proving that a C^*-module is a Σ^*-module, and we will use it for this purpose many times.

Proposition 3.1.4. *Let $\mathfrak{X}$ be a C^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$. The following are equivalent:*

(1) *$\mathfrak{X}$ is a Σ^*-module;*

(2) *whenever (x_n) is a sequence in $\mathfrak{X}$ such that $\langle x_n|y\rangle$ is WOT-convergent in $B(\mathcal{H})$ for all $y \in \mathfrak{X}$, then there is a (unique) $x \in \mathfrak{X}$ such that $\langle x_n|y\rangle \xrightarrow{WOT} \langle x|y\rangle$ for all $y \in \mathfrak{X}$;*

(3) *the space $\hat{\mathfrak{X}} := \{\langle x|\cdot\rangle : x \in \mathfrak{X}\}$ is point-WOT sequentially closed in $B_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{B})$.*

Proof. (1) $\implies$ (2). Assume $\mathfrak{X}$ is a Σ^*-module, and let (x_n) be a sequence in $\mathfrak{X}$ such that $\langle x_n|y\rangle$ is WOT-convergent in $B(\mathcal{H})$ for all $y \in \mathfrak{X}$. Then $\langle x_n(\zeta), y \otimes \eta\rangle = \langle \zeta, \langle x_n|y\rangle\eta\rangle$ is convergent for all $\zeta, \eta \in \mathcal{H}$ and $y \in \mathfrak{X}$. Since (x_n) is a bounded sequence (as in the proof of the forward direction of the previous lemma), it follows that $\langle x_n(\zeta), \xi\rangle$ converges for all $\zeta \in \mathcal{H}$ and $\xi \in \mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H}$. By Lemma 2.2.8, there is an operator $T \in B(\mathcal{H}, \mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$ with $x_n \xrightarrow{WOT} T$. So by assumption, $T \in \mathfrak{X}$. By the backward direction of the previous lemma, $\langle x_n|y\rangle \xrightarrow{WOT} \langle T|y\rangle$ for all $y \in \mathfrak{X}$. Uniqueness follows from the usual argument that the canonical map $\mathfrak{X} \to B_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{B})$ is one-to-one.

(2) $\implies$ (1). Assuming (2), let (x_n) be a sequence in $\mathfrak{X}$ such that $x_n \xrightarrow{WOT} T$ in $B(\mathcal{H}, \mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$. Then

$$\langle \zeta, \langle x_n|y\rangle\eta\rangle = \langle x_n(\zeta), y \otimes \eta\rangle \to \langle T(\zeta), y \otimes \eta\rangle$$

for all $\zeta, \eta \in \mathcal{H}$ and $y \in \mathfrak{X}$. It follows that $\langle x_n|y\rangle$ is WOT-convergent for all $y \in \mathfrak{X}$, so by assumption there is an $x \in \mathfrak{X}$ such that $\langle x_n|y\rangle \xrightarrow{WOT} \langle x|y\rangle$ for all $y \in \mathfrak{X}$. By the forward direction of the previous lemma, $x_n \xrightarrow{WOT} x$, so that $T = x \in \mathfrak{X}$.

The equivalence of (2) and (3) follows by noting that if (x_n) is a sequence in $\mathfrak{X}$ such that $\langle x_n|y\rangle$ is WOT-convergent in $B(\mathcal{H})$ for all $y \in \mathfrak{X}$, then $y \mapsto \text{WOT-}\lim_n\langle x_n|y\rangle$ defines an operator in $B_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{B})$. $\square$

Proposition 3.1.5. *If $\mathfrak{X}$ is a selfdual C^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, then $\mathfrak{X}$ is a Σ^*-module.*

Proof. Let (x_n) be a sequence in $\mathfrak{X}$ such that $\langle x_n|y\rangle$ is WOT-convergent for all $y \in \mathfrak{X}$. Define $\psi : \mathfrak{X} \to \mathfrak{B}$ by setting $\psi(y) = \text{WOT-}\lim_n\langle x_n|y\rangle$. It is easy to check that $\psi \in B_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{B})$, so by assumption $\psi = \langle x|\cdot\rangle$ for some $x \in \mathfrak{X}$. But this means $\langle x_n|y\rangle \xrightarrow{\text{WOT}} \langle x|y\rangle$ for all $y \in \mathfrak{X}$, and so by Proposition 3.1.4, $\mathfrak{X}$ is a Σ^*-module. $\square$

One of the most basic results in the theory of C^*-modules (and one that is crucial in the theory of Morita equivalence) is the fact that a right C^*-module X over a C^*-algebra A is a left C^*-module over $\mathbb{K}_A(X)$. Analogously, if Y is a right W^*-module over a W^*-algebra M, then $\mathbb{B}_M(Y)$ is a W^*-module, and Y is a left W^*-module over $\mathbb{B}_M(Y)$. The following proposition and theorem show that the obvious Σ^*-analogues of these statements are true. (Note that the following proposition generalizes the easy fact that the multiplier algebra and left multiplier algebra of a Σ^*-algebra are WOT sequentially closed. Indeed, in the special case $\mathfrak{X} = \mathfrak{B}$, we have by Proposition 2.1.1 that $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X}) = M(\mathfrak{B})$ and $B_{\mathfrak{B}}(\mathfrak{B}) = LM(\mathfrak{B})$.)

Proposition 3.1.6. *If $\mathfrak{X}$ is a right Σ^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, then $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ and $B_{\mathfrak{B}}(\mathfrak{X})$ are WOT sequentially closed in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$. For a sequence (T_n) and element T in $B_{\mathfrak{B}}(\mathfrak{X})$, $T_n \xrightarrow{WOT} T$ if and only if $T_n(x) \xrightarrow{WOT_{\mathfrak{B}}} T(x)$ for all $x \in \mathfrak{X}$.*

Proof. Let (T_n) be a sequence in $B_{\mathfrak{B}}(\mathfrak{X}) \subseteq B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$ such that $T_n \xrightarrow{WOT} T$ for some $T \in B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$. Then for $x, y \in \mathfrak{X}$ and $\zeta, \eta \in \mathcal{H}$,

$$\langle \zeta, \langle T_n(x)|y\rangle \eta\rangle = \langle T_n(x) \otimes \zeta, y \otimes \eta\rangle = \langle T_n(x \otimes \zeta), y \otimes \eta\rangle \to \langle T(x \otimes \zeta), y \otimes \eta\rangle.$$

Hence, fixing $x \in \mathfrak{X}$, we have $\langle T_n(x)|y\rangle$ is WOT-convergent for all $y \in \mathfrak{X}$. By Proposition 3.1.4, there is a unique element, call it $\tilde{T}(x)$, in $\mathfrak{X}$ such that

$$\langle T_n(x)|y\rangle \xrightarrow{WOT} \langle \tilde{T}(x)|y\rangle \text{ for all } y \in \mathfrak{X}.$$

Doing this for each $x \in \mathfrak{X}$ yields a map $\tilde{T} : \mathfrak{X} \to \mathfrak{X}$. Since $\|\tilde{T}(x)\| = \sup\{\|\langle \tilde{T}(x)|y\rangle\| : y \in \mathrm{Ball}(\mathfrak{X})\}$ and $\|\langle \tilde{T}(x)|y\rangle\| \leq (\sup_n \|T_n\|)\|x\|\|y\|$, we see that $\tilde{T}$ is bounded, and further direct arguments show that $\tilde{T} \in B_{\mathfrak{B}}(\mathfrak{X})$. That $\tilde{T}$ coincides with T in

$B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$ follows by combining the two displayed expressions. Hence $B_{\mathfrak{B}}(\mathfrak{X})$ is WOT sequentially closed.

Now we show that $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ is WOT sequentially closed. If (S_n) is a sequence in $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ converging weakly to $S \in B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$, then by what we just proved, $S \in B_{\mathfrak{B}}(\mathfrak{X})$. Since the adjoint is WOT-continuous, we also have $S^* \in B_{\mathfrak{B}}(\mathfrak{X})$, where S^* denotes the adjoint of S as a Hilbert space operator in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$. For $x, y \in \mathfrak{X}$ and $\zeta, \eta \in \mathcal{H}$, we have

$$\langle \zeta, \langle S(x)|y\rangle \eta\rangle = \langle S(x \otimes \zeta), y \otimes \eta\rangle = \langle x \otimes \zeta, S^*(y \otimes \eta)\rangle = \langle \zeta, \langle x|S^*(y)\rangle \eta\rangle.$$

Hence $\langle S(x)|y\rangle = \langle x|S^*(y)\rangle$, and so $S \in \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$.

For the final statement, we proved in the first paragraph above that if $T_n \xrightarrow{WOT} T$ in $B_{\mathfrak{B}}(\mathfrak{X}) \subseteq B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$, then $\langle T_n(x)|y\rangle \xrightarrow{WOT} \langle T(x)|y\rangle$ for all $x, y \in \mathfrak{X}$, which is the same as saying $T_n(x) \xrightarrow{WOT_{\mathfrak{B}}} T(x)$ for all $x \in \mathfrak{X}$. Conversely, if $\langle T_n(x)|y\rangle \xrightarrow{WOT} \langle T(x)|y\rangle$ for all $x, y \in \mathfrak{X}$, then (T_n) is bounded by the uniform boundedness principle, and $\langle T_n(x \otimes \zeta), y \otimes \eta\rangle \to \langle T(x \otimes \zeta), y \otimes \eta\rangle$ for all $\zeta, \eta \in \mathcal{H}$. A triangle inequality argument gives that $T_n \xrightarrow{WOT} T$ in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$. $\square$

Hence $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ is a Σ^*-algebra in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$, and so $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))$, the WOT sequential closure of $\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})$ in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$, is contained in $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$. Since $\mathfrak{X}$ is a left C^*-module over $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ with inner product taking values in $\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})$, $\mathfrak{X}$ is also a left C^*-module over the Σ^*-algebra $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))$. We show in Theorem 3.1.8 that $\mathfrak{X}$ is in fact a Σ^*-module over $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))$.

We will later show (Proposition 3.2.8), that $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) = \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ in the special case that $\mathfrak{X}$ is "$\Sigma^*_{\mathfrak{B}}$-countably generated." We do not know of any other example (outside the $\Sigma^*_{\mathfrak{B}}$-countably generated case) in which equality holds here, but note that equality does not hold in general—for example, if $\mathfrak{B}$ is a nonunital Σ^*-algebra, then $\mathfrak{B} \cong \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{B}))$ is not equal to $\mathbb{B}_{\mathfrak{B}}(\mathfrak{B})$ since the latter is unital.

Lemma 3.1.7. *Let $\mathfrak{X}$ is a right Σ^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$. For a sequence (x_n) and element x in $\mathfrak{X}$, $x_n \xrightarrow{\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))WOT} x$ if and only if $x_n \xrightarrow{WOT_{\mathfrak{B}}} x$.*

Proof. The claim is that $|x_n\rangle\langle w| \xrightarrow{WOT} |x\rangle\langle w|$ in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$ for all $w \in \mathfrak{X}$ if and only if $\langle x_n|z\rangle \xrightarrow{WOT} \langle x|z\rangle$ in $B(\mathcal{H})$ for all $z \in \mathfrak{X}$. Assuming the former, it follows from the uniform boundedness principle that (x_n) is a bounded sequence, and routine calculations give

$$\langle\langle w|y\rangle\zeta, \langle x_n|z\rangle\eta\rangle = \langle|x_n\rangle\langle w|(y\otimes\zeta), z\otimes\eta\rangle \longrightarrow \langle|x\rangle\langle w|(y\otimes\zeta), z\otimes\eta\rangle = \langle\langle w|y\rangle\zeta, \langle x|z\rangle\eta\rangle$$

for all $w, y, z \in \mathfrak{X}$ and $\zeta, \eta \in \mathcal{H}$. Our usual boundedness/density arguments show that if $P \in B(\mathcal{H})$ is the projection onto the closed subspace of $\mathcal{H}$ generated by $\{\langle x|y\rangle\zeta : x, y \in \mathfrak{X} \text{ and } \zeta \in \mathcal{H}\}$, then for any $\xi, \eta \in \mathcal{H}$ and $z \in \mathfrak{X}$, we have

$$\langle \xi, \langle x_n|z\rangle\eta\rangle = \langle P\xi, \langle x_n|z\rangle\eta\rangle \longrightarrow \langle P\xi, \langle x|z\rangle\eta\rangle = \langle \xi, \langle x|z\rangle\eta\rangle.$$

Hence $\langle x_n|z\rangle \xrightarrow{WOT} \langle x|z\rangle$ in $B(\mathcal{H})$ for all $z \in \mathfrak{X}$. The converse is similar. □

Theorem 3.1.8. *If $\mathfrak{X}$ is a right Σ^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, then $\mathfrak{X}$ is a left Σ^*-module over the Σ^*-algebra $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) \subseteq B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$.*

Proof. By the "left version" of Proposition 3.1.4, we need to show that if (x_n) is a sequence in $\mathfrak{X}$ such that $|x_n\rangle\langle y|$ is WOT-convergent in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$ for all $y \in \mathfrak{X}$, then there is an $x \in \mathfrak{X}$ such that $|x_n\rangle\langle y| \xrightarrow{WOT} |x\rangle\langle y|$ for all $y \in \mathfrak{X}$. If $|x_n\rangle\langle y|$ is WOT-convergent in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$ for all $y \in \mathfrak{X}$, then arguments from the first paragraph of the proof of Lemma 3.1.7 show that $\langle x_n|z\rangle$ is WOT-convergent for all $z \in \mathfrak{X}$. By Proposition 3.1.4, there is an $x \in \mathfrak{X}$ such that $\langle x_n|z\rangle \to \langle x|z\rangle$ for all $z \in \mathfrak{X}$, and by Lemma 3.1.7, $|x_n\rangle\langle y| \xrightarrow{WOT} |x\rangle\langle y|$ for all $y \in \mathfrak{X}$. $\square$

A *ternary ring of operators* (abbreviated *TRO*) is a norm-closed subspace $Z \subseteq B(\mathcal{H}, \mathcal{K})$, for Hilbert spaces $\mathcal{H}, \mathcal{K}$, such that $xy^*z \in Z$ for all $x, y, z \in Z$; and a *corner* of a C^*-algebra A is a subspace of the form pAq for projections $p, q \in M(A)$. (This is slightly different from the usual definition of a corner as a subspace of the form $pAp^\perp$, but every corner in our sense can be identified with a corner in the usual sense of a different C^*-algebra, so the two definitions are not essentially different.) Note that if Z is a TRO in $B(\mathcal{H}, \mathcal{K})$, then there is a canonical TRO-isomorphism (see Chapter 5, 5.2.2) identifying Z with a TRO in $B(\mathcal{H}, [Z\mathcal{H}])$. Similarly, letting $Z^\star = \{z^* \in B(\mathcal{K}, \mathcal{H}) : z \in Z\}$, there is a canonical TRO-isomorphism identifying Z with a TRO in $B([Z^\star\mathcal{K}], \mathcal{K})$ (let $\psi : Z^\star \hookrightarrow B(\mathcal{K}, [Z^\star\mathcal{K}])$ be the canonical isometric TRO-homomorphism (see Chapter 5, 5.2.2), and define $\varphi : Z \hookrightarrow B([Z^\star\mathcal{K}], \mathcal{K})$ by $\varphi(z) = \psi(z^*)^*$). Thus, just as for C^*-algebras, there is no real loss in assuming from the outset that a TRO is *nondegenerate*, i.e. that $[Z\mathcal{H}] = \mathcal{K}$ and $[Z^\star\mathcal{K}] = \mathcal{H}$.

In analogy with the situation in C^*-module theory and W^*-module theory, Σ^*-modules are essentially the same as WOT sequentially closed TROs, and essentially the same as corners of Σ^*-algebras. The next theorem gives the details for how to move from one of these "pictures" to another. To prepare for this, we first describe the Σ^*-version of the "linking algebra" of a C^*-module.

Proposition 3.1.9. *If $\mathfrak{X}$ is a Σ^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, then* $\mathcal{L}^{\mathscr{B}}(\mathfrak{X}) := \begin{bmatrix} \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) & \mathfrak{X} \\ \overline{\mathfrak{X}} & \mathfrak{B} \end{bmatrix}$ *is a Σ^*-algebra in $B((\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H}) \oplus^2 \mathcal{H})$.*

Proof. It is very easy to show that a sequence of 2×2 matrices in $\mathcal{L}^{\mathscr{B}}(\mathfrak{X})$ converges WOT to a 2×2 matrix $\xi \in B((\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H}) \oplus^2 \mathcal{H})$ if and only if each of the entries converges WOT to the corresponding entry in ξ. Since each of the four corners of $\mathcal{L}^{\mathscr{B}}(\mathfrak{X})$ is WOT sequentially closed, the result follows. □

In the following theorem, when we say "$\mathfrak{X} \cong (1-p)\mathfrak{C}p$ and $\mathfrak{B} \cong p\mathfrak{C}p$ under isomorphisms preserving all the Σ^*-module structure," we mean that there is an isometric isomorphism $\varphi : \mathfrak{X} \to (1-p)\mathfrak{C}p$ and a Σ^*-isomorphism $\psi : \mathfrak{B} \to p\mathfrak{C}p$ such that $\varphi(xb) = \varphi(x)\psi(b)$ and $\psi(\langle x|y\rangle) = \varphi(x)^*\varphi(y)$ for all $x, y \in \mathfrak{X}$ and $b \in \mathfrak{B}$. Note that these conditions imply that $\varphi(x_n) \xrightarrow{WOT_{p\mathfrak{C}p}} \varphi(x)$ whenever $x_n \xrightarrow{WOT_{\mathfrak{B}}} x$.

Theorem 3.1.10. (1) *If $\mathfrak{X}$ is a Σ^*-module over $\mathfrak{B} \subseteq B(\mathcal{H})$, then $\mathfrak{X}$ is a WOT sequentially closed TRO in $B(\mathcal{H}, \mathfrak{X} \otimes \mathcal{H})$. Conversely, if $\mathfrak{Z}$ is a nondegenerate WOT sequentially closed TRO in $B(\mathcal{K}_1, \mathcal{K}_2)$, then $\mathfrak{Z}$ is a Σ^*-module over $\mathscr{B}(\mathfrak{Z}^\star\mathfrak{Z})$ with the obvious module action and inner product $\langle z_1|z_2\rangle = z_1^* z_2$.*

(2) *If $\mathfrak{Y} = p\mathfrak{D}q$ is a corner of a Σ^*-algebra $\mathfrak{D}$, then $\mathfrak{Y}$ is canonically a Σ^*-module over $q\mathfrak{D}q$. Conversely, if $\mathfrak{X}$ is a Σ^*-module over $\mathfrak{B} \subseteq B(\mathcal{H})$, then there exists a Σ^*-algebra $\mathfrak{C} \subseteq B(\mathcal{K})$ and a projection $p \in M(\mathfrak{C})$ such that $\mathfrak{X} \cong (1-p)\mathfrak{C}p$ and $\mathfrak{B} \cong p\mathfrak{C}p$ under isomorphisms preserving all the Σ^*-module structure.*

Proof. (1) The forward direction follows immediately from the definition of Σ^*-modules. For the converse, we must first show that $\mathfrak{Z}$ is closed under right multiplication by elements in $\mathscr{B}(\mathfrak{Z}^\star\mathfrak{Z})$. Fixing $z \in \mathfrak{Z}$, the set $\mathscr{S}_z = \{b \in \mathscr{B}(\mathfrak{Z}^\star\mathfrak{Z}) : zb \in \mathfrak{Z}\}$ contains $\mathfrak{Z}^\star\mathfrak{Z}$ since $\mathfrak{Z}$ is a TRO, and an easy argument shows that $\mathscr{S}_z$ is WOT sequentially closed, so that $\mathscr{S}_z = \mathscr{B}(\mathfrak{Z}^\star\mathfrak{Z})$. So $\mathfrak{Z}$ is a right module over $\mathscr{B}(\mathfrak{Z}^\star\mathfrak{Z})$, and it is straightforward to show that it is a C^*-module over $\mathscr{B}(\mathfrak{Z}^\star\mathfrak{Z})$ with the canonical inner product. To prove that $\mathfrak{Z}$ is a Σ^*-module, note that under the canonical unitary $\mathfrak{Z} \otimes_{\mathscr{B}(\mathfrak{Z}^\star\mathfrak{Z})} \mathcal{K}_1 \cong [\mathfrak{Z}\mathcal{K}_1] = \mathcal{K}_2$, the embedding $\mathfrak{Z} \hookrightarrow B(\mathcal{K}_1, \mathfrak{Z} \otimes_{\mathscr{B}(\mathfrak{Z}^\star\mathfrak{Z})} \mathcal{K}_1)$ coincides with the inclusion $\mathfrak{Z} \subseteq B(\mathcal{K}_1, \mathcal{K}_2)$, so it follows from the definition that $\mathfrak{Z}$ is a Σ^*-module over $\mathscr{B}(\mathfrak{Z}^\star\mathfrak{Z})$.

(2) For the forward direction, first note the easy fact that if $\mathfrak{D}$ is a Σ^*-algebra in $B(\mathcal{K})$ and $q \in M(\mathfrak{D}) \subseteq B(\mathcal{K})$ is a projection, then $q\mathfrak{D}q$ is a Σ^*-algebra in $B(q\mathcal{K})$. It then follows easily either using the definition as in the proof of (1) or employing Proposition 3.1.4 that $\mathfrak{Y}$ is a Σ^*-module over $q\mathfrak{D}q$. The converse follows from Proposition 3.1.9 with $\mathfrak{C} = \mathcal{L}^{\mathscr{B}}(\mathfrak{X})$ and $p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. □

It is an interesting and useful fact that a C^*-module over a W^*-algebra always admits a "selfdual completion," that is, a unique W^*-module containing the original C^*-module as a weak*-dense submodule. Hamana in [23] and Lin in [27] also proved that a C^*-module X over a monotone complete C^*-algebra admits a selfdual completion, and Hamana proved uniqueness under the condition that $X^\perp = (0)$. The proposition below gives existence of a "Σ^*-module completion" analogous to the selfdual completion.

Note that an easy modification of Lemma 3.1.11 and Proposition 3.1.12 gives another proof of the existence of the selfdual completion of a C^*-module over a W^*-algebra (this is surely known to experts though).

For a C^*-module X over a nondegenerate C^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, recall a few canonical embeddings from Proposition 2.1.1:

$$X \cong \mathbb{K}_{\mathfrak{B}}(\mathfrak{B}, X) \hookrightarrow B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$$

$$B_{\mathfrak{B}}(X, \mathfrak{B}) \hookrightarrow B(X \otimes_{\mathfrak{B}} \mathcal{H}, \mathcal{H}).$$

In the following lemma, the definition of $\mathscr{S}$ implicitly uses the latter, and the last few statements use the former.

Lemma 3.1.11. *If X is a C^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, then*

$$\mathscr{S} := \{T \in B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H}) : T^* \in B_{\mathfrak{B}}(X, \mathfrak{B})\}$$

is WOT sequentially closed in $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$ and contains X. Hence we may view

$$X \subseteq \mathscr{B}(X) \subseteq \mathscr{S} \subseteq B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H}),$$

where by $\mathscr{B}(X)$ we mean the WOT sequential closure of X in $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$.

Proof. Suppose that (T_n) is a sequence in $\mathscr{S}$ and $T \in B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$ with $T_n \xrightarrow{WOT} T$ in $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$. Then $T_n^* \xrightarrow{WOT} T^*$ in $B(X \otimes_{\mathfrak{B}} \mathcal{H}, \mathcal{H})$, so

$$\langle T_n^*(x)\zeta, \eta\rangle = \langle T_n^*(x \otimes \zeta), \eta\rangle \longrightarrow \langle T^*(x \otimes \zeta), \eta\rangle \text{ for all } x \in X \text{ and } \zeta, \eta \in \mathcal{H},$$

and it follows that for each $x \in X$, the sequence $(T_n^*(x))$ converges WOT in $\mathfrak{B} \subseteq B(\mathcal{H})$. Define a map $\tau : X \to \mathfrak{B}$ by $\tau(x) = \text{WOT-}\lim_n T_n^*(x)$. It is direct to check that τ is in $B_{\mathfrak{B}}(X, \mathfrak{B})$, and since $\langle T^*(x \otimes \zeta), \eta \rangle = \langle \tau(x)\zeta, \eta \rangle = \langle \tau(x \otimes \zeta), \eta \rangle$ for all $x \in X$ and $\zeta, \eta \in \mathcal{H}$, we may conclude that T^* coincides with τ under the embedding $B_{\mathfrak{B}}(X, \mathfrak{B}) \hookrightarrow B(X \otimes_{\mathfrak{B}} \mathcal{H}, \mathcal{H})$. Hence $T \in \mathscr{S}$, and so $\mathscr{S}$ is WOT sequentially closed. All the other claims are evident. $\square$

It follows quickly from the definitions that $\mathscr{B}(X) = X$ if and only if X is a Σ^*-module. To see that $\mathscr{B}(X)$ and $\mathscr{S}$ may be different, take $X = \mathfrak{B}$ for a nonunital Σ^*-algebra $\mathfrak{B}$. Then $\mathscr{B}(X) = \mathfrak{B} \neq \mathscr{S}$ since $I_{\mathcal{H}} \in \mathscr{S}$.

To explain some terminology that appears in the following theorem and later on in this paper, a C^*-submodule X of a Σ^*-module $\mathfrak{X}$ is said to be *$WOT_{\mathfrak{B}}$ sequentially dense* if $\mathfrak{X}$ is the only subset of itself that contains X and is closed under limits of $\mathrm{WOT}_{\mathfrak{B}}$-convergent sequences. Note that this may be different from saying that every element in $\mathfrak{X}$ is the $\mathrm{WOT}_{\mathfrak{B}}$-limit of a sequence in X.

Theorem 3.1.12. *If X is a C^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, then in the notation of the preceding lemma, $\mathscr{B}(X)$ has a $\mathfrak{B}$-valued inner product making it into a Σ^*-module that contains X as a $WOT_{\mathfrak{B}}$ sequentially dense submodule and has*

$$\langle \tau | x \rangle = \tau^*(x) \textit{ for all } \tau \in \mathscr{B}(X), x \in X.$$

Moreover, the operator norm $\mathscr{B}(X)$ inherits from $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$ coincides with this C^*-module norm.

Proof. We first show that $\mathscr{B}(X)$ is a right $\mathfrak{B}$-module with the canonical module action coming from the inclusions $\mathscr{B}(X) \subseteq B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$ and $\mathfrak{B} \subseteq B(\mathcal{H})$. Fix $b \in \mathfrak{B}$, and let $\mathscr{S} = \{x \in \mathscr{B}(X) : xb \in \mathscr{B}(X)\}$. Then $\mathscr{S}$ is WOT sequentially closed and contains X, so $\mathscr{S} = \mathscr{B}(X)$. Since $b \in \mathfrak{B}$ was arbitrary, we have shown that $xb \in \mathscr{B}(X)$ for all $x \in \mathscr{B}(X)$ and $b \in \mathfrak{B}$.

Now note that for any $K, L \in \mathbb{K}_{\mathfrak{B}}(\mathfrak{B}, X)$, K^*L is in $\mathbb{K}_{\mathfrak{B}}(\mathfrak{B}) = \mathfrak{B} \subseteq B(\mathcal{H})$. Using this, arguments of the sort used in the previous paragraph (or in Proposition 2.2.4) shows that $S^*T \in \mathfrak{B}$ for all $S, T \in \mathscr{B}(X)$. Define a $\mathfrak{B}$-valued inner product on $\mathscr{B}(X)$ by $\langle S|T\rangle := S^*T$. With this inner product and the right $\mathfrak{B}$-module structure it inherits from $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$, it is easy to check that $\mathscr{B}(X)$ is a C^*-module over $\mathfrak{B}$. It is also straightforward to check that the centered equation in the claim holds.

To see that $\mathscr{B}(X)$ is a Σ^*-module, suppose that τ_n is a sequence in $\mathscr{B}(X)$ such that $\langle \tau_n|\sigma\rangle$ converges WOT in $\mathfrak{B}$ for all $\sigma \in \mathscr{B}(X)$. In particular, $\langle \tau_n|x\rangle = \tau_n^*(x)$ converges WOT to an element in $\mathfrak{B}$, call it $\tau^*(x)$, for each $x \in X$. Routine arguments show that $\tau^* : X \to \mathfrak{B}$ thus defined is in $B_{\mathfrak{B}}(X, \mathfrak{B})$ and that $\tau_n^* \xrightarrow{WOT} \tau^*$ in $B(X \otimes_{\mathfrak{B}} \mathcal{H}, \mathcal{H})$, so that $\langle \tau_n|\sigma\rangle = \tau_n^*\sigma \xrightarrow{WOT} \tau^*\sigma = \langle \tau|\sigma\rangle$ in $\mathfrak{B}$ for all $\sigma \in \mathscr{B}(X)$. Hence $\mathscr{B}(X)$ is a Σ^*-module by Proposition 3.1.4.

Note that we have demonstrated that $\mathrm{WOT}_{\mathfrak{B}}$-convergence of a sequence in $\mathscr{B}(X)$ is the same as WOT-convergence in $\mathscr{B}(X)$ considered as a subset of $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$. This fact combined with the definition of $\mathscr{B}(X)$ gives that X is $\mathrm{WOT}_{\mathfrak{B}}$ sequentially dense in $\mathscr{B}(X)$.

The last claim follows immediately from the definition of the inner product: $\|\langle \tau|\tau\rangle\|^2_{\mathscr{B}(X)} = \|\tau^*\tau\|_{B(\mathcal{H})} = \|\tau\|^2_{B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})}$. $\square$

Unfortunately, we were not able in general to prove uniqueness of the above construction with conditions as simple as those in the W^*-case or monotone complete case (but see Proposition 3.2.13 for a special case).

Definition 3.1.13. For a C^*-module X over a Σ^*-algebra $\mathfrak{B}$, a Σ^**-module completion* of X is any Σ^*-module $\mathfrak{X}$ over $\mathfrak{B} \subseteq B(\mathcal{H})$ such that:

(1) $\mathfrak{X}$ contains X as a $\mathrm{WOT}_{\mathfrak{B}}$ sequentially dense submodule;

(2) the $\mathfrak{B}$-valued inner product on $\mathfrak{X}$ extends that of X;

(3) $\|\xi\| = \sup\{\|\langle \xi | x \rangle\| : x \in \mathrm{Ball}(X)\}$ for all $\xi \in \mathfrak{X}$;

(4) if (ξ_n) is a sequence in $\mathfrak{X}$ such that $(\langle \xi_n | x \rangle)$ is WOT-convergent for all $x \in X$, then there is a $\xi \in \mathfrak{X}$ such that $\xi_n \xrightarrow{WOT_{\mathfrak{B}}} \xi$.

Proposition 3.1.14. *If X is a C^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, then the Σ^*-module $\mathscr{B}(X)$ of the previous theorem is the unique Σ^*-module completion of X (up to unitary isomorphism).*

Proof. It follows immediately from the previous theorem that $\mathscr{B}(X)$ satisfies (1) and (2) in the definition of a Σ^*-module completion. To see (3), for $\tau \in \mathscr{B}(X)$, we have

$$
\begin{aligned}
\|\tau\|_{\mathscr{B}(X)} &= \|\tau\|_{B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})} \\
&= \|\tau^*\|_{B(X \otimes_{\mathfrak{B}} \mathcal{H}, \mathcal{H})} \\
&= \|\tau^*\|_{B_{\mathfrak{B}}(X, \mathfrak{B})} \\
&= \sup\{\|\tau^*(x)\| : x \in \mathrm{Ball}(X)\} \\
&= \sup\{\|\langle \tau | x \rangle\| : x \in \mathrm{Ball}(X)\}.
\end{aligned}
$$

(The first equality here follows from the last claim in Theorem 3.1.12, the third equality follows from Lemma 3.1.11 and the isometric embedding $B_{\mathfrak{B}}(X, \mathfrak{B}) \hookrightarrow$

$B(X \otimes_{\mathfrak{B}} \mathcal{H}, \mathcal{H})$, and the final equality follows from the centered equation in Theorem 3.1.12.) The argument for (4) basically follows the second paragraph of the proof of the previous theorem.

To prove uniqueness, suppose that $\mathfrak{Y}$ is another Σ^*-module completion of X, and denote its $\mathfrak{B}$-valued inner product by $(\cdot|\cdot)$. Define maps $V : \mathfrak{Y} \to B_{\mathfrak{B}}(X, \mathfrak{B})$ and $U : \mathfrak{Y} \to B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$ by $V(\xi)(x) = (\xi|x)$ and $U(\xi) = V(\xi)^*$ for $\xi \in \mathfrak{Y}$ and $x \in X$. We will show that U is a $\mathfrak{B}$-linear isometry with range equal to $\mathscr{B}(X)$, and so the result follows Lance's result that every isometric, surjective module map between C^*-modules is a unitary ([25, Theorem 3.5]).

Note first the formula

$$\langle U(\xi)\eta, x \otimes \zeta\rangle = \langle \eta, V(\xi)(x \otimes \zeta)\rangle = \langle \eta, V(\xi)(x)\zeta\rangle = \langle \eta, (\xi|x)\zeta\rangle$$

for $\xi \in \mathfrak{Y}, x \in X$, and $\eta, \zeta \in \mathcal{H}$. An easy calculation from this shows that U is linear and $\mathfrak{B}$-linear.

By this, if $z \in X \subseteq \mathfrak{Y}$, then $\langle U(z)\eta, x \otimes \zeta\rangle = \langle \eta, (z|x)\zeta\rangle = \langle z \otimes \eta, x \otimes \zeta\rangle$ for all $\zeta, \eta \in \mathcal{H}$ and $x \in X$. Hence $U(z) = z$ in $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$, and we have shown that $X \subseteq U(\mathfrak{Y})$.

Let $\mathscr{T} = \{\xi \in \mathfrak{Y} : U(\xi) \in \mathscr{B}(X)\}$. We just showed that $X \subseteq \mathscr{T}$, so if we can show that $\mathscr{T}$ is WOT$_{\mathfrak{B}}$ sequentially closed, it will follow by sequential WOT$_{\mathfrak{B}}$-density of X in $\mathfrak{Y}$ that $\mathscr{T} = \mathfrak{Y}$. To this end, suppose that (ξ_n) is a sequence in $\mathscr{T}$ with $\xi_n \xrightarrow{WOT_{\mathfrak{B}}} \xi$ in $\mathfrak{Y}$. By the centered line above,

$$\langle U(\xi_n)\eta, x \otimes \zeta\rangle = \langle \eta, (\xi_n|x)\zeta\rangle \longrightarrow \langle \eta, (\xi|x)\eta\rangle = \langle U(\xi)\eta, x \otimes \zeta\rangle$$

for all $x \in X$ and $\eta, \zeta \in \mathcal{H}$, so that $U(\xi_n) \xrightarrow{WOT} U(\xi)$ in $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$. Thus $U(\xi) \in \mathscr{B}(X)$, and $\xi \in \mathscr{T}$. So we may conclude that $\mathscr{T} = \mathfrak{Y}$, which is to say $U(\mathfrak{Y}) \subseteq \mathscr{B}(X)$.

Combining the previous two paragraphs, we have $X \subseteq U(\mathfrak{Y}) \subseteq \mathscr{B}(X)$. So to have $U(\mathfrak{Y}) = \mathscr{B}(X)$, it remains to prove that $U(\mathfrak{Y})$ is $\mathrm{WOT}_{\mathfrak{B}}$ sequentially closed in $\mathscr{B}(X)$. Suppose that (ξ_n) is a sequence in $\mathfrak{Y}$ with $U(\xi_n) \xrightarrow{WOT_{\mathfrak{B}}} \tau$ in $\mathscr{B}(X)$. Then for $\zeta, \eta \in \mathcal{H}$ and $x \in X$,

$$\langle \zeta, (\xi_n|x)\eta \rangle = \langle U(\xi_n)(\zeta), x \otimes \eta \rangle \to \langle \tau(\zeta), x \otimes \eta \rangle = \langle \zeta, \tau^*(x)(\eta) \rangle = \langle \zeta, \langle \tau|x \rangle \eta \rangle,$$

so that $(\xi_n|x) \xrightarrow{WOT} \langle \tau|x \rangle$ in $\mathfrak{B}$. By assumption (4) in the definition above the proposition, $\xi_n \xrightarrow{WOT_{\mathfrak{B}}} \xi$ for some $\xi \in \mathfrak{Y}$, and the argument in the previous paragraph shows that $U(\xi_n) \xrightarrow{WOT_{\mathfrak{B}}} U(\xi)$, so that $\tau = U(\xi) \in U(\mathfrak{Y})$.

It remains to show that U is isometric. If $\xi \in \mathfrak{Y}$, $x \in X$, and $\eta, \zeta \in \mathcal{H}$, then

$$\langle \langle U(\xi)|x \rangle \zeta, \eta \rangle = \langle x \otimes \zeta, U(\xi)(\eta) \rangle = \langle V(\xi)(x \otimes \zeta), \eta \rangle = \langle (\xi|x)\zeta, \eta \rangle,$$

which gives that $\langle U(\xi)|x \rangle = (\xi|x)$ for all $x \in X$. That $\|U(\xi)\| = \|\xi\|$ now follows from assumption (3) in the definition above. □

To close this section, we provide a result that is used in the next section and seems interesting when one dwells upon the similarities between Σ^*-modules and W^*-modules. For W^*-modules Y and Z over M, we have $\{\langle y|\cdot\rangle : y \in Y\} = B_M(Y, M)$ and $\mathbb{B}_M(Y, Z) = B_M(Y, Z)$ and all the maps in both of these spaces are weak*-continuous. The following result is a Σ^*-analogue of this fact, but with an additional condition that may be taken as a weak type of the assumption of being "countably generated" (indeed, we will see in the next section that all $\Sigma^*_{\mathfrak{B}}$-countably generated Σ^*-modules meet this condition). This condition cannot be removed in general—for example, if $\mathfrak{B}$ is a nonunital Σ^*-algebra considered as a Σ^*-module over itself, then $\mathrm{id}_{\mathfrak{B}}$ is in the latter set in (1) below, but is not in the former.

Proposition 3.1.15. *If $\mathfrak{X}$ is a Σ^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$ such that $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) = \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$, and $\mathfrak{Y}$ is any other Σ^*-module over $\mathfrak{B}$, then*

(1) $\{\langle z|\cdot\rangle : z \in \mathfrak{X}\} = \{\xi \in B_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{B}) : x_n \xrightarrow{WOT_{\mathfrak{B}}} x \implies \xi(x_n) \xrightarrow{WOT} \xi(x)\}$;

(2) $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{Y}) = \{T \in B_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{Y}) : x_n \xrightarrow{WOT_{\mathfrak{B}}} x \implies T(x_n) \xrightarrow{WOT_{\mathfrak{B}}} T(x)\}$.

Proof. The forward inclusion of (1) is evident from the definitions. For the other inclusion, fix a ξ as in the latter set in (1). Note that the condition $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) = \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ is equivalent to saying that $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ is generated as a Σ^*-algebra by the "finite-rank operators," that is, operators of the form $\sum_{i=1}^n |x_i\rangle\langle y_i|$ for $x_i, y_i \in \mathfrak{X}$. Let

$$\mathscr{T} = \{T \in \mathbb{B}_{\mathfrak{B}}(\mathfrak{X}) : \xi \circ T = \langle z|\cdot\rangle \text{ and } \xi \circ T^* = \langle w|\cdot\rangle \text{ for some } z, w \in \mathfrak{X}\}.$$

To see that $\mathscr{T}$ is WOT sequentially closed, suppose that (T_n) is a sequence in $\mathscr{T}$ with $T_n \xrightarrow{WOT} T$ in $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X}) \subseteq B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$. For $x \in \mathfrak{X}$, by Proposition 3.1.6 we have $T_n(x) \xrightarrow{WOT_{\mathfrak{B}}} T(x)$, so that $\xi \circ T_n(x) \xrightarrow{WOT} \xi \circ T(x)$ by the assumption on ξ. Writing $\xi \circ T_n(x) = \langle z_n|x\rangle$, we may conclude by Proposition 3.1.4 that $\xi \circ T = \langle z|\cdot\rangle$ for some $z \in \mathfrak{X}$. Since $T_n \xrightarrow{WOT} T$ implies $T_n^* \xrightarrow{WOT} T^*$, the same argument shows that $\xi \circ T^* = \langle w|\cdot\rangle$ for some $w \in \mathfrak{X}$. So $\mathscr{T}$ is WOT sequentially closed. It is easy to check that $\mathscr{T}$ is a $*$-subalgebra of $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ containing all the finite-rank operators; hence $\mathscr{T} = \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$. Since $I \in \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$, we conclude that $\xi = \langle z|\cdot\rangle$ for some $z \in \mathfrak{X}$.

For the forward inclusion of (2), let $S \in \mathbb{B}_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{Y})$, and suppose $x_n \xrightarrow{WOT_{\mathfrak{B}}} x$ in $\mathfrak{X}$. Then

$$\langle S(x_n)|y\rangle = \langle x_n|S^*(y)\rangle \xrightarrow{WOT} \langle x|S^*(y)\rangle = \langle S(x)|y\rangle$$

for all $y \in \mathfrak{X}$, so that $S(x_n) \xrightarrow{WOT_{\mathfrak{B}}} S(x)$. For the other inclusion, suppose that T is in the latter set in (2). Then for any $y \in \mathfrak{Y}$, the map $\langle y|T(\cdot)\rangle$ is in the latter set in (1), so there is a $z \in \mathfrak{X}$ such that $\langle z|x\rangle = \langle y|T(x)\rangle$ for all $x \in \mathfrak{X}$. Hence T is adjointable. $\square$

Note 3.1.16. In principle, one could work out analogous theories to that presented above for many different classes of C^*-algebras. For example, one could define a *Borel module* to be a C^*-module $\mathfrak{X}$ over a Borel $*$-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$ such that $\begin{bmatrix} \mathscr{B}^m(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) & \mathfrak{X} \\ \overline{\mathfrak{X}} & \mathfrak{B} \end{bmatrix}$ is monotone sequentially closed in $B((\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H}) \oplus^2 \mathcal{H})$, where $\mathscr{B}^m(\cdot)$ denotes monotone sequential closure. It would be interesting to try to work out the appropriate Borel analogues of the results for Σ^*-modules we have proved, but it does not seem clear how to do this even for the first few of our results.

3.2 Countably generated Σ^*-modules

Many of the most interesting results in C^*-module theory require some type of "smallness" condition on either the module or the coefficient C^*-algebra; e.g., that the module is countably generated or that the C^*-algebra is separable or σ-unital. In this section, we study a weak sequential analogue of the condition of being a countably generated module. The more elegant results we obtain in this section indicate that Σ^*-modules meeting this countably generated condition are more similar to W^*-modules than are general Σ^*-modules.

The main highlights of this section are Proposition 3.2.7 (which is analogous to some well-known equivalent conditions to being a (norm) countably generated C^*-module (see [9, 8.2.5])), Theorem 3.2.10 (which says that in the class of "$\Sigma^*_{\mathfrak{B}}$-sequentially countably generated" C^*-modules over Σ^*-algebras, the Σ^*-modules coincide with the selfdual C^*-modules); Proposition 3.2.17 (an interesting result about column spaces donated to us by David Blecher); Theorem 3.2.19 (our analogue of Kasparov's stabilization theorem); and Proposition 3.2.21.

We now explain some potentially confusing terminology in the first bullet in the definition following. If S is a subset of right C^*-module X over a C^*-algebra $A \subseteq B(\mathcal{H})$, the *relative WOT_A sequential closure of S in X* is the smallest set $T \subseteq X$ such that: (1) $S \subseteq T$, and (2) if (x_n) is a sequence in T and $x \in X$ with $\langle x_n|y\rangle \xrightarrow{WOT} \langle x|y\rangle$ in $B(\mathcal{H})$ for all $y \in X$, then $x \in T$.

Definition 3.2.1.

- A right C^*-module X over a C^*-algebra $A \subseteq B(\mathcal{H})$ is *Σ^*_A-countably generated* if there is a countable set $\{x_i\}_{i=1}^\infty$ such that the relative WOT_A sequential closure of $\{\sum_{i=1}^N x_i b_i : b_i \in A, N \in \mathbb{N}\}$ in X is all of X.

- For a nondegenerate C^*-algebra $\mathfrak{C} \subseteq B(\mathcal{K})$, say that a sequence (e_n) in $\mathrm{Ball}(\mathfrak{C})$ is a *sequential weak cai for* $\mathfrak{C}$ if $e_n c \xrightarrow{\mathrm{WOT}} c$ for all $c \in \mathfrak{C}$. Since (e_n) is bounded and $[\mathfrak{C}\mathcal{K}] = \mathcal{K}$, a triangle inequality argument shows that (e_n) is a sequential weak cai if and only if $e_n \xrightarrow{\mathrm{WOT}} I_{\mathcal{K}}$. So in this case we also have $ce_n \xrightarrow{\mathrm{WOT}} c$ for all $c \in \mathfrak{C}$.

- A Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$ is Σ^*-*countably generated* (resp. Σ^*-*singly generated*) if there is a countable (resp. singleton) subset B of $\mathfrak{B}$ such that B generates $\mathfrak{B}$ as a Σ^*-algebra, that is, $\mathscr{B}(C^*(B)) = \mathfrak{B}$, where $C^*(B)$ is the C^*-algebra generated by B and $\mathscr{B}(\cdot)$ denotes WOT sequential closure (see notation above Proposition 2.2.4).

Example 3.2.2. (1) If $\mathfrak{C} \subseteq B(\mathcal{K})$ is a Σ^*-countably generated Σ^*-algebra (e.g., the Σ^*-envelope of a separable C^*-algebra), and $p \in M(\mathfrak{C})$, then $(1-p)\mathfrak{C}p$ is a right Σ^*-module over $p\mathfrak{C}p$ (see Theorem 3.1.10 (2)), and $(1-p)\mathfrak{C}p$ is $\Sigma^*_{p\mathfrak{C}p}$-countably generated. Indeed, one may deduce this quickly from the following observation (which uses and is analogous to the fact that countably generated C^*-algebras are separable): if a Σ^*-algebra $\mathfrak{B}$ is Σ^*-countably generated, then there is a countable subset D of $\mathfrak{B}$ such that $\mathscr{B}(D) = \mathfrak{B}$.

(2) It is immediate that if a Σ^*-algebra $\mathfrak{B}$, considered as a Σ^*-module over itself, is unital, then it is $\Sigma^*_{\mathfrak{B}}$-countably generated. We will show in Corollary 3.2.11 that the converse of this is also true.

(3) For a unital Σ^*-algebra $\mathfrak{B}$, the column Σ^*-module $C^w(\mathfrak{B})$ described above Corollary 3.2.15 is $\Sigma^*_{\mathfrak{B}}$-countably generated.

The von Neumann algebra analogue of the following proposition is well known, and since the spectral theorem still holds in Σ^*-algebras (by Proposition 2.2.7), the proof is virtually the same. We thank David Blecher for pointing this result out, and for the example following.

Proposition 3.2.3. *If $\mathfrak{B}$ is a Σ^*-countably generated commutative Σ^*-algebra, then it is Σ^*-singly generated by a selfadjoint element.*

Note 3.2.4. Related to (2) in Example 3.2.2, it is easy to see that every Σ^*-countably generated Σ^*-algebra is unital (since a countable subset will generate a σ-unital, WOT sequentially dense C^*-subalgebra), but the converse is not necessarily true. Take for example the von Neumann algebra $\ell^\infty(I) \subseteq B(\ell^2(I))$ for a set I with cardinality strictly greater than that of $\mathbb{R}$. If $\ell^\infty(I)$ were Σ^*-countably generated, then by Proposition 3.2.3 it would be Σ^*-singly generated by a selfadjoint element $x = (x_i)_{i\in I}$. However, as the map $I \to \mathbb{R}$, $i \mapsto x_i$, cannot be one-to-one, there must be $k, l \in I$, $k \neq l$, with $x_k = x_l$. Since the set $\mathscr{S}$ of $(y_i)_{i\in I}$ in $\ell^\infty(I)$ such that $y_k = y_l$ is WOT sequentially closed in $B(\ell^2(I))$ and contains x, we have the contradiction $\ell^\infty(I) \subseteq \mathscr{S}$.

The following simple lemma is a weak sequential version of some well-known characterizations of σ-unital C^*-algebras (cf. [34, 3.10.5]).

Lemma 3.2.5. *If $A \subseteq B(\mathcal{H})$ is a C^*-algebra, the following are equivalent:*

(1) *A has an element a such that $\psi(a) > 0$ for all nonzero WOT sequentially continuous positive functionals ψ on $\mathscr{B}(A)$;*

(2) *A has a positive element a such that $\overline{a(\mathcal{H})} = \mathcal{H}$;*

(3) *A has a positive increasing sequential weak cai.*

Proof. (1) $\implies$ (2). Let $a \in A$ be as in (1). Then $\langle a\zeta, \zeta\rangle > 0$ for all nonzero $\zeta \in \mathcal{H}$, and hence $\mathrm{Ker}(a) = \mathrm{Ran}(a)^\perp = (0)$, so that $\overline{\mathrm{Ran}(a)} = \mathcal{H}$.

(2) $\implies$ (3). Assume (2), and set $e_n = a^{1/n}$. Then $e_n \nearrow s(a)$ in $B(\mathcal{H})$, where $s(a)$ denotes the support projection of a. Since a has dense range, $s(a)$ is the identity operator on $\mathcal{H}$. So $e_n \xrightarrow{\text{WOT}} I_{\mathcal{H}}$.

(3) $\implies$ (1). Let (e_n) be a positive increasing sequential weak cai in A. As mentioned in the definition, this means that $e_n \nearrow I_{\mathcal{H}}$. Set $a = \sum_{n=1}^{\infty} 2^{-n} e_n$. If ψ is a WOT sequentially continuous positive functional on $\mathscr{B}(A)$ with $\psi(a) = 0$, then $\psi(e_n) = 0$ for all n since $e_n \leq a$. But since $\psi(e_n) \nearrow \psi(I_{\mathcal{H}}) = \|\psi\|$, we have $\psi = 0$. $\square$

Lemma 3.2.6. *If X is a right C^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$ that is $\Sigma^*_{\mathfrak{B}}$-countably generated by a subset $\{x_i\}$, then $\mathscr{B}(\{\sum_{i,j=1}^{n} |x_i b_{ij}\rangle\langle x_j| : b_{ij} \in \mathfrak{B}\}) = \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X))$.*

Proof. Set $\mathcal{A} := \{\sum_{i,j=1}^{n} |x_i b_{ij}\rangle\langle x_j| : b_{ij} \in \mathfrak{B}, n \in \mathbb{N}\}$. Clearly $\mathscr{B}(\mathcal{A}) \subseteq \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X))$. It is easy to check that $\mathcal{A}$ is a $*$-subalgebra of $\mathbb{B}_{\mathfrak{B}}(X)$, so $\mathscr{B}(\mathcal{A}) = \mathscr{B}(\overline{\mathcal{A}})$ is a C^*-algebra, and thus the inclusion $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X)) \subseteq \mathscr{B}(\mathcal{A})$ will follow if we can show that $|x\rangle\langle z| \in \mathscr{B}(\mathcal{A})$ for all $x, z \in X$. Fix $k \in \mathbb{N}$ and $b \in \mathfrak{B}$, and set $\mathscr{T} = \{x \in X : |x\rangle\langle x_k b| \in \mathscr{B}(\mathcal{A})\}$. An easy calculation shows that $\sum_{i=1}^{N} x_i b_i \in \mathscr{T}$ for all $b_i \in \mathfrak{B}$ and $N \in \mathbb{N}$, and it follows from Lemma 3.1.7 that $\mathscr{T}$ is WOT sequentially closed in X, so $\mathscr{T} = X$. A similar argument show that $\{x \in X : |x\rangle\langle z| \in \mathscr{B}(\mathcal{A})\} = X$ for all $z \in X$, and this proves the result. $\square$

Proposition 3.2.7. *Let X be a right C^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$. Then the following are equivalent:*

(1) *X is $\Sigma^*_{\mathfrak{B}}$-countably generated;*

(2) $\mathbb{K}_{\mathfrak{B}}(X)$ *has an element* T *such that* $\psi(T) > 0$ *for all nonzero WOT sequentially continuous positive functionals* ψ *on* $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X))$*;*

(3) $\mathbb{K}_{\mathfrak{B}}(X)$ *has a positive element* T *with* $\overline{T(X \otimes_{\mathfrak{B}} \mathcal{H})} = X \otimes_{\mathfrak{B}} \mathcal{H}$*;*

(4) $\mathbb{K}_{\mathfrak{B}}(X)$ *has a positive increasing sequential weak cai.*

If additionally X *is a* Σ^**-module, these conditions imply that* $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X)) = \mathbb{B}_{\mathfrak{B}}(X)$.

Proof. The equivalence of (2), (3), and (4) follows from Lemma 3.2.5.

(1) $\implies$ (2) Suppose that X is $\Sigma^*_{\mathfrak{B}}$-countably generated by $\{x_i\}_{i=1}^{\infty}$, and that these are scaled so that the series $\sum_{i=1}^{\infty} |x_i\rangle\langle x_i|$ converges in norm to a positive element T in $\mathbb{K}_{\mathfrak{B}}(X)$. Let $\mathcal{A} \subseteq \mathbb{K}_{\mathfrak{B}}(\mathfrak{X})$ be as in the proof of Lemma 3.2.6. By a calculation in the proof of [11, Theorem 7.13], if φ is a positive functional on $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X)) \subseteq B(X \otimes_{\mathfrak{B}} \mathcal{H})$ such that $\varphi(T) = 0$, then $\varphi(a) = 0$ for all $a \in \mathcal{A}$. Let ψ be a WOT sequentially continuous positive functional on $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X))$ such that $\psi(T) = 0$. By the calculation just mentioned, $\mathrm{Ker}(\psi)$ contains $\mathcal{A}$, and evidently $\mathrm{Ker}(\psi)$ is sequentially WOT-closed. By Lemma 3.2.6, $\mathrm{Ker}(\psi) = \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X))$, so that $\psi = 0$.

(4) $\implies$ (1) Let $\{e_n\}_{n=1}^{\infty}$ be a weak cai for $\mathbb{K}_{\mathfrak{B}}(X)$. For each $n \in \mathbb{N}$, pick $x_i^n, y_i^n \in X$ for $i = 1, ..., m_n$ such that $\| \sum_{i=1}^{m_n} |x_i^n\rangle\langle y_i^n| - e_n\| < \frac{1}{n}$ and $\| \sum_{i=1}^{m_n} |x_i^n\rangle\langle y_i^n|\| \leq 1$. We claim that $f_n := \sum_{i=1}^{m_n} |x_i^n\rangle\langle y_i^n|$ is also a weak cai for $\mathbb{K}_{\mathfrak{B}}(X)$. To this end, let $K \in \mathbb{K}_{\mathfrak{B}}(X)$. Take two nonzero vectors $h, k \in X \otimes_{\mathfrak{B}} \mathcal{H}$, let $\epsilon > 0$, and pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2(\|K\|+1)\|h\|\|k\|}$ and $|\langle e_n K h, k\rangle - \langle K h, k\rangle| < \frac{\epsilon}{2}$ for all $n \geq N$. Then for

$n \geq N$,

$$|\langle f_n Kh, k\rangle - \langle Kh, k\rangle| \leq |\langle f_n Kh, k\rangle - \langle e_n Kh, k\rangle| + |\langle e_n Kh, k\rangle - \langle Kh, k\rangle|$$
$$\leq \|f_n - e_n\|\|K\|\|h\|\|k\| + \frac{\epsilon}{2} < \epsilon.$$

Hence $f_n K \xrightarrow{\text{WOT}} K$, and so $\{f_n\}$ is a weak cai for $\mathbb{K}_{\mathfrak{B}}(X)$. By the final assertion in Proposition 3.1.6, $f_n|x\rangle\langle y|(z) \xrightarrow{WOT_{\mathfrak{B}}} |x\rangle\langle y|(z) = x\langle y|z\rangle$ for all $x, y, z \in X$. But $f_n|x\rangle\langle y|(z) = \sum_{i=1}^{m_n} x_i^n \langle y_i^n|x\rangle\langle y|z\rangle$, and so we have shown that every element in X of the form $x\langle y|z\rangle$ is a $\text{WOT}_{\mathfrak{B}}$-limit of a sequence of elements from $\text{Span}\{x_i^n b : b \in \mathfrak{B}, n \in \mathbb{N}, i = 1, ..., m_n\}$. Since the span of elements of the form $x\langle y|z\rangle$ is dense in X ([9, 8.1.4 (2)]), it follows that X is $\text{WOT}_{\mathfrak{B}}$-generated by the countable set $\{x_i^n : n \in \mathbb{N}, i = 1, ..., m_n\}$.

For the last assertion, it follows directly from (4) that $I \in \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X))$, and the assumption that X is a Σ^*-module gives that $\mathbb{B}_{\mathfrak{B}}(X)$ is a Σ^*-algebra in $B(X \otimes_{\mathfrak{B}} \mathcal{H})$, so that $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X)) \subseteq \mathbb{B}_{\mathfrak{B}}(X)$. Since $\mathbb{K}_{\mathfrak{B}}(X)$ is an ideal in $\mathbb{B}_{\mathfrak{B}}(X)$, it follows that $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X))$ is also an ideal in $\mathbb{B}_{\mathfrak{B}}(X)$, and so $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(X)) = \mathbb{B}_{\mathfrak{B}}(X)$. □

Proposition 3.2.8. *Let $\mathfrak{X}$ be a $\Sigma^*_{\mathfrak{B}}$-countably generated Σ^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$. Then $B_{\mathfrak{B}}(\mathfrak{X}) = \mathbb{B}_{\mathfrak{B}}(\mathfrak{X}) = \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))$.*

Proof. Suppose that $T \in B_{\mathfrak{B}}(\mathfrak{X})$, and let (e_n) be a sequence in $\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})$ with $e_n \nearrow I$ in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$. Viewing T as an operator in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$, we have $Te_n \xrightarrow{\text{WOT}} T$ in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$. Since each Te_n is adjointable, and $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ is WOT sequentially closed in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$ by Proposition 3.1.6, we have proved that T is adjointable. The last equality is the last assertion in Proposition 3.2.7. □

Lemma 3.2.9. *If $\mathfrak{X}$ is a right Σ^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, then $\mathfrak{X}$ is selfdual as a $\mathfrak{B}$-module if and only if it is selfdual as a $\mathscr{B}(\langle X|X\rangle)$-module.*

Proof. This follows directly from the general fact that for a C^*-module X over A, $B_A(X, A) = B_J(X, J)$ for any ideal J in A containing $\langle X|X\rangle$ (see [9, Lemma 8.5.2]). □

Theorem 3.2.10. *Let $\mathfrak{X}$ be a $\Sigma^*_{\mathfrak{B}}$-countably generated C^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$. Then $\mathfrak{X}$ is a Σ^*-module over $\mathfrak{B}$ if and only if $\mathfrak{X}$ is selfdual.*

Proof. ($\Longrightarrow$) By Lemma 3.2.9, we may assume without loss of generality that $\mathscr{B}(\langle X|X\rangle) = \mathfrak{B}$. Let $\varphi \in B_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{B})$. Fix $x_0 \in X$, and define $T : \mathfrak{X} \to \mathfrak{X}$ by $T(x) = x_0\varphi(x)$ for $x \in \mathfrak{X}$. It is easily checked that $T \in B_{\mathfrak{B}}(\mathfrak{X})$, so by Proposition 3.2.8, T is adjointable, and by the easy direction of Proposition 3.1.15 (2), if $x_n \xrightarrow{WOT_{\mathfrak{B}}} x$ in $\mathfrak{X}$ and $y \in \mathfrak{X}$, then

$$\langle y|x_0\rangle\varphi(x_n) = \langle y|T(x_n)\rangle \xrightarrow{\text{WOT}} \langle y|T(x)\rangle = \langle y|x_0\rangle\varphi(x).$$

Since x_0 was arbitrary, we have shown that for any $\zeta, \eta \in \mathcal{H}$ and $y, z \in \mathfrak{X}$,

$$\langle \varphi(x_n)\zeta, \langle z|y\rangle\eta\rangle \to \langle \varphi(x)\zeta, \langle z|y\rangle\eta\rangle.$$

Hence $\varphi(x_n) \xrightarrow{\text{WOT}} \varphi(x)$, and so $\varphi = \langle y_0|\cdot\rangle$ for some $y_0 \in \mathfrak{X}$ by Proposition 3.1.15 (1).

($\Longleftarrow$) Proposition 3.1.5. □

Corollary 3.2.11. *Let $\mathfrak{B} \subseteq B(\mathcal{H})$ be a Σ^*-algebra considered as a Σ^*-module over itself. If $\mathfrak{B}$ is $\Sigma^*_{\mathfrak{B}}$-countably generated, then $\mathfrak{B}$ is unital.*

Proof. By Theorem 3.2.10, $\mathfrak{B}$ is selfdual. Hence the identity map on $\mathfrak{B}$ is equal to $x \mapsto y^* x$ for some $y \in \mathfrak{B}$, so that y is a unit for $\mathfrak{B}$. $\square$

Lemma 3.2.12. *If X is a $WOT_{\mathfrak{B}}$ sequentially dense subset of a Σ^*-module $\mathfrak{X}$, then $X^{\perp} = (0)$.*

Proof. If $w \in X^{\perp}$, then $\mathscr{S} = \{\xi \in \mathfrak{X} : \langle \xi | w \rangle = 0\}$ is $\mathrm{WOT}_{\mathfrak{B}}$ sequentially closed and contains X, so $\mathscr{S} = \mathfrak{X}$. Hence $w \in \mathscr{S}$, so $w = 0$. $\square$

Proposition 3.2.13. *If X is a (norm) countably generated C^*-module over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, then the Σ^*-module completion $\mathscr{B}(X)$ from Theorem 3.1.12 is the unique Σ^*-module containing X as a $WOT_{\mathfrak{B}}$ sequentially dense submodule.*

Proof. (Cf. [23], proof of uniqueness in Theorem 2.2) Let $\mathfrak{Y}$ be another such Σ^*-module, and denote the $\mathfrak{B}$-valued inner product of $\mathfrak{Y}$ by $(\cdot|\cdot)$. As in the proof of Proposition 3.1.14, define $U : \mathfrak{Y} \to B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$ by $U(\xi)^*(x) = (\xi | x)$ for $\xi \in \mathfrak{Y}$ and $x \in X$. It follows just as in that proof that U is linear and $\mathfrak{B}$-linear and that $X \subseteq U(\mathfrak{Y}) \subseteq \mathscr{B}(X)$. Also note that U is bounded by the calculation

$$\begin{aligned} \|U(\xi)\| = \|U(\xi)^*\| &= \sup\{\|U(\xi)^*(x)\| : x \in \mathrm{Ball}(X)\} \\ &= \sup\{\|(\xi|x)\| : x \in \mathrm{Ball}(X)\} \leq \|\xi\| \end{aligned}$$

for $\xi \in \mathfrak{Y}$. Now fix $\xi \in \mathfrak{Y}$, and consider the map $\mathfrak{Y} \to \mathfrak{B}$, $\eta \mapsto \langle U(\xi) | U(\eta) \rangle$, which is easily seen to be in $B_{\mathfrak{B}}(\mathfrak{Y}, \mathfrak{B})$. Since X is countably generated and $\mathrm{WOT}_{\mathfrak{B}}$ sequentially dense in $\mathfrak{Y}$, $\mathfrak{Y}$ is a $\Sigma^*_{\mathfrak{B}}$-countably generated Σ^*-module, and so is selfdual by Theorem 3.2.10. Hence there is a $y_\xi \in \mathfrak{Y}$ such that

$$(y_\xi | \eta) = \langle U(\xi) | U(\eta) \rangle \text{ for all } \eta \in \mathfrak{Y}.$$

Define $T : \mathfrak{Y} \to \mathfrak{Y}$ by $T(\xi) = y_\xi$, which is easily seen to be in $B_{\mathfrak{B}}(\mathfrak{Y}) = \mathbb{B}_{\mathfrak{B}}(\mathfrak{Y})$. Consider $\mathrm{Ker}(\mathrm{id}_{\mathfrak{Y}} - T) = \{y \in \mathfrak{Y} : T(y) = y\}$. This set contains X and is WOT$_{\mathfrak{B}}$ sequentially closed since $\mathrm{id}_{\mathfrak{Y}} - T$ is adjointable, so $\mathrm{id}_{\mathfrak{Y}} = T$. Thus $(\xi|\eta) = \langle U(\xi)|U(\eta)\rangle$ for all $\xi, \eta \in \mathfrak{Y}$.

To prove that U is a unitary between $\mathfrak{Y}$ and $\mathscr{B}(X)$, it remains to show that $U(\mathfrak{Y})$ is WOT$_{\mathfrak{B}}$ sequentially closed in $\mathscr{B}(X)$. To this end, suppose that (ξ_n) is a sequence in $\mathfrak{Y}$ such that $U(\xi_n) \xrightarrow{WOT_{\mathfrak{B}}} \tau$ in $\mathscr{B}(X)$. By what we just proved,

$$(\xi_n|\eta) = \langle U(\xi_n)|U(\eta)\rangle \xrightarrow{WOT} \langle \tau|U(\eta)\rangle \text{ for all } \eta \in \mathfrak{Y}.$$

By Proposition 3.1.4, there exists a $\xi \in \mathfrak{Y}$ such that $(\xi_n|\eta) \xrightarrow{WOT} (\xi|\eta)$ for all $\eta \in \mathfrak{Y}$. Thus $\langle \tau|U(\eta)\rangle = (\xi|\eta) = \langle U(\xi)|U(\eta)\rangle$; hence $\langle \tau - U(\xi)|U(\eta)\rangle = 0$ for all $\eta \in \mathfrak{Y}$. So $\tau - U(\xi) \in U(\mathfrak{Y})^\perp \subseteq X^\perp$. By Lemma 3.2.12, $X^\perp = (0)$, so $\tau = U(\xi)$. $\square$

In preparation for our analogue of Kasparov's stabilization theorem, we now present a direct sum construction for Σ^*-modules. Fix a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, and let $\{\mathfrak{X}_n\}$ be a countable collection of Σ^*-algebras over $\mathfrak{B}$. Define the *direct sum Σ^*-module* to be the set

$$\oplus^w \mathfrak{X}_n := \{(x_n) \in \prod_n \mathfrak{X}_n : \sum_n \langle x_n|x_n\rangle \text{ is WOT-convergent in } \mathfrak{B}\},$$

with the inner product $\langle (x_n)|(y_n)\rangle = \sum_n \langle x_n|y_n\rangle$ and obvious $\mathfrak{B}$-module action. (Note that in the definition above, it does not matter whether we consider $\sum_n \langle x_n|x_n\rangle$ as a sequence of partial sums or as a net over finite subsets of $\mathbb{N}$. Note also that "$\sum_n \langle x_n|x_n\rangle$ is WOT-convergent in $\mathfrak{B}$" can be replaced with "$\sum_n \langle x_n|x_n\rangle$ is norm-bounded" or "$\sum_n \langle x_n|x_n\rangle$ is bounded above in $\mathfrak{B}$.") The proof below that this is a Σ^*-module closely follows [9, 8.5.26].

Lemma 3.2.14. *The space $\mathfrak{X} := \oplus^w \mathfrak{X}_n$ defined above is a Σ^*-module over $\mathfrak{B}$.*

Proof. It follows from the operator inequality $(y + i^k x)^*(y + i^k x) \leq 2(x^*x + y^*y)$ and the polarization identity $x^*y = \frac{1}{4}\sum_{k=0}^{3} i^k(y + i^k x)^*(y + i^k x)$ that $\langle (x_n)|(y_n)\rangle := \sum_n \langle x_n|y_n\rangle$ does indeed define a $\mathfrak{B}$-valued inner product on $\mathfrak{X}$. It is easy to check that $\langle\cdot|\cdot\rangle$ satisfies all the axioms of a C^*-module inner product.

Define $\mathcal{K}_n := \mathfrak{X}_n \otimes_{\mathfrak{B}} \mathcal{H}$ and $\mathcal{K} := \oplus_n^2 \mathcal{K}_n$, and let $P_n : \mathcal{K} \to \mathcal{K}_n$ be the canonical projection. Since each $\mathfrak{X}_n$ is WOT sequentially closed in $B(\mathcal{H}, \mathcal{K}_n)$, it follows immediately that the space

$$W := \{T \in B(\mathcal{H}, \mathcal{K}) : P_n T \in \mathfrak{X}_n \text{ for all } n\}$$

is a WOT sequentially closed TRO. By Theorem 3.1.10, W is a Σ^*-module over $\mathscr{B}(W^*W) \subseteq \mathfrak{B}$. Hence W is a Σ^*-module over $\mathfrak{B}$. Define a $\mathfrak{B}$-module map $U : \mathfrak{X} \to W$ by sending $(x_n) \in \mathfrak{X}$ to the SOT-convergent sum $\sum_n P_n^* x_n$ (indeed, for $\zeta \in \mathcal{H}$ and $N, M \in \mathbb{N}$ with $N \geq M$, a short calculation gives $\|\sum_{n=M}^{N} P_n^* x_n(\zeta)\|^2 = \sum_{n=M}^{N}\langle \zeta, \langle x_n|x_n\rangle\zeta\rangle$, and so by Cauchy's convergence test, the series $\sum_n P_n^* x_n(\zeta)$ converges). To check surjectivity of U, note that $\{P_n^* P_n\}_{n=1}^{\infty}$ is a family of mutually orthogonal projections in $B(\mathcal{K})$ with $\sum_n P_n^* P_n = I_{\mathcal{K}}$. If $T \in W$, then $\sum_{n=1}^{N}\langle P_n T|P_n T\rangle = T^*(\sum_{n=1}^{N} P_n^* P_n)T \leq T^*T$. Thus $(P_n T) \in \mathfrak{X}$, and $U((P_n T)) = \sum_n P_n^* P_n T = T$. Finally, it is easy to verify that the formula

$$U(x)^*U(y) = \langle x|y\rangle$$

holds for all $x, y \in \mathfrak{X}$ (by first checking this when both x and y are "finitely supported," then extending via WOT-limits to the general case).

So we have established the existence of a surjective inner-product-preserving $\mathfrak{B}$-module map $U : \mathfrak{X} \to W$, where W is a Σ^*-module over $\mathfrak{B}$. It follows immediately that $\mathfrak{X}$ is complete, and a straightforward application of Proposition 3.1.4 shows that $\mathfrak{X}$ is also a Σ^*-module over $\mathfrak{B}$. $\square$

Letting $\mathfrak{X}_n = \mathfrak{B}$ for all n, we obtain a Σ^*-module over $\mathfrak{B}$ which we denote as $C^w(\mathfrak{B})$.

Corollary 3.2.15. *If $\mathfrak{B}$ is a unital Σ^*-algebra, then the Σ^*-module $C^w(\mathfrak{B})$ is selfdual.*

Proof. Since $\mathfrak{B}$ is unital, $C^w(\mathfrak{B})$ is $\Sigma^*_{\mathfrak{B}}$-countably generated. $\square$

Early in this investigation, David Blecher proved an interesting generalization of Corollary 3.2.15, and we thank him for allowing it to be included here.

For a cardinal number I, a C^*-algebra $B \subseteq B(\mathcal{H})$ is said to be *I-additively weakly closed* if whenever $\sum_{k\in I} x_k^* x_k$ is bounded in $B(\mathcal{H})$ for a collection $\{x_k\}_{k\in I}$ in B, then the WOT-limit of this sum is an element in B. For an I-additively weakly closed $B \subseteq B(\mathcal{H})$, define

$$C_I^w(B) = \{(x_k) \in \prod_{k\in I} B : \sum_k x_k^* x_k \text{ is WOT-convergent in } B\}.$$

One may then show as in the first part of the proof of Lemma 3.2.14 that $\langle (x_k)|(y_k)\rangle := \sum_{k\in I} x_k^* y_k$ defines a B-valued inner product on $C_I^w(B)$. It is easy to argue that this satisfies all the axioms of a C^*-module inner product. Completeness of $C_I^w(B)$ follows as in the second to last paragraph of [9, 1.2.26], since $C_I^w(B)$ clearly coincides with the (underlying Banach space of the) operator space of the same notation there.

To see that Proposition 3.2.17 below generalizes Corollary 3.2.15, note that an easy "telescoping series" argument shows that B is $\mathbb{N}$-additively weakly closed if and only if B is a Borel $*$-algebra (that is, closed under weak limits in $B(\mathcal{H})$ of bounded monotone sequences of selfadjoint elements). Hence every Σ^*-algebra is $\mathbb{N}$-additively weakly closed.

To set some notation for the following lemma and proposition, let $B \subseteq B(\mathcal{H})$ be a nondegenerate I-additively weakly closed C^*-algebra. For each $j \in I$, denote by $\varepsilon_j : \mathcal{H} \to \mathcal{H}^{(I)}$ the canonical inclusion into the j^{th} summand, and by $P_j : \mathcal{H}^{(I)} \to \mathcal{H}$ the canonical projection from the j^{th} summand (so $\varepsilon_j^* = P_j$). For $b \in B$ and $j \in I$, denote by $e_j b$ the element in $C_I^w(B)$ with b in the j^{th} slot and 0's elsewhere.

Lemma 3.2.16. *If $B \subseteq B(\mathcal{H})$ is an I-additively weakly closed C^*-algebra, then $C_I^w(B) \otimes_B \mathcal{H} \cong \mathcal{H}^{(I)}$ via a unitary $U : \mathcal{H}^{(I)} \to C_I^w(B) \otimes_B \mathcal{H}$ such that*

$$U(\varepsilon_j(b\zeta)) = e_j b \otimes \zeta \text{ for all } b \in B,\ j \in I,\ \text{and } \zeta \in \mathcal{H}$$

and

$$U^*((b_i) \otimes \zeta) = (b_i \zeta) \text{ for all } (b_i) \in C_I^w(B) \text{ and } \zeta \in \mathcal{H}.$$

Proof. By Cohen's factorization theorem ([9, A.6.2]), every element in $\mathcal{H}$ can be expressed in the form $b\zeta$ for some $b \in B$ and $\zeta \in \mathcal{H}$. Using this, define a map $U_0 : \mathcal{F} \to C_I^w(B) \otimes_B \mathcal{H}$ on the dense subspace $\mathcal{F}$ of finitely supported columns in $\mathcal{H}^{(I)}$ by

$$U_0(\sum_{j \in F} \varepsilon_j(b_j \zeta_j)) = \sum_{j \in F} e_j b_j \otimes \zeta_j$$

for $(b_j \zeta_j) \in \mathcal{F}$ supported on a finite subset $F \subseteq I$. To see that this is well-defined, suppose that $b, b' \in B$ and $\zeta, \zeta' \in \mathcal{H}$ with $b\zeta = b'\zeta'$. Then for any $(c_i) \otimes \eta \in$

$C_I^w(B) \otimes_B \mathcal{H}$,

$$\langle e_j b \otimes \zeta - e_j b' \otimes \zeta', (c_i) \otimes \eta \rangle = \langle \zeta, b^* c_j \eta \rangle - \langle \zeta', (b')^* c_j \eta \rangle = \langle b\zeta - b'\zeta', c_j \eta \rangle = 0.$$

By totality of the simple tensors in $C_I^w(B) \otimes_B \mathcal{H}$, $e_j b \otimes \zeta - e_j b' \otimes \zeta' = 0$. It follows that U_0 is well-defined. A direct calculation shows that U_0 is isometric, hence extends to an isometry $U : \mathcal{H}^{(I)} \to C_I^w(B) \otimes_B \mathcal{H}$. To see that U is surjective, let $(b_i) \in C_I^w(B)$, $\zeta \in \mathcal{H}$, take $F \subseteq I$ to be finite, and denote by $(b_i)_F$ the "restriction" of (b_i) to F. Then

$$\begin{aligned} \|(b_i) \otimes \zeta - \sum_{i \in F} e_i b_i \otimes \zeta\|^2 &= \langle \zeta, \langle (b_i) - (b_i)_F | (b_i) - (b_i)_F \rangle \zeta \rangle \\ &= \langle \zeta, (\sum_{i \in I} b_i^* b_i - \sum_{i \in F} b_i^* b_i) \zeta \rangle. \end{aligned}$$

If we interpret $(\sum_{i \in I} b_i^* b_i - \sum_{i \in F} b_i^* b_i)$ as a net indexed by the collection of finite subsets F of I, the last displayed quantity converges to 0. So

$$U(\sum_{i \in F} \varepsilon_i (b_i \zeta)) = \sum_{i \in F} e_i b_i \otimes \zeta \longrightarrow (b_i) \otimes \zeta \text{ in norm},$$

where $\sum_{i \in F} e_i b_i \otimes \zeta$ is considered to be a net indexed by the collection of finite subsets F of I. Since the set of simple tensors in $C_I^w(B) \otimes_{\mathfrak{B}} \mathcal{H}$ spans a dense subset, it follows that U is surjective. The first displayed equation in the statement is obvious from the first displayed equation in this proof. For the second, we need to show

$$\langle (b_i) \otimes \zeta, U((\zeta_i)) \rangle = \langle (b_i \zeta), (\zeta_i) \rangle$$

for all $(b_i) \in C_I^w(B)$, $\zeta \in \mathcal{H}$, and $(\zeta_i) \in \mathcal{H}^{(I)}$. Straightforward calculations verify this formula in the case that $(\zeta_i) = \sum_{j \in F} \varepsilon_j (c_j \eta_j)$ for a finite set $F \subseteq I$, $c_j \in B$, and $\eta_j \in \mathcal{H}$. By Cohen's theorem again, this covers the case of finitely supported

(ζ_i). The formula for general (ζ_i) follows by norm-density of the finitely supported elements in $\mathcal{H}^{(I)}$. $\square$

Proposition 3.2.17 (David Blecher)**.** *If $B \subseteq B(\mathcal{H})$ is a unital and I-additively weakly closed C^*-algebra, then the C^*-module $C_I^w(B)$ is selfdual.*

Proof. We first fix some notation. Following [9, 1.2.26], denote by $\mathbb{M}_I(B(\mathcal{H}))$ the space of $I \times I$ matrices over $B(\mathcal{H})$ whose finite submatrices have uniformly bounded norm, and equip this space with the norm

$$\|u\| = \sup\{\|u_F\| : u_F \text{ is a finite submatrix of } u\}.$$

It is well known (see, e.g., the section in [9] just mentioned) that this is a Banach space that is canonically isometrically isomorphic to $B(\mathcal{H}^{(I)})$. Denote by $\mathbb{M}_I(B)$ the subspace of $\mathbb{M}_I(B(\mathcal{H}))$ consisting of matrices with entries in B.

Fixing an index $j_0 \in I$, there is a canonical isometric embedding of $C_I^w(B)$ onto the subspace of $\mathbb{M}_I(B)$ consisting of matrices supported on the j_0^{th} column (we omit the routine details of this), and a canonical embedding of B onto the subspace of matrices in $\mathbb{M}_I(B)$ supported on the (j_0, j_0)-entry. Write

$$\rho : C_I^w(B) \hookrightarrow \mathbb{M}_I(B)$$

$$\sigma : B \hookrightarrow \mathbb{M}_I(B)$$

for these embeddings.

We show that there is also a canonical embedding

$$\pi : B_B(C_I^w(B)) \hookrightarrow \mathbb{M}_I(B).$$

Indeed, by Proposition 2.1.1, Lemma 3.2.16, and [9] (1.19), we have the following canonical embedding and isomorphisms:

$$B_B(C_I^w(B)) \hookrightarrow B(C_I^w(B) \otimes_B \mathcal{H}) \cong B(\mathcal{H}^{(I)}) \cong \mathbb{M}_I(B(\mathcal{H})).$$

Using the unitary U from Lemma 3.2.16, we have

$$P_i \circ U^* T U \circ \varepsilon_j(\zeta) = P_i(U^* T(e_j \otimes \zeta)) = P_i(U^*(T(e_j) \otimes \zeta)) = (Te_j)_i(\zeta) = \langle e_i | T(e_j) \rangle (\zeta)$$

for all $i, j \in I$ and $\zeta \in \mathcal{H}$. That is, under the embedding and isomorphisms just mentioned, $T \in B_B(C_I^w(B))$ corresponds to the matrix $[T_{ij}] \in \mathbb{M}_I(B(\mathcal{H}))$ with $T_{ij} = \langle e_i | T(e_j) \rangle \in B$.

It is straightforward (using the definitions of ρ, π, and σ as composite maps involving the unitary U from Lemma 3.2.16) to show that for $x, y \in C_I^w(B)$ and $T \in B_B(C_I^w(B))$, we have

$$\rho(Tx) = \pi(T)\rho(x),$$
$$\sigma(\langle x|y \rangle) = \rho(x)^* \rho(y).$$

Note also that $\pi(T)^* \rho(y)$ is a matrix in $\mathbb{M}_I(B)$ supported on the j_0^{th} column, so $\pi(T)^* \rho(y) = \rho(z)$ for some $z \in C_I^w(B)$. Hence

$$\sigma(\langle Tx|y \rangle) = \rho(Tx)^* \rho(y) = (\pi(T)\rho(x))^* \rho(y) = \rho(x)^* \pi(T)^* \rho(y) = \rho(x)^* \rho(z) = \sigma(\langle x|z \rangle).$$

So $\langle Tx|y \rangle = \langle x|z \rangle$, and this is enough to prove that T is adjointable.

Thus $B_B(C_I^w(B)) = \mathbb{B}_B(C_I^w(B))$. To prove selfduality, let $\tau \in B_B(C_I^w(B), B)$ and fix an index $k \in I$. Define $T \in B_B(C_I^w(B))$ by $T(x) = e_k \tau(x)$. Then

$$\tau(x) = \langle e_k | e_k \tau(x) \rangle = \langle e_k | T(x) \rangle = \langle T^*(e_k) | x \rangle$$

for all $x \in C_I^w(B)$, so that $\tau = \langle T^*(e_k)|\cdot\rangle$. $\square$

The following lemma is a Σ^*-analogue of Lemma 2.34 from [36] or Proposition 3.8 in [25]. Note that the simple proof presented in these books does not seem to work in our setting, since it is unclear how to extend an isometry from a $\mathrm{WOT}_{\mathfrak{B}}$ sequentially dense subspace to the whole space. In the proof below, we write $\mathscr{B}(S)$ to denote the $\mathrm{WOT}_{\mathfrak{B}}$ sequential closure of a subset S of a Σ^*-module over $\mathfrak{B}$. (Recall from Note 3.1.3 that for a sequence in a Σ^*-module $\mathfrak{X}$ over $\mathfrak{B}$, $\mathrm{WOT}_{\mathfrak{B}}$-convergence coincides with WOT-convergence in $B(\mathcal{H}, \mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$, so this notation does not clash with our previous meaning of $\mathscr{B}(\cdot)$ as WOT sequential closure of subsets of $B(\mathcal{H})$.)

Lemma 3.2.18. *Let $\mathfrak{X}, \mathfrak{Y}$ be Σ^*-modules over $\mathfrak{B} \subseteq B(\mathcal{H})$. If T is an operator in $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{Y})$ such that $T(\mathfrak{X})$ is $WOT_{\mathfrak{B}}$ sequentially dense in $\mathfrak{Y}$ and $T^*(\mathfrak{Y})$ is $WOT_{\mathfrak{B}}$ sequentially dense in $\mathfrak{X}$, then $\mathfrak{X}$ and $\mathfrak{Y}$ are unitarily equivalent.*

Proof. Consider T as an element in the Σ^*-algebra

$$\mathbb{B}_{\mathfrak{B}}(\mathfrak{X} \oplus \mathfrak{Y}) \cong \begin{bmatrix} \mathbb{B}_{\mathfrak{B}}(\mathfrak{X}) & \mathbb{B}_{\mathfrak{B}}(\mathfrak{Y}, \mathfrak{X}) \\ \mathbb{B}_{\mathfrak{B}}(\mathfrak{X}, \mathfrak{Y}) & \mathbb{B}_{\mathfrak{B}}(\mathfrak{Y}) \end{bmatrix} \subseteq B((\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H}) \oplus (\mathfrak{Y} \otimes_{\mathfrak{B}} \mathcal{H})),$$

and take the polar decomposition

$$\begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix} = U \begin{bmatrix} |T| & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} |T| & 0 \\ 0 & 0 \end{bmatrix}.$$

By Proposition 2.2.6, $U \in \mathbb{B}_{\mathfrak{B}}(\mathfrak{X} \oplus \mathfrak{Y})$. We see that $U_{11}|T| = 0$, and the formula $U^*T = |T|$ shows that $U_{22}^*T = 0$. Consider the set $\mathscr{T} = \{y \in \mathfrak{Y} : T^*(y) \in \mathscr{B}(|T|\mathfrak{X})\}$, where $\mathscr{B}(S)$ denotes the $\mathrm{WOT}_{\mathfrak{B}}$ sequential closure of a subset $S \subseteq \mathfrak{X}$. Since T^*

is adjointable, it is $\mathrm{WOT}_{\mathfrak{B}}$ sequentially continuous, so $\mathscr{T}$ is a $\mathrm{WOT}_{\mathfrak{B}}$ sequentially closed subset of $\mathfrak{Y}$ containing $T(\mathfrak{X})$. Hence $\mathscr{T} = \mathfrak{Y}$, i.e. $T^*(\mathfrak{Y}) \subseteq \mathscr{B}(|T|\mathfrak{X})$, so $\mathfrak{X} = \mathscr{B}(T^*(\mathfrak{Y})) \subseteq \mathscr{B}(|T|\mathfrak{X})$ (using the notation mentioned above the statement of the lemma). Since U_{11} is $\mathrm{WOT}_{\mathfrak{B}}$ sequentially continuous, its kernel in $\mathfrak{X}$ is $\mathrm{WOT}_{\mathfrak{B}}$ sequentially closed, and since $\mathrm{Ker}(U_{11})$ contains the $\mathrm{WOT}_{\mathfrak{B}}$ sequentially dense set $|T|(\mathfrak{X})$, we have $U_{11} = 0$. A similar but shorter argument shows that $U_{22} = 0$ as well.

Since U is a partial isometry, it follows now that U_{21} is as well. The relation $U_{21} = U_{21}U_{21}^*U_{21}$ implies that $U_{21}(\mathfrak{X})$ is $\mathrm{WOT}_{\mathfrak{B}}$ sequentially closed in $\mathfrak{Y}$. (Indeed, suppose $U_{21}(x_n) \xrightarrow{WOT} y$ in $\mathfrak{Y}$. Then $U_{21}(x_n) = U_{21}U_{21}^*U_{21}(x_n) \xrightarrow{WOT} U_{21}U_{21}^*(y)$ since U_{21} and U_{21}^* are adjointable, hence $\mathrm{WOT}_{\mathfrak{B}}$ sequentially continuous. Thus $y = U_{21}U_{21}^*(y) \in U_{21}(\mathfrak{X})$.) Since $T = U_{21}|T|$, $U_{21}(\mathfrak{X})$ contains the $\mathrm{WOT}_{\mathfrak{B}}$ sequentially dense set $T(\mathfrak{X})$, and so U_{21} is surjective. Similarly, U_{21}^* is a partial isometry with $U_{21}^*(\mathfrak{Y}) = \mathfrak{X}$, and it follows that $U_{21} : \mathfrak{X} \to \mathfrak{Y}$ is a unitary. $\square$

It is quite surprising that (given Lemma 3.2.18) the obvious Σ^*-analogue of Kasparov's stabilization theorem now follows from only a slight modification of the proof presented in [25, Theorem 6.2] and [36, Theorem 5.49] for the original stabilization theorem.

Theorem 3.2.19. *If $\mathfrak{B} \subseteq B(\mathcal{H})$ is a Σ^*-algebra and $\mathfrak{X}$ is a $\Sigma^*_{\mathfrak{B}}$-countably generated Σ^*-module over $\mathfrak{B}$, then $\mathfrak{X} \oplus C^w(\mathfrak{B}) \cong C^w(\mathfrak{B})$ unitarily.*

Proof. Using the second comment in the paragraph under Definition 3.1.1 to make sense of the reduction to the unital case, apply the argument in [25, Theorem 6.2] or [36, Theorem 5.49], changing "generating set" to "$\Sigma^*_{\mathfrak{B}}$-generating set," "dense" to “WOT$_{\mathfrak{B}}$ sequentially dense,” and the C^*-module direct sum of countably many copies of $\mathfrak{B}$ to $C^w(\mathfrak{B})$. Finish off the argument by invoking Lemma 3.2.18. $\square$

We now discuss more generally orthogonally complemented submodules of Σ^*-modules, and then make a connection between a Σ^*-analogue of the C^*-module theory of quasibases and orthogonally complemented submodules of the Σ^*-module $C^w(\mathfrak{B})$.

Definition 3.2.20.

- A closed submodule X of a C^*-module Y over a C^*-algebra A is said to be *orthogonally complemented in* Y if there is another closed submodule W in Y such that $W + X = Y$ and $\langle w|x\rangle = 0$ for all $w \in W$ and $x \in X$. It is easy to see that this happens exactly when X is the range of a projection $P \in \mathbb{B}_A(Y)$. (For one direction of this, check that if X is orthogonally complemented in Y with W as above, then each element in Y has a unique representation as a sum $x + w$ with $x \in X$, $w \in W$. Then show that the map $P : Y \to Y$ defined $P(w + x) = x$ for $w \in W$ and $x \in X$, satisfies $\langle P(x + w)|x' + w'\rangle = \langle x + w|P(x' + w')\rangle$ for $w, w' \in W$, $x, x' \in X$.)

- A closed submodule $\mathfrak{X}$ of a Σ^*-module $\mathfrak{Y}$ over $\mathfrak{B}$ will be called a Σ^*-*submodule of* $\mathfrak{Y}$ if $\mathfrak{X}$ is a Σ^*-module with the C^*-module structure it inherits from $\mathfrak{Y}$, and if $\mathfrak{X}$ satisfies the following additional condition: whenever (x_n) is a sequence in $\mathfrak{X}$ and $x \in \mathfrak{X}$ such that $\langle x_n|z\rangle \xrightarrow{WOT} \langle x|z\rangle$ for all $z \in \mathfrak{X}$, then $\langle x_n|y\rangle \xrightarrow{WOT} \langle x|y\rangle$ for all $y \in \mathfrak{Y}$ (in other words, if a sequence converges WOT$_{\mathfrak{B}}$ in $\mathfrak{X}$, then it converges WOT$_{\mathfrak{B}}$ in $\mathfrak{Y}$ to the same limit).

Proposition 3.2.21. *Let $\mathfrak{X}$ be a closed submodule of a Σ^*-module $\mathfrak{Y}$ over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$. Consider the following conditions:*

(1) *$\mathfrak{X}$ is orthogonally complemented in $\mathfrak{Y}$;*

(2) *$\mathfrak{X}$ is a Σ^*-submodule of $\mathfrak{Y}$;*

(3) *$\mathfrak{X}$ is $WOT_{\mathfrak{B}}$ sequentially closed in $\mathfrak{Y}$.*

We have $(1) \implies (2) \iff (3)$. *If $\mathfrak{X}$ satisfies* (2) *and* $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) = \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$, *then* (1) *holds.*

Proof. $(1) \implies (2)$. Let $P \in \mathbb{B}_{\mathfrak{B}}(\mathfrak{Y})$ be a projection with range $\mathfrak{X}$. Suppose (x_n) is a sequence in $\mathfrak{X}$ such that $\langle x_n|w\rangle$ is WOT-convergent for all $w \in \mathfrak{X}$. Since $\mathfrak{Y}$ is a Σ^*-module and $\langle x_n|y\rangle = \langle x_n|Py\rangle$ for all $y \in \mathfrak{Y}$, it follows from Proposition 3.1.4 that there is an $x \in \mathfrak{Y}$ such that $\langle x_n|y\rangle \xrightarrow{WOT} \langle x|y\rangle$ for all $y \in \mathfrak{Y}$. But then $\langle x_n|y\rangle = \langle x_n|Py\rangle \xrightarrow{WOT} \langle x|Py\rangle = \langle Px|y\rangle$ for all $y \in \mathfrak{Y}$, so $x = Px \in \mathfrak{X}$. This proves that $\mathfrak{X}$ is a Σ^*-module. To see that $\mathfrak{X}$ is a Σ^*-submodule of $\mathfrak{Y}$, suppose that $(x_n), x \in \mathfrak{X}$ with $\langle x_n|w\rangle \xrightarrow{WOT} \langle x|w\rangle$ for all $w \in \mathfrak{X}$. Then $\langle x_n|y\rangle = \langle x_n|Py\rangle \xrightarrow{WOT} \langle x|Py\rangle = \langle x|y\rangle$ for all $y \in \mathfrak{Y}$.

$(2) \implies (3)$. To show that $\mathfrak{X}$ is $\mathrm{WOT}_{\mathfrak{B}}$ sequentially closed in $\mathfrak{Y}$, suppose that (x_n) is a sequence in $\mathfrak{X}$ converging $\mathrm{WOT}_{\mathfrak{B}}$ to y in $\mathfrak{Y}$, i.e. $\langle x_n|w\rangle \xrightarrow{WOT} \langle y|w\rangle$ for all $w \in \mathfrak{Y}$. In particular, $\langle x_n|z\rangle$ is WOT-convergent for all $z \in \mathfrak{X}$, so by Proposition 3.1.4, there is an $x \in \mathfrak{X}$ such that $\langle x_n|z\rangle \xrightarrow{WOT} \langle x|z\rangle$ for all $z \in \mathfrak{X}$. By the "additional condition" in the definition of Σ^*-submodule, $\langle x_n|w\rangle \xrightarrow{WOT} \langle x|w\rangle$ for all $w \in \mathfrak{Y}$. Hence $y = x \in \mathfrak{X}$.

$(3) \implies (2)$. Assume (3). Note that we can canonically identify $\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H}$ with a closed subspace of $\mathfrak{Y}\otimes_{\mathfrak{B}}\mathcal{H}$. Indeed, the canonical inclusion of $\{\sum_{i=1}^{n} x_i\otimes\zeta_i \in \mathfrak{X}\otimes_{\mathfrak{B}}\mathcal{H} : x_i \in \mathfrak{X}, \zeta_i \in \mathcal{H}\}$ into $\mathfrak{Y}\otimes_{\mathfrak{B}}\mathcal{H}$ is isometric, hence extends to an isometry from $\mathfrak{X}\otimes_{\mathfrak{B}}\mathcal{H}$ into $\mathfrak{Y}\otimes_{\mathfrak{B}}\mathcal{H}$. To see that $\mathfrak{X}$ is a Σ^*-module with the inherited C^*-module structure, suppose that (x_n) is a sequence in $\mathfrak{X}$ viewed in $B(\mathcal{H}, \mathfrak{X}\otimes_{\mathfrak{B}}\mathcal{H})$ with $x_n \xrightarrow{WOT} T$ in $B(\mathcal{H}, \mathfrak{X}\otimes_{\mathfrak{B}}\mathcal{H})$. Note that there is a canonical WOT-continuous embedding of $B(\mathcal{H}, \mathfrak{X}\otimes_{\mathfrak{B}}\mathcal{H})$ into $B(\mathcal{H}, \mathfrak{Y}\otimes_{\mathfrak{B}}\mathcal{H})$ making the following diagram commute:

$$\begin{array}{ccc} B(\mathcal{H}, \mathfrak{X}\otimes_{\mathfrak{B}}\mathcal{H}) & \longrightarrow & B(\mathcal{H}, \mathfrak{Y}\otimes_{\mathfrak{B}}\mathcal{H}) \\ \uparrow & & \uparrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{Y} \end{array}$$

So $x_n \xrightarrow{WOT} T$ in $B(\mathcal{H}, \mathfrak{Y}\otimes_{\mathfrak{B}}\mathcal{H})$, and since $\mathfrak{Y}$ is WOT sequentially closed in the latter, $T \in \mathfrak{Y}$ and $x_n \xrightarrow{WOT_{\mathfrak{B}}} T$ in $\mathfrak{Y}$. By the assumption that $\mathfrak{X}$ is WOT$_{\mathfrak{B}}$ sequentially closed in $\mathfrak{Y}$, $T \in \mathfrak{X}$. Hence $\mathfrak{X}$ is WOT sequentially closed in $B(\mathcal{H}, \mathfrak{X}\otimes_{\mathfrak{B}}\mathcal{H})$, and so by definition, $\mathfrak{X}$ is a Σ^*-module.

Now suppose that $(x_n), x$ are in $\mathfrak{X}$ and $\langle x_n|z\rangle \xrightarrow{WOT} \langle x|z\rangle$ for all $z \in \mathfrak{X}$. Fixing $\zeta, \eta \in \mathcal{H}$, we have

$$\langle x_n\otimes\zeta, z\otimes\eta\rangle = \langle\zeta, \langle x_n|z\rangle\eta\rangle \to \langle\zeta, \langle x_n|z\rangle\eta\rangle = \langle x_n\otimes\zeta, z\otimes\eta\rangle$$

for all $z \in \mathfrak{X}$. Take $y \in \mathfrak{Y}$, and let $\epsilon > 0$. Denote by P the projection in $B(\mathfrak{Y}\otimes_{\mathfrak{B}}\mathcal{H})$ with range $\mathfrak{X}\otimes_{\mathfrak{B}}\mathcal{H}$. By the principle of uniform boundedness, there is a $K > 0$ such that $\|x_n\| \leq K$ for all n and $\|x\| \leq K$. Pick $\sum_{i=1}^{k} z_i\otimes\zeta_i \in \mathfrak{X}\otimes_{\mathfrak{B}}\mathcal{H}$ with

$$\|P(y\otimes\eta) - \sum_{i=1}^{k} z_i\otimes\zeta_i\| < \frac{\epsilon}{3K(\|\zeta\|+1)},$$

and pick $N \in \mathbb{N}$ with

$$|\langle x_n\otimes\zeta, \sum_{i=1}^{k} z_i\otimes\zeta_i\rangle - \langle x\otimes\zeta, \sum_{i=1}^{k} z_i\otimes\zeta_i\rangle| < \frac{\epsilon}{3}$$

for all $n \geq N$. A triangle inequality argument then gives

$$|\langle \zeta, \langle x_n|y\rangle \eta\rangle - \langle \zeta, \langle x|y\rangle \eta\rangle| = |\langle x_n \otimes \zeta, P(y \otimes \eta)\rangle - \langle x \otimes \zeta, P(y \otimes \eta)\rangle < \epsilon$$

for all $n \geq N$. Since $\zeta, \eta \in \mathcal{H}$ were arbitrary, we have shown that $\langle x_n|y\rangle \xrightarrow{WOT} \langle x|y\rangle$ for all $y \in \mathfrak{Y}$.

Now we prove the final claim in the statement of the proposition. Suppose that $\mathfrak{X}$ is a Σ^*-submodule of $\mathfrak{Y}$ and that $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) = \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$. By definition of Σ^*-submodule, the inclusion $\iota : \mathfrak{X} \hookrightarrow \mathfrak{Y}$ is $\mathrm{WOT}_{\mathfrak{B}}$ sequentially continuous, so by Proposition 3.1.15 (2), ι is adjointable. Since $\langle \iota^*\iota x|x'\rangle = \langle \iota x|\iota x'\rangle = \langle x|x'\rangle$ for all $x, x' \in \mathfrak{X}$, we have $\iota^*\iota = \mathrm{id}_{\mathfrak{X}}$. It follows that $P = \iota\iota^* \in \mathbb{B}_{\mathfrak{B}}(\mathfrak{Y})$ is a projection with range $\mathfrak{X}$. □

Note 3.2.22. (We thank David Blecher for pointing this out.) To show that (2)/(3) does not imply (1) in Proposition 3.2.21 in general, let $\mathfrak{B} \subseteq B(\mathcal{H})$ be a nonunital Σ^*-algebra, and take $\mathfrak{X}$ to be $\mathfrak{B}$ and $\mathfrak{Y}$ to be the unitization $\mathfrak{B}^1 \subseteq B(\mathcal{H})$, where we view these both as Σ^*-modules over $\mathfrak{B}^1$. Clearly $\mathfrak{X}$ satisfies (3), but $\mathfrak{X}$ is not orthogonally complemented in $\mathfrak{Y}$ since it is a proper subset and $\{y \in \mathfrak{Y} : \langle y|x\rangle = 0 \text{ for all } x \in \mathfrak{X}\} = \{c + \mu I_{\mathcal{H}} \in \mathfrak{B}^1 : b^*c + \mu b^* = 0 \text{ for all } b \in \mathfrak{B}\} = (0)$ by an approximate identity argument.

We now define a Σ^*-analogue of "quasibases" for C^*-modules (see [9] 8.2.5 and relevant notes in 8.7).

Definition 3.2.23. For a Σ^*-module $\mathfrak{X}$ over a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, a countable subset $\{x_k\}$ of $\mathfrak{X}$ is a called a *weak quasibasis* for $\mathfrak{X}$ if for any $x \in \mathfrak{X}$, the sequence of finite sums $\sum_{k=1}^{n} x_k\langle x_k|x\rangle$ WOT$_{\mathfrak{B}}$-converges to x. In other words, $\{x_k\}$ is a weak quasibasis iff $\sum_{k=1}^{n} |x_k\rangle\langle x_k| \nearrow I$ in $B(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$.

Remark 3.2.24. Quasibases are also called "frames" (it appears that "quasibasis" is the older term for these and "frame" is the term most commonly employed in recent literature). Frank and Larson in [22] initiated a systematic study of quasibases/frames for Hilbert C^*-modules, and what we have called "weak quasibases" are essentially equivalent to "non-standard normalized tight frames" in the terminology of their paper (see [22, Definition 2.1]). We also remark that Frank and Larson followed a similar approach to ours in using Kasparov's stabilization theorem to deduce the existence of quasibases/frames. (We thank the anonymous referee of the published version of this chapter for drawing our attention to these points.)

Proposition 3.2.25. *Let $\mathfrak{B} \subseteq B(\mathcal{H})$ be a Σ^*-algebra, and let $\mathfrak{X}$ be a right Banach module over $\mathfrak{B}$.*

If $\mathfrak{X}$ is a Σ^*-module over $\mathfrak{B}$ with a weak quasibasis, then $\mathfrak{X}$ is isometrically $\mathfrak{B}$-isomorphic to an orthogonally complemented submodule of $C^w(\mathfrak{B})$.

Conversely, if $\mathfrak{X}$ is isometrically $\mathfrak{B}$-isomorphic to an orthogonally complemented submodule of a Σ^*-module $\mathfrak{Y}$ over $\mathfrak{B}$ such that $\mathfrak{Y}$ has a weak quasibasis, then $\mathfrak{X}$ is a Σ^*-module over $\mathfrak{B}$ with the canonically induced inner product, and $\mathfrak{X}$ has a weak quasibasis.

Proof. For the first statement, if $\mathfrak{X}$ is a Σ^*-module over $\mathfrak{B}$ with a weak quasibasis, then clearly $\mathfrak{X}$ is $\Sigma^*_{\mathfrak{B}}$-countably generated, and the result now follows from Theorem 3.2.19.

For the converse, note first that if $\mathfrak{X}$ is isometrically $\mathfrak{B}$-isomorphic to any Σ^*-module $\mathfrak{X}_0$ over $\mathfrak{B}$ via an isometric $\mathfrak{B}$-isomorphism $U : \mathfrak{X} \to \mathfrak{X}_0$, then defining $\langle x|y\rangle := \langle Ux|Uy\rangle$ for $x, y \in \mathfrak{X}$ makes U a unitary between C^*-modules and makes $\mathfrak{X}$ a Σ^*-module over $\mathfrak{B}$ (this can be seen either by applying Proposition 3.1.4 or invoking Definition 3.1.1, noting that U induces a canonical unitary $\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H} \cong \mathfrak{X}_0 \otimes_{\mathfrak{B}} \mathcal{H}$). Since every orthogonally complemented submodule of $\mathfrak{Y}$ is a Σ^*-module over $\mathfrak{B}$ by Proposition 3.2.21, and since unitaries between Σ^*-modules preserve weak quasibases, we may assume without loss of generality that $\mathfrak{X}$ is actually an orthogonally complemented submodule of $\mathfrak{Y}$. In that case, let $P \in \mathbb{B}_{\mathfrak{B}}(\mathfrak{Y})$ be a projection with range $\mathfrak{X}$, and let $\{e_k\}$ be a weak quasibasis for $\mathfrak{Y}$. Then for any $x \in \mathfrak{X}$,

$$\sum_{k=1}^{n} P(e_k)\langle P(e_k)|x\rangle = \sum_{k=1}^{n} P(e_k)\langle e_k|x\rangle = P(\sum_{k=1}^{n} e_k\langle e_k|x\rangle) \xrightarrow{WOT_{\mathfrak{B}}} P(x) = x,$$

so that $\{P(e_k)\}$ is a weak quasibasis for $\mathfrak{X}$. $\square$

We close by coalescing some of the main results of this section in the case of a unital coefficient Σ^*-algebra:

Theorem 3.2.26. *Let $\mathfrak{B} \subseteq B(\mathcal{H})$ be a unital Σ^*-algebra, and let $\mathfrak{X}$ be a Banach module over $\mathfrak{B}$. The following are equivalent:*

(1) *$\mathfrak{X}$ is a $\Sigma^*_{\mathfrak{B}}$-countably generated Σ^*-module over $\mathfrak{B}$;*

(2) *$\mathfrak{X}$ is a $\Sigma^*_{\mathfrak{B}}$-countably generated selfdual C^*-module over $\mathfrak{B}$;*

(3) *$\mathfrak{X}$ is a Σ^*-module with a weak quasibasis;*

(4) *$\mathfrak{X}$ is isometrically $\mathfrak{B}$-isomorphic to an orthogonally complemented submodule of $C^w(\mathfrak{B})$;*

(5) $\mathfrak{X} \oplus C^w(\mathfrak{B}) \cong C^w(\mathfrak{B})$.

Proof. (1) $\iff$ (2). Theorem 3.2.10.

(1) $\implies$ (5). Theorem 3.2.19.

(5) $\implies$ (4). Easy.

(4) $\implies$ (3). Proposition 3.2.25, noting that if $\mathfrak{B}$ is unital, then $C^w(\mathfrak{B})$ has a canonical weak quasibasis.

(3) $\implies$ (1). Easy. $\square$

Chapter 4

Morita Equivalence for Σ^*-algebras

4.1 Introduction

Strong Morita equivalence for C^*-algebras is an equivalence relation coarser than $*$-isomorphism, but fine enough to preserve many distinguishing structures and properties a C^*-algebra can have (e.g., ideal structure, representation theory, and K-theory in the σ-unital case). Originally a concept in pure ring theory, Morita equivalence was imported into operator algebra theory by M. Rieffel in [37, 38], where he defined and initiated the study of C^*-algebraic Morita equivalence and the corresponding version for W^*-algebras. It has since taken a central role in operator algebra theory, and appears in a number of the most important classification results for C^*-algebras (e.g., the Kirchberg-Phillips classification for Kirchberg algebras [39, Theorem 8.4.1], the Dixmier-Douady classification for continuous-trace C^*-algebras [36, Theorem 5.29], and the classification of unital graph C^*-algebras recently discovered by Eilers, Restorff, Ruiz, and Sørensen [20]). The main goal of the present work is to define and study a version of Morita equivalence for Σ^*-algebras.

The organization of this chapter is as follows. In Section 4.2, we provide some background material on Morita equivalence for C^*-algebras and W^*-algebras, mostly as motivation for the analogues we prove later, but some of this material is actually used. In Section 4.3, we define strong Σ^*-Morita equivalence and Σ^*-imprivitivity bimodules, and prove a few basic results that are used throughout the rest of the chapter. In Section 4.4, we give an account of the Σ^*-module interior tensor product of Σ^*-imprivitivity bimodules, with the main goal being to prove that Σ^*-Morita equivalence is transitive (hence an equivalence relation). Section 4.5 contains a discussion of "weak Morita equivalence" for C^*-algebras and Σ^*-algebras (in the W^*-case, the analogous definition coincides with the "strong" version). In Section 4.6, we prove a version of the "full corners" characterization of strong Morita equivalence and some consequences. In Section 4.7, we develop the Σ^*-module exterior tensor product, which allows us in Section 4.8 to prove an analogue of the Brown-Green-Rieffel stable isomorphism theorem.

4.2 Background

Again, since most of this is well known, we will be terse. We generally refer to [9, Chapter 8] for notation and results; other references include [25, 36], and [5, Section II.7.6].

If X is a left or right C^*-module over A, denote by $\langle X|X\rangle$ the linear span of the range of the A-valued inner product, i.e.

$$\langle X|X\rangle := \text{span}\{\langle x|y\rangle : x, y \in X\}.$$

Note that $\langle X|X\rangle$ is also the (not necessarily closed) ideal in A generated by $\{\langle x|y\rangle : x, y \in X\}$. We say that X is a *full* C^*-module over A if $\langle X|X\rangle$ is norm-dense in A. Similarly, a W^*-module Y over a W^*-algebra M is said to be *w^*-full* if $\langle Y|Y\rangle$ is weak*-dense in M.

Definition 4.2.1. Two C^*-algebras A and B are said to be *strongly C^*-Morita equivalent* if there exists an $A - B$ bimodule X that is a full left C^*-module over A and a full right C^*-module over B, and such that the two inner products and module actions on X are related according to the following formula:

$${}_A\langle x|y\rangle z = x\langle y|z\rangle_B \text{ for all } x, y, z \in X.$$

In this case X is called an *$A - B$ C^*-imprivitivity bimodule.*

Similarly, two W^*-algebras M and N are *W^*-Morita equivalent* if there exists an $M - N$ bimodule Y that is a w^*-full left W^*-module over M and a w^*-full right W^*-module over N, and such that ${}_M\langle x|y\rangle z = x\langle y|z\rangle_N$ for all $x, y, z \in Y$. In this case Y is called an *$M - N$ W^*-imprivitivity bimodule.*

4.2.2 (Interior tensor product of C^*-modules)**.** We sketch the construction of the interior tensor product of C^*-modules (see [25, pgs. 39–41] for details). Suppose A and B are C^*-algebras, X is a right C^*-module over A, and Y is a right C^*-module over B such that there is a $*$-homomorphism $\rho : A \to \mathbb{B}_B(Y)$. The algebraic module tensor product $X \odot_A Y$ is then a right B-module which admits a B-valued inner product determined by the formula

$$\langle x_1 \otimes y_1 | x_2 \otimes y_2 \rangle_B = \langle y_1 | \rho(\langle x_1 | x_2 \rangle_A) y_2 \rangle$$

for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. The completion of $X \odot_A Y$ in the norm induced by this inner product yields a right C^*-module $X \otimes_A Y$ over B, called the *interior tensor product* of X and Y. (Note that this construction does not require A to act nondegenerately on Y via ρ. Some authors require this though, as it yields the sensible formula $A \otimes_A Y = Y$. At any rate, we will only apply this construction to certain "imprivitivity bimodules," which are automatically nondegenerate.)

4.2.3 (Exterior tensor product of C^*-modules)**.** We now sketch the other main tensor product for C^*-modules—the exterior tensor product (see [25, pgs. 34–38] for these details). The set-up here is: let X be a right C^*-module over A, and let Y be a right C^*-module over B. The algebraic (vector space) tensor product $X \odot Y$ is canonically a right module over the algebraic tensor product $A \odot B$, and there is an $A \odot B$-valued inner product on $X \odot Y$ determined by the rule

$$\langle x_1 \otimes y_2 | x_2 \otimes y_2 \rangle = \langle x_1 | x_2 \rangle \otimes \langle y_1 | y_2 \rangle$$

on simple tensors. A "double-completion" process on the inner product $A \odot B$-module $X \odot Y$ then yields a C^*-module $X \otimes Y$ over the minimal C^*-algebra tensor product $A \otimes B$. The C^*-module $X \otimes Y$ is called the *exterior tensor product* of X and Y.

Now we record a couple of the most important results in basic C^*- and W^*-algebraic Morita equivalence theory. We will prove Σ^*-analogies for each of these in this work. Following the theorems is a note on some aspects of these theorems relevant to the present work, as well as explanations for some of the undefined terms.

Theorem 4.2.4. *Strong C^*-Morita equivalence and W^*-Morita equivalence are equivalence relations strictly coarser than $*$-isomorphism.*

Theorem 4.2.5. *Two C^*-algebras (resp. W^*-algebras) A and B are strongly C^*-Morita equivalent (resp. W^*-Morita equivalent) if and only if they are $*$-isomorphic to complementary full (resp. w^*-full) corners of a C^*-algebra (resp. W^*-algebra) C.*

Theorem 4.2.6. (1) *Two σ-unital C^*-algebras A and B are strongly C^*-Morita equivalent if and only if A and B are stably isomorphic (i.e. $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$, where $\mathbb{K}$ is the space of compact operators on a separable Hilbert space).*

(2) *Two W^*-algebras M and N are W^*-Morita equivalent if and only if there exists a Hilbert space $\mathcal{H}$ such that $M \overline{\otimes} B(\mathcal{H}) \cong N \overline{\otimes} B(\mathcal{H})$ (where $\overline{\otimes}$ denotes the spatial von Neumann algebra tensor product).*

Note 4.2.7. The C^*-algebra versions of all these are in the standard texts [25, 36]. For the W^*-versions, see [9, Section 8.5].

The only non-trivial part of Theorem 4.2.4 is transitivity, which is typically proved in the C^*-algebra case using the interior tensor product of C^*-modules outlined above. The proof of transitivity in the W^*-case may be done in many different ways, including via an analogue of the interior tensor product (see [9, end of chapter notes for 8.5] for a different way). In this paper, we develop a Σ^*-analogue of the interior tensor product to prove the corresponding result for strong Σ^*-Morita equivalence.

To explain the undefined terms in Theorem 4.2.5, *complementary corners* of a C^*-algebra C are C^*-subalgebras of the form pCp and $(1-p)C(1-p)$ for a projection p in the multiplier algebra $M(C)$. A corner of a C^*-algebra (resp. W^*-algebra) is *full* (resp. w^*-full) if the closed (resp. w^*-closed) ideal it generates is all of C. One direction of Theorem 4.2.5 is proved using an important construction called the *linking algebra*, and we prove our analogue with the obvious Σ^*-version of the linking algebra.

Part (1) of Theorem 4.2.6 is the incredible Brown-Green-Rieffel stable isomorphism theorem. A C^*-algebra is σ-unital if and only if it has a sequential contractive approximate identity, so in particular every separable C^*-algebra is σ-unital. There are examples of pairs of C^*-algebras that are strongly C^*-Morita equivalent but not stably isomorphic—the easiest example is $\mathbb{C}$ and $\mathbb{K}(\mathcal{H})$ for non-separable $\mathcal{H}$ (and see [5, II.7.6.10] for mention of a fancier and probably more satisfying example). Note the interesting fact that in Theorem 4.2.6 (2), there are no restrictions on the W^*-algebras involved.

Note 4.2.8. In this chapter, we usually find it more convenient to work with abstract Σ^*-algebras rather than concrete Σ^*-algebras as in Chapter 3. As such, the notation for convergence in this chapter changes a little: we will generally write $b_n \xrightarrow{\sigma} b$ to mean that $((b_n), b)$ is in the σ-convergence system of a Σ^*-algebra.

Also, if $\mathfrak{X}$ is a right (resp. left) Σ^*-module over a Σ^*-algebra $\mathfrak{B}$ and $(x_n), x \in \mathfrak{X}$, we write

$$x_n \xrightarrow{\sigma_{\mathfrak{B}}} x \quad (\text{resp. } x_n \xrightarrow{{}_{\mathfrak{B}}\sigma} x)$$

to mean that $\langle x_n|y\rangle \xrightarrow{\sigma} \langle x|y\rangle$ for all $y \in \mathfrak{X}$, and we call this convergence $\sigma_{\mathfrak{B}}$-*convergence* (resp. ${}_{\mathfrak{B}}\sigma$*-convergence*). Using Lemma 2.2.8, it is easy to check that $x_n \xrightarrow{\sigma_{\mathfrak{B}}} x$ (resp. $x_n \xrightarrow{{}_{\mathfrak{B}}\sigma} x$) if and only if $x_n \xrightarrow{WOT} x$ in $B(\mathcal{H}, \mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$ (resp. in $B(\overline{\mathfrak{X}} \otimes_{\mathfrak{B}} \mathcal{H}, \mathcal{H})$) for any (hence every) faithful Σ^*-representation $\mathfrak{B} \hookrightarrow B(\mathcal{H})$.

4.3 Strong Σ^*-Morita equivalence

In this section, $\mathfrak{A}$ and $\mathfrak{B}$ are always taken to be Σ^*-algebras.

Definition 4.3.1. A C^*-module $\mathfrak{X}$ over a Σ^*-algebra $\mathfrak{B}$ is *σ-full over* $\mathfrak{B}$ if $\mathscr{B}(\langle\mathfrak{X}|\mathfrak{X}\rangle) = \mathfrak{B}$.

Definition 4.3.2. An $\mathfrak{A}-\mathfrak{B}$ bimodule $\mathfrak{X}$ is called an $\mathfrak{A}-\mathfrak{B}$ *Σ^*-imprivitivity bimodule* if:

(1) $\mathfrak{X}$ is a σ-full left Σ^*-module over $\mathfrak{A}$ and a σ-full right Σ^*-module over $\mathfrak{B}$;

(2) ${}_{\mathfrak{A}}\langle x|y\rangle z = x\langle y|z\rangle_{\mathfrak{B}}$ for all $x, y, z \in \mathfrak{X}$;

(3) for a sequence $(x_n) \in \mathfrak{X}$ and $x \in \mathfrak{X}$, $x_n \xrightarrow{{}_{\mathfrak{A}}\sigma} x$ if and only if $x_n \xrightarrow{\sigma_{\mathfrak{B}}} x$.

If there exists an $\mathfrak{A}-\mathfrak{B}$ Σ^*-imprivitivity bimodule, we say that $\mathfrak{A}$ and $\mathfrak{B}$ are *strongly Σ^*-Morita equivalent.*

Note 4.3.3. In the C^*- and W^*-settings, the analogue of condition (3) above is automatic, but we do not know if (3) is automatic in our case. One way to show that (3) does not always hold would be to exhibit two Σ^*-algebras that are isomorphic as C^*-algebras but are not Σ^*-isomorphic. (The existence of such algebras is an open question.) Indeed, suppose $\mathfrak{A}$ and $\mathfrak{B}$ are Σ^*-algebras that are $*$-isomorphic, but not Σ^*-isomorphic. If $\mathfrak{X}$ is the underlying C^*-algebra, then $\mathfrak{X}$ evidently satisfies conditions (1) and (2), but not (3).

Example 4.3.4. (Examples of Σ^*-imprivitivity bimodules.)

(1) If $\mathfrak{X}$ is any right Σ^*-module over $\mathfrak{B}$, then by Lemma 3.1.7 and Theorem 3.1.8, $\mathfrak{X}$ is a $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) - \mathscr{B}(\langle\mathfrak{X}|\mathfrak{X}\rangle)$ Σ^*-imprivitivity bimodule.

(2) Suppose that $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a Σ^*-isomorphism. Then $\mathfrak{B}$ may be viewed as a right Σ^*-module over itself, and also as a left Σ^*-module over $\mathfrak{A}$ via the module action $a \cdot b = \varphi(a)b$ and inner product $\langle b|c\rangle_{\mathfrak{A}} = \varphi^{-1}(bc^*)$ for $a \in \mathfrak{A}$ and $b, c \in \mathfrak{B}$. It is straightforward to check that $\mathfrak{B}$ is an $\mathfrak{A} - \mathfrak{B}$ Σ^*-imprivitivity bimodule.

(3) It is direct to show that for any $n \in \mathbb{N}$, $\mathfrak{B}$ is Σ^*-Morita equivalent to $M_n(\mathfrak{B})$ via the Σ^*-module $C_n(\mathfrak{B})$. We show below (Corollary 4.6.5) that there is an infinite version of this, analogous to the facts that for any Hilbert space $\mathcal{H}$ of dimension I, any C^*-algebra A, and any W^*-algebra M, A is strongly C^*-Morita equivalent to $\mathbb{K}(\mathcal{H}) \otimes A$ via $C_I(A)$, and M is W^*-Morita equivalent to $B(\mathcal{H})\overline{\otimes}M$ via $C_I^w(M)$ (here, $C_I(A)$ is the C^*-module direct sum of I copies of A, and $C_I^w(M)$ is the W^*-module direct sum of I copies of M; see [9, 8.1.9 and 8.5.15]).

We now show that condition (3) in Definition 4.3.2 may be replaced with the condition that $\mathfrak{A}$ canonically embeds as a Σ^*-subalgebra of $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$.

Lemma 4.3.5. *If $\mathfrak{X}$ is a σ-full left Σ^*-module over $\mathfrak{A}$ and a right Σ^*-module over $\mathfrak{B}$*

such that ${}_{\mathfrak{A}}\langle x|y\rangle z = x\langle y|z\rangle_{\mathfrak{B}}$ *for all* $x, y, z \in \mathfrak{X}$*, then there is a canonical isometric* $*$*-homomorphism* $\lambda : \mathfrak{A} \hookrightarrow \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$*.*

Proof. It is well known from basic C^*-module theory (cf. the first few lines in the proof of [9, Lemma 8.1.15]) that the inner product condition on $\mathfrak{X}$ implies that the canonical map $\lambda : \mathfrak{A} \to B(\mathfrak{X})$, defined $\lambda(a)(x) = ax$ for $a \in \mathfrak{A}$ and $x \in \mathfrak{X}$, maps into $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ and is a $*$-homomorphism.

We now use the assumption that $\mathfrak{X}$ is σ-full over $\mathfrak{A}$ to prove that λ is injective, hence isometric. For this, suppose that $a \in \mathfrak{A}$ with $\lambda(a) = 0$. Then $\lambda(a)(x) = ax = 0$ for all $x \in \mathfrak{X}$, so $\langle ax|y\rangle_{\mathfrak{A}} = a\langle x|y\rangle_{\mathfrak{A}} = 0$ for all $x, y \in \mathfrak{X}$. With $\mathscr{S} = \{j \in \mathfrak{A} : aj = 0\}$, it is straightforward to check that $\langle \mathfrak{X}|\mathfrak{X}\rangle_{\mathfrak{A}} \subseteq \mathscr{S}$ and that $\mathscr{S}$ is σ-closed. Since $\langle \mathfrak{X}|\mathfrak{X}\rangle_{\mathfrak{A}}$ is σ-dense in $\mathfrak{A}$, $\mathscr{S} = \mathfrak{A}$. Hence $aa^* = 0$, so $a = 0$. □

Lemma 4.3.6. *If* S *is any subset of* $B(\mathcal{H})$*, then* $[S\mathcal{H}] = [\mathscr{B}(S)\mathcal{H}]$*. In particular, if* S *is a* σ*-dense subset of a* Σ^**-algebra* $\mathfrak{B} \subseteq B(\mathcal{H})$*, then* $[S\mathcal{H}] = \mathcal{H}$*.*

Proof. Obviously $[S\mathcal{H}] \subseteq [\mathscr{B}(S)\mathcal{H}]$. For the other inclusion, fix $\zeta \in \mathcal{H}$, and define $\mathscr{S} := \{b \in \mathscr{B}(S) : b\zeta \in [S\mathcal{H}]\}$. Clearly $S \subseteq \mathscr{S}$, and since $[S\mathcal{H}]$ is weakly closed in $\mathcal{H}$ (by the general fact that every norm-closed convex set in a normed space is weakly closed), it follows that $\mathscr{S}$ is WOT sequentially closed. Thus $\mathscr{B}(S) = \mathscr{S}$, and the other inclusion follows. The last statement follows immediately (recalling our convention that concrete Σ^*-algebras are taken to be nondegenerately acting). □

Note that if $\mathfrak{A} \subseteq B(\mathcal{K})$ is a Σ^*-algebra, then the copy of the multiplier algebra $M(\mathfrak{A})$ in $B(\mathcal{K})$ is WOT sequentially closed, hence is also a Σ^*-algebra. If $\mathfrak{X}$ is a left Σ^*-module over $\mathfrak{A}$, then $\mathfrak{X}$ is canonically a left C^*-module over $M(\mathfrak{A})$ (see [9, 8.1.4(4)]), and it is easy to check that $\mathfrak{X}$ is a left Σ^*-module over $M(\mathfrak{A})$ in this case.

Lemma 4.3.7. *Let $\mathfrak{A} \subseteq B(\mathcal{K})$ be a Σ^*-algebra, and let (ξ_n) be a sequence in the copy of the multiplier algebra $M(\mathfrak{A})$ in $B(\mathcal{K})$. Then (ξ_n) is WOT-convergent if and only if $(\xi_n a)$ is σ-convergent in $\mathfrak{A}$ for all $a \in \mathfrak{A}$. For $\xi \in M(\mathfrak{A})$, we have $\xi_n \xrightarrow{WOT} \xi$ if and only if $\xi_n a \xrightarrow{\sigma} \xi a$ for all $a \in \mathfrak{A}$. Hence $M(\mathfrak{A})$ has a unique structure as a Σ^*-algebra in which $\mathfrak{A}$ is a Σ^*-subalgebra.*

Proof. The forward direction of both of the first two claims is direct consequence of separate WOT-continuity of the product in $B(\mathcal{K})$.

For the converse of the first claim, suppose that $(\xi_n a)$ is σ-convergent in $\mathfrak{A}$ for all $a \in \mathfrak{A}$. Then $(\xi_n a)$ is bounded for each $a \in \mathfrak{A}$, so it follows from the principle of uniform boundedness that (ξ_n) is bounded. We also have that $(\langle \xi_n a\zeta, \eta\rangle)$ converges for all $a \in \mathfrak{A}$ and $\zeta, \eta \in \mathcal{H}$. Since vectors of the form $a\zeta$ for $a \in \mathfrak{A}$ and $\zeta \in \mathcal{H}$ are total in $\mathcal{H}$, Lemma 2.2.8 gives that (ξ_n) is σ-convergent in $\mathfrak{A}$.

For the backward direction of the second claim, suppose (ξ_n), $\xi \in M(\mathfrak{A})$ such that $\xi_n a \xrightarrow{\sigma} \xi a$ for all $a \in \mathfrak{A}$. By the first claim, there is a $\xi' \in M(\mathfrak{A})$ such that $\xi_n \xrightarrow{WOT} \xi'$. Then $\xi a = \xi' a$ for all $a \in \mathfrak{A}$, which implies that $\xi = \xi'$ since $\mathfrak{A}$ is an essential ideal in $M(\mathfrak{A})$. Hence $\xi_n \xrightarrow{WOT} \xi$.

For the final statement, suppose $\mathscr{S}$ is a σ-convergence system on $M(\mathfrak{A})$ such that $(M(\mathfrak{A}), \mathscr{S})$ is a Σ^*-algebra containing $\mathfrak{A}$ as a Σ^*-subalgebra. Let $\pi : (M(\mathfrak{A}), \mathscr{S}) \hookrightarrow B(\mathcal{H})$ be a faithful Σ^*-representation. Then $\pi|_{\mathfrak{A}}$ is a faithful Σ^*-representation of $\mathfrak{A}$ since every nondegenerate representation of $M(\mathfrak{A})$ restricts to a nondegenerate representation of $\mathfrak{A}$. (Indeed, to see the latter claim, suppose $\varphi : M(\mathfrak{A}) \to B(\mathcal{H})$ is nondegenerate and (e_λ) is a cai for $\mathfrak{A}$. Then $\varphi(e_\lambda)\varphi(\eta)\zeta \to \varphi(\eta)\zeta$ for all $\eta \in M(\mathfrak{A})$ and $\zeta \in \mathcal{H}$. By Cohen's factorization theorem, $\{\varphi(\eta)\zeta : \eta \in M(\mathfrak{A}),\, \zeta \in \mathcal{H}\} = \mathcal{H}$, so the claim follows.) Then $\xi_n \xrightarrow{\mathscr{S}} \xi$ in $M(\mathfrak{A})$ if and only if $\pi(\xi_n) \xrightarrow{WOT} \pi(\xi)$ in $B(\mathcal{H})$, which by what we proved above is equivalent to: $\pi(\xi_n a) \xrightarrow{WOT} \pi(\xi a)$ in $B(\mathcal{H})$ for all $a \in \mathfrak{A}$. Since π restricts to a faithful Σ^*-representation on $\mathfrak{A}$, the latter occurs if and only if $\xi_n a \xrightarrow{\sigma} \xi a$ in $\mathfrak{A}$ for all $a \in \mathfrak{A}$, which by the above again is equivalent to $\xi_n \xrightarrow{WOT} \xi$ in $B(\mathcal{K})$. So every such σ-convergence system $\mathscr{S}$ on $M(\mathfrak{A})$ coincides with the one induced by the faithful Σ^*-representation $\mathfrak{A} \subseteq B(\mathcal{K})$. $\square$

The next lemma is a slight generalization of the previous lemma.

Lemma 4.3.8. *Let $\mathfrak{X}$ be a σ-full left Σ^*-module over a Σ^*-algebra $\mathfrak{A}$, give $M(\mathfrak{A})$ the Σ^*-algebra structure as in Lemma 4.3.7, and let (ξ_n) be a sequence in $M(\mathfrak{A})$. Then (ξ_n) is σ-convergent in $M(\mathfrak{A})$ if and only if $(\xi_n x)$ is ${}_{\mathfrak{A}}\sigma$-convergent for all $x \in \mathfrak{X}$. If $\xi \in M(\mathfrak{A})$, $\xi_n \xrightarrow{\sigma} \xi$ in $M(\mathfrak{A})$ if and only if $\xi_n x \xrightarrow{\mathfrak{A}\sigma} \xi x$ for all $x \in \mathfrak{X}$.*

Proof. Suppose (ξ_n) is σ-convergent in $M(\mathfrak{A})$. Since $\xi_n\langle x|y\rangle = \langle \xi_n x|y\rangle$ for $x, y \in \mathfrak{X}$, the sequence $(\langle \xi_n x|y\rangle)$ is σ-convergent in $\mathfrak{A}$ for all $x, y \in \mathfrak{X}$. By Proposition 3.1.4, $(\xi_n x)$ is ${}_{\mathfrak{A}}\sigma$-convergent for all $x \in \mathfrak{X}$.

Conversely, suppose $(\xi_n x)$ is ${}_{\mathfrak{A}}\sigma$-convergent for all $x \in \mathfrak{X}$, and fix a faithful Σ^*-representation $\mathfrak{A} \subseteq M(\mathfrak{A}) \subseteq B(\mathcal{K})$. A short calculation shows that $(\langle \xi_n j\zeta, \eta\rangle)$ converges for all $j \in \langle \mathfrak{X}|\mathfrak{X}\rangle$ and $\zeta, \eta \in \mathcal{K}$. The desired result follows by combining Lemma 4.3.6 and Lemma 2.2.8.

The final statement follows similarly. $\square$

Recall from Proposition 3.1.6 that if $\mathfrak{X}$ is a right Σ^*-module over $\mathfrak{B}$, then $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ is canonically a Σ^*-algebra in which $T_n \xrightarrow{\sigma} T$ if and only if $T_n(x) \xrightarrow{\sigma_{\mathfrak{B}}} T(x)$ for all $x \in \mathfrak{X}$. It is this Σ^*-module structure on $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ to which the results below refer.

Proposition 4.3.9. *If $\mathfrak{X}$ is a Σ^*-Morita equivalence bimodule between $\mathfrak{A}$ and $\mathfrak{B}$, then the canonical isometric $*$-homomorphism $\lambda : \mathfrak{A} \hookrightarrow \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ from Lemma 4.3.5 is a Σ^*-embedding.*

Proof. We first show that λ is σ-continuous. Suppose that $a_n \xrightarrow{\sigma} a$ in $\mathfrak{A}$. By Lemma 4.3.8, we have $\lambda(a_n)(x) = a_n x \xrightarrow{\mathfrak{A}\sigma} ax = \lambda(a)(x)$ for all $x \in \mathfrak{X}$, so that $\lambda(a_n)(x) \xrightarrow{\sigma_{\mathfrak{B}}} \lambda(a)(x)$ for all $x \in \mathfrak{X}$ by Definition 4.3.2 (3). Hence $\lambda(a_n) \xrightarrow{\sigma} \lambda(a)$ by the paragraph above the statement of the current proposition.

Now let (a_n) be a sequence in $\mathfrak{A}$ such that $\lambda(a_n) \xrightarrow{\sigma} T$ in $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$. Then $a_n x = \lambda(a_n)(x) \xrightarrow{\sigma_{\mathfrak{B}}} T(x)$, so $a_n x \xrightarrow{\mathfrak{A}\sigma} T(x)$ for all $x \in \mathfrak{X}$. By Lemma 4.3.8, (a_n) is σ-convergent to some $a \in \mathfrak{A}$. Thus λ is a Σ^*-embedding by Lemma 2.2.2. $\square$

Proposition 4.3.10. *If $\mathfrak{X}$ is an $\mathfrak{A} - \mathfrak{B}$ bimodule satisfying (1) and (2) in Definition 4.3.2 and such that the canonical isometric $*$-homomorphism $\lambda : \mathfrak{A} \hookrightarrow \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ from Lemma 4.3.5 is a Σ^*-embedding, then $\mathfrak{X}$ is a Σ^*-imprivitivity bimodule.*

Proof. We need only to check (3) in Definition 4.3.2. Suppose that $x_n \xrightarrow{\mathfrak{A}\sigma} x$ in $\mathfrak{X}$. Then for any $y, z \in \mathfrak{X}$,

$$y\langle x_n|z\rangle_{\mathfrak{B}} = {}_{\mathfrak{A}}\langle y|x_n\rangle z = \lambda({}_{\mathfrak{A}}\langle y|x_n\rangle)(z) \xrightarrow{\sigma_{\mathfrak{B}}} \lambda({}_{\mathfrak{A}}\langle y|x\rangle)(z) = {}_{\mathfrak{A}}\langle y|x\rangle z = y\langle x_n|z\rangle_{\mathfrak{B}}.$$

By the "right version" of Lemma 4.3.8, $\langle x_n|z\rangle_{\mathfrak{B}} \xrightarrow{\sigma} \langle x|z\rangle_{\mathfrak{B}}$ for all $z \in \mathfrak{X}$, i.e. $x_n \xrightarrow{\sigma_{\mathfrak{B}}} x$. The other direction of (3) in the definition of Σ^*-imprivitivity bimodules follows by symmetry. $\square$

Theorem 4.3.11. *Let $\mathfrak{X}$ be an $\mathfrak{A} - \mathfrak{B}$ bimodule satisfying (1) and (2) in Definition 4.3.2. Then $\mathfrak{X}$ is a Σ^*-imprivitivity bimodule if and only if the canonical isometric $*$-homomorphism $\lambda : \mathfrak{A} \hookrightarrow \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ from Lemma 4.3.5 is a Σ^*-embedding.*

In this case, $\mathfrak{A} \cong \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))$ Σ^*-isomorphically.

Proof. Proposition 4.3.9 does one direction of the first claim, and Proposition 4.3.10 does the other. For the final claim, note that for $x, y \in \mathfrak{X}$, $\lambda({}_{\mathfrak{A}}\langle x|y\rangle)$ coincides with $|x\rangle\langle y|$ in $\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})$. Since the spans of these two types of elements generate $\lambda(\mathfrak{A})$ and $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))$ as Σ^*-algebras respectively, we have that $\lambda(\mathfrak{A}) = \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))$. Hence the claim is proved since λ is a Σ^*-embedding by Proposition 4.3.9. $\square$

4.4 The Σ^*-module interior tensor product of Σ^*-imprivitivity bimodules

Let $\mathfrak{X}$ be a right Σ^*-module over $\mathfrak{B}$, let $\mathfrak{Y}$ be a right Σ^*-module over $\mathfrak{C}$, and suppose there is a σ-continuous $*$-homomorphism $\lambda : \mathfrak{B} \to \mathbb{B}_{\mathfrak{C}}(\mathfrak{Y})$ with σ-closed range (although the definition below works if λ is just a $*$-homomorphism). Recall the C^*-module interior tensor product $\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$ discussed in 4.2.2 above—this is a right C^*-module over $\mathfrak{C}$. In direct analogy with the W^*-module case (see [7, Section 3]), we define the Σ^*-*module interior tensor product* $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ to be the Σ^*-module completion (see 3.1.11–3.1.14) of $\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$. In symbols,

$$\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y} := \mathscr{B}(\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}),$$

where the σ-closure here is taken in $B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$ for any faithful Σ^*-representation $\mathfrak{C} \hookrightarrow B(\mathcal{H})$.

For the rest of this section, let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ be Σ^*-algebras, let $\mathfrak{C} \hookrightarrow B(\mathcal{H})$ be a faithful Σ^*-representation, let $\mathfrak{X}$ be a Σ^*-imprivitivity bimodule between $\mathfrak{A}$ and $\mathfrak{B}$, and let $\mathfrak{Y}$ be a Σ^*-imprivitivity bimodule between $\mathfrak{B}$ and $\mathfrak{C}$. By Proposition 4.3.9, there is a canonical Σ^*-embedding $\mathfrak{B} \hookrightarrow \mathbb{B}_{\mathfrak{C}}(\mathfrak{Y})$, so we may take the Σ^*-module interior tensor product $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$, which makes sense at least as a right Σ^*-module over $\mathfrak{C}$. The reader should keep in mind the inclusions

$$\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y} \subseteq \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y} = \mathscr{B}(\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \subseteq B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H}).$$

We now prove that $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ can be equipped with the structure of a left Σ^*-module over $\mathfrak{A}$ so that it becomes a Σ^*-imprivitivity bimodule between $\mathfrak{A}$ and $\mathfrak{C}$. The strategy is to show: (1) there is a canonical faithful Σ^*-representation $\pi : \mathfrak{A} \hookrightarrow B((\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$, so that $B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$ has a canonical left $\mathfrak{A}$-action; (2) $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ is an $\mathfrak{A}$-submodule of $B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$; (3) $\xi\eta^* \in \pi(\mathfrak{A})$ for all $\xi, \eta \in \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$; (4) the $\mathfrak{A}$-valued map $\langle \xi | \eta \rangle := \pi^{-1}(\xi\eta^*)$ makes $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ into a left Σ^*-module over $\mathfrak{A}$; (5) with this structure, $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ is a Σ^*-imprivitivity bimodule between $\mathfrak{A}$ and $\mathfrak{C}$.

Lemma 4.4.1. *With* $\mathcal{J} := \langle \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y} | \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y} \rangle_{\mathfrak{A}}$ *and* $\mathcal{I} := \langle \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y} | \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y} \rangle_{\mathfrak{C}}$, *we have* $\mathscr{B}(\mathcal{J}) = \mathfrak{A}$ *and* $\mathscr{B}(\mathcal{I}) = \mathfrak{C}$.

Proof. We prove $\mathscr{B}(\mathcal{J}) = \mathfrak{A}$; the other claim is similar. For $x, x' \in \mathfrak{X}$, and $y, y' \in \mathfrak{Y}$, recall the formula $\langle x \otimes y | x' \otimes y' \rangle_{\mathfrak{A}} = \langle x \langle y|y' \rangle_{\mathfrak{B}} | x' \rangle_{\mathfrak{A}}$. It follows easily from this that $\mathcal{J} = \langle \mathfrak{X} \langle \mathfrak{Y}|\mathfrak{Y} \rangle_{\mathfrak{B}} | \mathfrak{X} \rangle_{\mathfrak{A}}$, where by the latter we mean the span in $\mathfrak{A}$ of elements of the form $\langle x \langle y|y' \rangle_{\mathfrak{B}} x' \rangle_{\mathfrak{A}}$. With $x, x' \in \mathfrak{X}$ fixed, let $\mathscr{R} := \{b \in \mathfrak{B} : \langle xb|x' \rangle_{\mathfrak{A}} \in \mathscr{B}(\mathcal{J})\}$. By the fact just mentioned, we easily see that $\langle \mathfrak{Y}|\mathfrak{Y} \rangle_{\mathfrak{B}} \subseteq \mathscr{R}$, and it follows from Lemma 4.3.8 that $\mathscr{R}$ is σ-closed. Since $\mathfrak{Y}$ is σ-full over $\mathfrak{B}$, we have $\mathscr{R} = \mathfrak{B}$. Since by Cohen's factorization theorem ([9, A.6.2]) we can write any $x \in \mathfrak{X}$ as $x = x'b$ for some $x' \in \mathfrak{X}$, $b \in \mathfrak{B}$, it follows that $\langle \mathfrak{X}|\mathfrak{X} \rangle_{\mathfrak{A}} \subseteq \mathscr{B}(\mathcal{J})$. Thus

$$\mathfrak{A} = \mathscr{B}(\langle \mathfrak{X}|\mathfrak{X} \rangle_{\mathfrak{A}}) \subseteq \mathscr{B}(\mathcal{J}) \subseteq \mathfrak{A},$$

so that $\mathscr{B}(\mathcal{J}) = \mathfrak{A}$. $\square$

Lemma 4.4.2. *There is a canonical faithful Σ^*-representation $\pi : \mathfrak{A} \hookrightarrow B((\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$.*

Proof. By basic C^*-module theory, $\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$ is a left C^*-module over $\mathfrak{A}$ and right C^*-module over $\mathfrak{C}$, and a straightforward calculation (first checking on simple tensors) shows that $\langle \xi|\zeta \rangle_{\mathfrak{A}} \eta = \xi \langle \zeta|\eta \rangle_{\mathfrak{C}}$ for all $\xi, \zeta, \eta \in \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$. It follows as in the first few lines of Lemma 4.3.5 there is a canonical $*$-homomorphism $\pi : \mathfrak{A} \to \mathbb{B}_{\mathfrak{C}}(\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \subseteq B((\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$ To see that π is injective, hence isometric, suppose that $a \in \mathfrak{A}$ with $\pi(a) = 0$. Then

$$a \langle x \otimes y | x' \otimes y' \rangle_{\mathfrak{A}} = \langle a(x \otimes y) | x' \otimes y' \rangle_{\mathfrak{A}} = \langle ax \otimes y | x' \otimes y' \rangle_{\mathfrak{A}} = 0$$

for all $x, x' \in \mathfrak{X}$ and $y, y' \in \mathfrak{Y}$. By taking linear combinations, we have $aj = 0$ for all $j \in \mathcal{J} := \langle \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y} | \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y} \rangle_{\mathfrak{A}}$. The set $\mathscr{T} = \{b \in \mathfrak{A} : ab = 0\}$ is evidently σ-closed and contains $\mathcal{J}$, so $\mathscr{T} = \mathfrak{A}$ since $\mathscr{B}(\mathcal{J}) = \mathfrak{A}$ by Lemma 4.4.1. Hence $aa^* = 0$, so $a = 0$.

To see that π is σ-continuous, suppose $a_n \xrightarrow{\sigma} a$ in $\mathfrak{A}$. By Lemma 4.3.8, $a_n x \xrightarrow{\sigma_{\mathfrak{B}}} ax$ for all $x \in \mathfrak{X}$, hence by Lemma 4.3.8 again,

$$\langle a_n x | x' \rangle_{\mathfrak{B}} y' \xrightarrow{\sigma_{\mathfrak{C}}} \langle ax | x' \rangle_{\mathfrak{B}} y'$$

for all $x, x' \in \mathfrak{X}$ and $y' \in \mathfrak{Y}$. A couple of easy calculations then give that for any $x, x' \in \mathfrak{X}$, $y, y' \in \mathfrak{Y}$, and $k, k' \in \mathcal{H}$,

$$\begin{aligned} \langle \pi(a_n)((x \otimes y) \otimes k), (x' \otimes y') \otimes k' \rangle &= \langle k, \langle y | \langle a_n x | x' \rangle_{\mathfrak{B}} y' \rangle_{\mathfrak{C}} k' \rangle \\ \to \langle k, \langle y | \langle ax | x' \rangle_{\mathfrak{B}} y' \rangle_{\mathfrak{C}} k' \rangle &= \langle \pi(a)((x \otimes y) \otimes k), (x' \otimes y') \otimes k' \rangle. \end{aligned}$$

It is easy to check that elements of the form $(x \otimes y) \otimes k$ are total in $(\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H}$, so by Lemma 2.2.8 we get $\pi(a_n) \xrightarrow{WOT} \pi(a)$.

We omit these, but a couple of routine arguments using Lemma 4.3.6 and Lemma 2.2.8 show that π has WOT sequentially closed range and that π^{-1} is σ-continuous. $\square$

Lemma 4.4.3. *Let π be as in the previous lemma, and view $B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$ as a left $\mathfrak{A}$-module via the action $a \cdot T = \pi(a)T$. Then $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ is an $\mathfrak{A}$-submodule of $B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$.*

Proof. Fix $a \in \mathfrak{A}$. Then, as mentioned at the beginning of the proof of the previous lemma, $\pi(a)$ lies in the canonical copy of $\mathbb{B}_{\mathfrak{C}}(\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y})$ in $B((\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{B}} \mathcal{H})$, so it follows that for any η in $\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$, $\pi(a)\eta$ remains in $\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$. We now employ our usual trick: setting $\mathscr{R} = \{\eta \in \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y} : \pi(a)\eta \in \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}\}$, we have $\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y} \subseteq \mathscr{R}$, and the latter is WOT sequentially closed since $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ is. So $\mathscr{R} = \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$. $\square$

Lemma 4.4.4. *Let π be as in the previous lemma. If $\xi, \eta \in \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y} \subseteq B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$, then $\xi\eta^* \in \pi(\mathfrak{A})$.*

Proof. First, suppose that $\xi, \eta \in \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$, which as we mentioned at the beginning of the proof of Lemma 4.4.2 is a left C^*-module over $\mathfrak{A}$ and right C^*-module over $\mathfrak{C}$ satisfying the inner product relation $\langle \xi | \zeta \rangle_{\mathfrak{A}} \eta = \xi \langle \zeta | \eta \rangle_{\mathfrak{C}}$. Then for any $\zeta, \zeta' \in \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$ and $k, k' \in \mathcal{H}$, we have

$$\begin{aligned}
\langle \xi\eta^*(\zeta \otimes k), \zeta' \otimes k' \rangle &= \langle \eta^*(\zeta \otimes k), \xi^*(\zeta' \otimes k') \rangle \\
&= \langle \langle \eta | \zeta \rangle_{\mathfrak{C}} k, \langle \xi | \zeta' \rangle_{\mathfrak{C}} k' \rangle \\
&= \langle \xi \otimes \langle \eta | \zeta \rangle_{\mathfrak{C}} k, \zeta' \otimes k' \rangle \\
&= \langle \xi \langle \eta | \zeta \rangle_{\mathfrak{C}} \otimes k, \zeta' \otimes k' \rangle \\
&= \langle \langle \xi | \eta \rangle_{\mathfrak{A}} \zeta \otimes k, \zeta' \otimes k' \rangle \\
&= \langle \pi(\langle \xi | \eta \rangle_{\mathfrak{A}})(\zeta \otimes k), \zeta' \otimes k' \rangle.
\end{aligned}$$

Since the simple tensors are total, we conclude that $\xi\eta^* = \pi(\langle \xi | \eta \rangle_{\mathfrak{A}}) \in \pi(\mathfrak{A})$.

Keep $\xi \in \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$, and define $\mathscr{R} := \{\eta \in \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y} : \xi\eta^* \in \pi(\mathfrak{A})\}$ (recalling again that we are viewing $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ as the WOT sequential closure of $\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$ in $B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$). By what we just proved, $\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y} \subseteq \mathscr{R}$, and since $\pi(\mathfrak{A})$ is WOT sequentially closed by Lemma 4.4.2, it follows that $\mathscr{R}$ is WOT sequentially closed, so $\mathscr{R} = \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$.

A similar argument using this shows the full claim. $\square$

Theorem 4.4.5. *With $\mathfrak{A}$-module action $a \cdot \xi = \pi(a)\xi$ and $\mathfrak{A}$-valued inner product defined $\langle \xi|\eta\rangle_{\mathfrak{A}} := \pi^{-1}(\xi\eta^*)$, $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ is a Σ^*-imprivitivity bimodule between $\mathfrak{A}$ and $\mathfrak{C}$.*

Proof. We have already mentioned that $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ is a right Σ^*-module over $\mathfrak{C}$ by construction since it is a Σ^*-module completion.

We now describe why $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ is a left C^*-module over $\mathfrak{A}$. Most of the axioms defining a C^*-module tensor product are easily checked for $\langle\cdot|\cdot\rangle_{\mathfrak{A}}$ using the fact that π^{-1} is a homomorphism. Completeness of $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ in the norm induced by $\langle\cdot|\cdot\rangle_{\mathfrak{A}}$ follows since $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ is complete in the norm it inherits from $B((\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$, and since these two norms coincide:

$$\|\xi\|^2_{B((\mathfrak{X}\otimes_{\mathfrak{B}}\mathfrak{Y})\otimes_{\mathfrak{C}}\mathcal{H})} = \|\xi\xi^*\|_{B((\mathfrak{X}\otimes_{\mathfrak{B}}\mathfrak{Y})\otimes_{\mathfrak{C}}\mathcal{H})} = \|\pi^{-1}(\xi\xi^*)\|_{\mathfrak{A}} = \|\langle\xi|\xi\rangle_{\mathfrak{A}}\|$$

for $\xi \in \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$, using the fact that π^{-1} is isometric.

To see that $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ is a left Σ^*-module over $\mathfrak{A}$, suppose that (ξ_n) is a sequence in $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ such that $\langle \xi_n|\eta\rangle_{\mathfrak{A}}$ is σ-convergent for all $\eta \in \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$. Then $\xi_n\eta^* = \pi(\langle\xi_n|\eta\rangle)$ is WOT-convergent in $B((\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$ since π is a Σ^*-representation. By Lemma 4.4.1, $\mathscr{B}(\langle\mathfrak{X}\otimes_{\mathfrak{B}}\mathfrak{Y}|\mathfrak{X}\otimes_{\mathfrak{B}}\mathfrak{Y}\rangle_{\mathfrak{C}}\mathcal{H}) = \mathfrak{C}$. So by Lemma 4.3.6, $[(\mathfrak{X}\otimes_{\mathfrak{B}}\mathfrak{Y})^*((\mathfrak{X}\otimes_{\mathfrak{B}}\mathfrak{Y})\otimes_{\mathfrak{C}}\mathcal{H})] = [\langle\mathfrak{X}\otimes_{\mathfrak{B}}\mathfrak{Y}|\mathfrak{X}\otimes_{\mathfrak{B}}\mathfrak{Y}\rangle_{\mathfrak{C}}\mathcal{H}] = \mathcal{H}$, which implies that elements of the form η^*h, with $\eta \in \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$ and $h \in (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H}$, are total in $\mathcal{H}$. Since $\langle \xi_n\eta^*h, k\rangle$ converges for all $\eta \in \mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$ and $h, k \in (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H}$, we may conclude by Lemma 2.2.8 that (ξ_n) WOT-converges to some $\xi \in B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$. This ξ is in $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ since the latter is WOT sequentially closed, and we have

$$\langle\xi_n|\eta\rangle_{\mathfrak{A}} = \pi^{-1}(\xi_n\eta^*) \xrightarrow{\sigma} \pi^{-1}(\xi\eta^*) = \langle\xi|\eta\rangle_{\mathfrak{A}} \text{ for all } \eta \in \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$$

since π^{-1} is σ-continuous. Hence $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ is a Σ^*-module.

We now check condition (3) in Definition 4.3.2. We showed in the previous paragraph that if $\xi_n \xrightarrow{\mathfrak{A}\sigma} \xi$, then $\xi_n \xrightarrow{WOT} \xi$ in $B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$, but the latter is equivalent to $\xi_n \xrightarrow{\sigma\mathfrak{C}} \xi$. Conversely, if $\xi_n \xrightarrow{\sigma\mathfrak{C}} \xi$, then $\xi_n \eta^* \xrightarrow{WOT} \xi\eta^*$ in $B((\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$ for all $\eta \in B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}) \otimes_{\mathfrak{C}} \mathcal{H})$. In particular, since π^{-1} is σ-continuous, $\langle \xi_n | \eta \rangle_{\mathfrak{A}} = \pi^{-1}(\xi_n \eta^*) \xrightarrow{\sigma} \pi^{-1}(\xi\eta^*) = \langle \xi | \eta \rangle_{\mathfrak{A}}$ for all $\eta \in \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$. Hence $\xi_n \xrightarrow{\mathfrak{A}\sigma} \xi$.

Lemma 4.4.1 shows that $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ is Σ^*-full over both $\mathfrak{A}$ and $\mathfrak{B}$ (the former since the $\mathfrak{A}$-valued inner product on $\mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$ extends the one on $\mathfrak{X} \otimes_{\mathfrak{B}} \mathfrak{Y}$ by construction), so it only remains to check the inner product formula in Definition 4.3.2 (3). This is straightforward from the definitions of the $\mathfrak{A}$-module action and inner product:

$$\langle \xi | \eta \rangle_{\mathfrak{A}} \zeta = \pi^{-1}(\xi\eta^*) \cdot \zeta = (\xi\eta^*)\zeta = \xi(\eta^*\zeta) = \xi \langle \eta | \zeta \rangle_{\mathfrak{C}}$$

for $\xi, \eta, \zeta \in \mathfrak{X} \otimes_{\mathfrak{B}}^{\Sigma^*} \mathfrak{Y}$. □

Corollary 4.4.6. *Strong Σ^*-Morita equivalence is an equivalence relation strictly coarser than Σ^*-isomorphism.*

Proof. Reflexivity follows from (2) in Example 4.3.4.

Symmetry follows just as in C^*-Morita equivalence—if $\mathfrak{X}$ is an $\mathfrak{A}-\mathfrak{B}$ Σ^*-imprivitivity bimodule, then the adjoint C^*-module $\overline{\mathfrak{X}}$ (see [9, 8.1.1]) is easily checked to be a $\mathfrak{B}-\mathfrak{A}$ Σ^*-imprivitivity bimodule.

Transitivity was proved in Proposition 4.4.5.

The "coarser" claim was shown in Example 4.3.4 (2), and the "strictly" part may be seen from Example 4.3.4 (3). $\square$

4.5 Weak Σ^*-Morita equivalence

In his original paper [38] on the subject, Rieffel proved that two W^*-algebras are W^*-Morita equivalent if and only if their categories of normal Hilbert space representations are equivalent (the latter is actually his definition of Morita equivalence, see [38, Definition 7.4]; the result is [38, Theorem 7.9]). The corresponding statement for C^*-algebras is however not true, and so there are two different notions of C^*-algebraic Morita equivalence—the "strong" version we have already defined, and a "weak" version described below. After describing weak C^*-Morita equivalence, we define and study the Σ^*-version, which we call "weak Σ^*-Morita equivalence." This turns out (as in the C^*-case and differing from the W^*-case) to be strictly coarser than strong Σ^*-Morita equivalence.

Let A be a C^*-algebra. A *Hilbert A-module* is a Hilbert space H carrying an A-module structure via a nondegenerate $*$-homomorphism $A \to B(H)$. Let ${}_AHMOD$ be the category of Hilbert A-modules with morphisms the bounded A-module maps. For C^*-algebras A and B, a *$*$-functor* from ${}_AHMOD$ to ${}_BHMOD$ is a linear functor $F : {}_AHMOD \to {}_BHMOD$ such that $F(T^*) = F(T)^*$ whenever $T : H \to K$ is a morphism of Hilbert A-modules. Recall that two categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* if there are functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ with natural isomorphisms $FG \cong \mathrm{id}_{\mathscr{C}}$ and $GF \cong \mathrm{id}_{\mathscr{D}}$ (see [28, IV.4]).

Definition 4.5.1. Two C^*-algebras A and B are *weakly C^*-Morita equivalent* if ${}_AHMOD$ and ${}_BHMOD$ are equivalent via $*$-functors.

Note 4.5.2. Strongly C^*-Morita equivalent C^*-algebras are automatically weakly C^*-Morita equivalent ([37, Theorem 6.23]). As mentioned above though, these notions do not coincide—two weakly C^*-Morita equivalent C^*-algebras need not be strongly C^*-Morita equivalent. For example, $C([0,1])$ and $C(\mathbb{T})$ are weakly but not strongly C^*-Morita equivalent. In fact, two commutative C^*-algebras are strongly C^*-Morita equivalent if and only if they are $*$-isomorphic ([37, Corollary 6.27]), but any two separable commutative C^*-algebras whose spectra have the same cardinality are weakly C^*-Morita equivalent by [38, Proposition 8.18]. See also Beer's paper [4] for further interesting results about weak and strong C^*-Morita equivalence. A word of warning on the terminology: in Beer's and Rieffel's works, weak C^*-Morita equivalence is called simply "Morita equivalence."

We now investigate the Σ^*-analogue of weak C^*-Morita equivalence, showing among other things that weak Σ^*-Morita equivalence is an equivalence relation strictly coarser than strong Σ^*-Morita equivalence.

Let $\mathfrak{A}$ be a Σ^*-algebra. If H is a Hilbert space carrying an $\mathfrak{A}$-module structure via a Σ^*-representation $\mathfrak{A} \to B(H)$, say that H is a *Σ^*-Hilbert $\mathfrak{A}$-module*. Let ${}_{\mathfrak{A}}\Sigma^*HMOD$ be the category of Σ^*-Hilbert A-modules with morphisms the bounded $\mathfrak{A}$-module maps.

Definition 4.5.3. Two Σ^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are *weakly Σ^*-Morita equivalent* if ${}_{\mathfrak{A}}\Sigma^*HMOD$ and ${}_{\mathfrak{B}}\Sigma^*HMOD$ are equivalent via $*$-functors.

If A is a C^*-algebra, its *Davies-Baire envelope*, denoted $\Sigma(A)$, is the WOT sequential closure of A in its universal representation. (Note that $\Sigma(A)$ may be identified with the weak* sequential closure of A in A^{**}.)

Proposition 4.5.4. *If A and B are C^*-algebras, the following are equivalent:*

1. *A and B are weakly C^*-Morita equivalent;*

2. *$\Sigma(A)$ and $\Sigma(B)$ are weakly Σ^*-Morita equivalent;*

3. *A^{**} and B^{**} are W^*-Morita equivalent.*

Proof. Equivalence of the first two follows directly from the correspondences between representations of a C^*-algebra and Σ^*-representations of its Davies-Baire envelope (see [16], Theorem 3.1 and its proof). Equivalence of the first and last follows similarly ([38, Proposition 8.18]). $\square$

Lemma 4.5.5. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are Σ^*-algebras and $\mathfrak{X}$ is an $\mathfrak{A}-\mathfrak{B}$ Σ^*-imprivitivity bimodule. If $\mathcal{K}$ is a Σ^*-Hilbert $\mathfrak{A}$-module, then $\overline{\mathfrak{X}} \otimes_{\mathfrak{A}} \mathcal{K}$ is a Σ^*-Hilbert $\mathfrak{B}$-module. Similarly, if $\mathcal{H}$ is a Σ^*-Hilbert $\mathfrak{B}$-module, then $\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H}$ is a Σ^*-Hilbert $\mathfrak{A}$-module.*

Proof. Routine calculations show that if $\mathcal{K}$ is a Σ^*-Hilbert $\mathfrak{A}$-module, then there is a $*$-homomorphism $\pi : \mathfrak{B} \to B(\overline{\mathfrak{X}} \otimes_{\mathfrak{A}} \mathcal{K})$ determined by the formula

$$\pi(b)(\overline{x} \otimes \zeta) = \overline{xb^*} \otimes \zeta$$

on simple tensors. To see that π is σ-continuous, suppose $b_n \xrightarrow{\sigma} b$ in $\mathfrak{B}$. Then $(\pi(b_n))$ is a bounded sequence, and for any $x, y \in \mathfrak{X}$ and $\zeta, \eta \in \mathcal{K}$,

$$\langle \pi(b_n)(\overline{x} \otimes \zeta), \overline{y} \otimes \eta \rangle = \langle \zeta, {}_{\mathfrak{A}}\langle xb_n | y \rangle \eta \rangle \to \langle \zeta, {}_{\mathfrak{A}}\langle xb | y \rangle \eta \rangle = \langle \pi(b)(\overline{x} \otimes \zeta), \overline{y} \otimes \eta \rangle$$

since $xb_n \xrightarrow{\mathfrak{A}\sigma} xb$ by Lemma 4.3.8 and since $\mathfrak{X}$ is a Σ^*-imprivitivity bimodule. Hence $\pi(b_n) \xrightarrow{WOT} \pi(b)$ by Lemma 2.2.8.

The other claim follows similarly. $\square$

Proposition 4.5.6. *If $\mathfrak{A}$ and $\mathfrak{B}$ are strongly Σ^*-Morita equivalent Σ^*-algebras, then they are weakly Σ^*-Morita equivalent.*

Proof. Let $\mathfrak{X}$ be an $\mathfrak{A} - \mathfrak{B}$ Σ^*-imprivitivity bimodule. Define a functor

$$F : {}_{\mathfrak{A}}\Sigma^* HMOD \to {}_{\mathfrak{B}}\Sigma^* HMOD$$

by $F(\mathcal{K}) = \overline{\mathfrak{X}} \otimes_{\mathfrak{A}} \mathcal{K}$ for a Σ^*-Hilbert $\mathfrak{A}$-module $\mathcal{K}$ and $F(T) = \mathrm{id}_{\overline{\mathfrak{X}}} \otimes T$ for a bounded $\mathfrak{A}$-module map $T : \mathcal{K} \to \mathcal{K}'$ between Σ^*-Hilbert $\mathfrak{A}$-modules. (The existence of $\mathrm{id}_{\overline{\mathfrak{X}}} \otimes T$ follows from functoriality of the C^*-module interior tensor product—see [9, 8.2.12(1)].) Similarly, define a functor

$$G : {}_{\mathfrak{B}}\Sigma^* HMOD \to {}_{\mathfrak{A}}\Sigma^* HMOD$$

by $G(\mathcal{H}) = \mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H}$ and $G(S) = \mathrm{id}_{\mathfrak{X}} \otimes S$.

By Lemma 4.5.5, these functors do indeed map into the desired categories, and it is straightforward to check that they are $*$-functors. To see that FG is naturally isomorphic to $\mathrm{id}_{{}_{\mathfrak{B}}\Sigma^* HMOD}$, take a Σ^*-Hilbert $\mathfrak{B}$-module $\mathcal{H}$. Then by standard facts about the C^*-module interior tensor product (or see [9, 8.2.19]), we have canonical $\mathfrak{B}$-module isomorphisms

$$\begin{aligned} FG(\mathcal{H}) &= \overline{\mathfrak{X}} \otimes_{\mathfrak{A}} (\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H}) \\ &\cong (\overline{\mathfrak{X}} \otimes_{\mathfrak{A}} \mathfrak{X}) \otimes_{\mathfrak{B}} \mathcal{H} \end{aligned}$$

$$\cong {}_{\mathfrak{A}}\mathbb{K}(\mathfrak{X}) \otimes_{\mathfrak{B}} \mathcal{H}$$

$$\cong \langle \mathfrak{X}|\mathfrak{X}\rangle_{\mathfrak{B}} \otimes_{\mathfrak{B}} \mathcal{H}$$

$$\cong [\langle \mathfrak{X}|\mathfrak{X}\rangle_{\mathfrak{B}}\mathcal{H}]$$

$$= \mathcal{H},$$

where the last line holds by Lemma 4.3.6. (To see the fourth line, note that since $\mathfrak{X}$ is a $\langle \mathfrak{X}|\mathfrak{X}\rangle_{\mathfrak{A}} - \langle \mathfrak{X}|\mathfrak{X}\rangle_{\mathfrak{B}}$ C^*-imprivitivity bimodule, ${}_{\langle \mathfrak{X}|\mathfrak{X}\rangle_{\mathfrak{A}}}\mathbb{K}(\mathfrak{X}) = {}_{\mathfrak{A}}\mathbb{K}(\mathfrak{X})$ with its canonical C^*-algebra structure is canonically $*$-isomorphic to $\langle \mathfrak{X}|\mathfrak{X}\rangle_{\mathfrak{B}}$, and it is easy to check that this $*$-isomorphism is a $\mathfrak{B}$-$\mathfrak{B}$ bimodule map.)

The induced transformation from FG to $\mathrm{id}_{\mathfrak{B}\Sigma^*HMOD}$ is easily check to be a natural isomorphism. That GH is naturally isomorphic to $\mathrm{id}_{\mathfrak{A}\Sigma^*HMOD}$ may be checked similarly. $\square$

To show that weak Σ^*-Morita equivalence is strictly weaker than strong Σ^*-Morita equivalence, it suffices as in the C^*-case to look at commutative algebras, but one has to look a little deeper.

The proof of the following is essentially the same as one that works in the C^*-case (see [9], note on top of pg. 352).

Proposition 4.5.7. *Commutative Σ^*-algebras are strongly Σ^*-Morita equivalent if and only if they are Σ^*-isomorphic.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be commutative Σ^*-algebras, and let $\mathfrak{X}$ be an $\mathfrak{A} - \mathfrak{B}$ Σ^*-imprivitivity bimodule. We will show that there is a Σ^*-isomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ determined by the formula $ax = x\varphi(a)$ for $x \in \mathfrak{X}$, $a \in \mathfrak{A}$.

Consider $\mathfrak{X}$ as a left Σ^*-module over $M(\mathfrak{A})$ and a right Σ^*-module over $M(\mathfrak{B})$ (see discussion above Lemma 4.3.7). We claim that for any fixed $\eta \in M(\mathfrak{A})$, there is a unique $\zeta \in M(\mathfrak{B})$ such that $\eta x = x\zeta$ for all $x \in \mathfrak{X}$. A short calculation using commutativity of $M(\mathfrak{A})$ shows that the map $x \mapsto \eta x$ is in ${}_{\mathfrak{A}}\mathbb{B}(\mathfrak{X}) = M({}_{\mathfrak{A}}\mathbb{K}(\mathfrak{X})) \subseteq M(\mathscr{B}({}_{\mathfrak{A}}\mathbb{K}(\mathfrak{X})))$, so the canonical identification of the latter with $M(\mathfrak{B})$ from the left-module version of Theorem 4.3.11 gives existence. For uniqueness, suppose $\zeta_1, \zeta_2 \in M(\mathfrak{B})$ satisfy $x\zeta_1 = x\zeta_2$ for all $x \in \mathfrak{X}$. Fixing $b \in \mathfrak{B}$, we have $xb(\zeta_1 - \zeta_2) = 0$ for all $x \in \mathfrak{X}$, so that $\langle y|x\rangle b(\zeta_1 - \zeta_2) = 0$ for all $y, x \in \mathfrak{X}$. Hence $\mathscr{J} := \{c \in \mathfrak{B} : cb(\zeta_1 - \zeta_2) = 0\}$ is a WOT sequentially closed subset of $\mathfrak{B}$ containing $\langle \mathfrak{X}|\mathfrak{X}\rangle_{\mathfrak{B}}$, so $\mathscr{J} = \mathfrak{B}$ and $b(\zeta_1 - \zeta_2) = 0$. Thus $\mathfrak{B}(\zeta_1 - \zeta_2) = 0$, and since $\mathfrak{B}$ is an essential ideal in $M(\mathfrak{B})$, $\zeta_1 = \zeta_2$.

We have proved the existence of a map $\theta : M(\mathfrak{A}) \to M(\mathfrak{B})$ determined by the formula $\eta x = x\theta(\eta)$ for $x \in \mathfrak{X}$, $\eta \in M(\mathfrak{A})$. By symmetry, θ has an inverse, so θ is injective and surjective. It is straightforward to check that θ is in fact a $*$-isomorphism (part of this uses commutativity of $M(\mathfrak{A})$ again). To see that θ is a Σ^*-isomorphism, suppose that (η_n) is a sequence in $M(\mathfrak{A})$ with $\eta_n \xrightarrow{\sigma} \eta$. Lemma 4.3.8 shows that $x\theta(\eta_n) = \eta_n x \xrightarrow{{}_{\mathfrak{A}}\sigma} \eta x = x\theta(\eta)$ for all $x \in \mathfrak{X}$. Hence $x\theta(\eta_n) \xrightarrow{\sigma_{\mathfrak{B}}} x\theta(\eta)$ for all $x \in \mathfrak{X}$ since $\mathfrak{X}$ is a Σ^*-imprivitivity bimodule. By the "right version" of Lemma 4.3.8 again, we have $\theta(\eta_n) \xrightarrow{\sigma} \theta(\eta)$. By symmetry, θ^{-1} is also σ-continuous.

It remains to show that $\theta(\mathfrak{A}) = \mathfrak{B}$. (This part of the proof is essentially from the end of the proof of 8.6.5 in [9].) For $x, y \in \mathfrak{X}$, we have

$$\begin{aligned} {}_{\mathfrak{A}}\langle y|y\rangle^2 x &= y\langle y|y\rangle_{\mathfrak{B}}\langle y|x\rangle_{\mathfrak{B}} = y\langle y|x\rangle_{\mathfrak{B}}\langle y|y\rangle_{\mathfrak{B}} \\ &= {}_{\mathfrak{A}}\langle y|y\rangle_{\mathfrak{A}}\langle x|y\rangle y = {}_{\mathfrak{A}}\langle x|y\rangle_{\mathfrak{A}}\langle y|y\rangle y \end{aligned}$$

$$= x\langle y|y\rangle_{\mathfrak{B}}^2$$

by Definition 4.3.2 (2) and commutativity of $\mathfrak{A}$ and $\mathfrak{B}$. Hence

$$\theta({}_{\mathfrak{A}}\langle y|y\rangle)^2 = \theta({}_{\mathfrak{A}}\langle y|y\rangle^2) = \langle y|y\rangle_{\mathfrak{B}}^2,$$

so $\theta({}_{\mathfrak{A}}\langle y|y\rangle) = \langle y|y\rangle_{\mathfrak{B}}$. By polarization, we have $\theta({}_{\mathfrak{A}}\langle y|z\rangle) = \langle z|y\rangle_{\mathfrak{B}}$ for all $y, z \in \mathfrak{X}$, and it follows that $\theta({}_{\mathfrak{A}}\langle \mathfrak{X}|\mathfrak{X}\rangle) = \langle \mathfrak{X}|\mathfrak{X}\rangle_{\mathfrak{B}}$. Since θ is σ-continuous, $\theta(\mathfrak{A}) \subseteq \mathfrak{B}$. A symmetric argument shows that $\theta^{-1}(\mathfrak{B}) \subseteq \mathfrak{A}$. Hence $\theta(\mathfrak{A}) = \mathfrak{B}$ and $\varphi = \theta|_{\mathfrak{A}}$ is a Σ^*-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. $\square$

Example 4.5.8. Let L denote the long line (see [29, Section 24, Exercise 12]). By [4, pg. 257], $C([0,1])$ and $C_0(L)$ are weakly C^*-Morita equivalent. Hence by Proposition 4.5.4, Baire$([0,1])$ and Baire(L) are weakly Σ^*-Morita equivalent. On the other hand, L is not σ-compact (an exercise in basic topology shows that L is not even Lindelöf), so Baire(L) is non-unital by fact (b) at the end of Example 2.2.5 (4). Thus Baire$([0,1])$ and Baire(L) are not $*$-isomorphic. Hence they are not strongly Σ^*-Morita equivalent by Proposition 4.5.7.

4.6 The "full corners" characterization

Definition 4.6.1. Let $\mathfrak{C}$ be a Σ^*-algebra. Two C^*-subalgebras $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathfrak{C}$ are called *complementary σ-full corners of* $\mathfrak{C}$ if there is a projection $p \in M(\mathfrak{C})$ such that $p\mathfrak{C}p \cong \mathfrak{A}$ Σ^*-isomorphically, $p^\perp\mathfrak{C}p^\perp \cong \mathfrak{B}$ Σ^*-isomorphically, and $\mathscr{B}(\mathfrak{C}p\mathfrak{C}) = \mathfrak{C} = \mathscr{B}(\mathfrak{C}p^\perp\mathfrak{C})$.

Theorem 4.6.2. *Two Σ^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are strongly Σ^*-Morita equivalent if and only if they are Σ^*-isomorphic to complementary σ-full corners of a Σ^*-algebra $\mathfrak{C}$. If $p\mathfrak{C}p$ and $p^\perp\mathfrak{C}p^\perp$ are complementary σ-full corners in a Σ^*-algebra $\mathfrak{C}$, then $p\mathfrak{C}p^\perp$ is a $p\mathfrak{C}p - p^\perp\mathfrak{C}p^\perp$ Σ^*-imprivitivity bimodule.*

Proof. ($\Longrightarrow$) Let $\mathfrak{X}$ be an $\mathfrak{A} - \mathfrak{B}$ Σ^*-imprivitivity bimodule. By Proposition 3.1.9,

$$\mathcal{L}^{\mathscr{B}}(\mathfrak{X}) = \begin{bmatrix} \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) & \mathfrak{X} \\ \overline{\mathfrak{X}} & \mathfrak{B} \end{bmatrix}$$

is a Σ^*-algebra. With $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we have $\mathfrak{B} = p^\perp\mathcal{L}^{\mathscr{B}}(\mathfrak{X})p^\perp$ and $\mathfrak{A} \cong \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) = p\mathcal{L}^{\mathscr{B}}(\mathfrak{X})p$ Σ^*-isomorphically by the last statement in Theorem 4.3.11. It is easy to check using Cohen's factorization theorem and σ-fullness of $\mathfrak{X}$ that $\mathscr{B}(\mathcal{L}^{\mathscr{B}}(\mathfrak{X})p\mathcal{L}^{\mathscr{B}}(\mathfrak{X})) = \mathcal{L}^{\mathscr{B}}(\mathfrak{X})$ and $\mathscr{B}(\mathcal{L}^{\mathscr{B}}(\mathfrak{X})p^\perp\mathcal{L}^{\mathscr{B}}(\mathfrak{X})) = \mathcal{L}^{\mathscr{B}}(\mathfrak{X})$.

($\Longleftarrow$) Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are Σ^*-isomorphic to complementary σ-full corners of a Σ^*-algebra $\mathfrak{C}$, and set $\mathfrak{X} := p\mathfrak{C}p^\perp$. We will show that $\mathfrak{X}$ is a Σ^*-imprivitivity bimodule between $p\mathfrak{C}p$ and $p^\perp\mathfrak{C}p^\perp$ (thus also proving the final statement). As observed in Theorem 3.1.10 (2), for a faithful Σ^*-representation $\mathfrak{C} \hookrightarrow B(\mathcal{H})$, $\mathfrak{X}$ is a left (resp. right) Σ^*-module over the Σ^*-algebra $p\mathfrak{C}p \subseteq B(p\mathcal{H})$ (resp. $p^\perp\mathfrak{C}p^\perp \subseteq B(p^\perp\mathcal{H})$). To see that $\mathfrak{X}$ is σ-full over these, note that $\mathscr{B}_{p\mathcal{H}}(\langle\mathfrak{X}|\mathfrak{X}\rangle_{p\mathfrak{C}p}) = \mathscr{B}_{p\mathcal{H}}(p\mathfrak{C}p^\perp\mathfrak{C}p) = p\mathscr{B}_{\mathcal{H}}(\mathfrak{C}p^\perp\mathfrak{C})p = p\mathfrak{C}p$ and similarly $\mathscr{B}_{p^\perp\mathcal{H}}(\langle\mathfrak{X}|\mathfrak{X}\rangle_{p^\perp\mathfrak{C}p^\perp}) = p^\perp\mathfrak{C}p^\perp$ (using the easy fact that $\mathscr{B}_{p\mathcal{H}}(pAp) = p\mathscr{B}_{\mathcal{H}}(A)p$ for any C^*-algebra $A \subseteq B(\mathcal{H})$). So we have checked condition (1) in Definition 4.3.2. Condition (2) is obvious, so it remains to check condition (3), i.e. that for $(x_n), x \in \mathfrak{X}$, $x_n \xrightarrow{p\mathfrak{C}p\sigma} x$ if and only if $x_n \xrightarrow{\sigma_{p^\perp\mathfrak{C}p^\perp}} x$. Note first that $[p^\perp \mathfrak{C} p \mathcal{H}] = p^\perp \mathcal{H}$ and $[p \mathfrak{C} p^\perp \mathcal{H}] = p\mathcal{H}$. Indeed, since $\mathscr{B}(p^\perp \mathfrak{C} p \mathfrak{C}) = p^\perp \mathfrak{C}$, we have

$$[p^\perp \mathfrak{C} p \mathcal{H}] = [p^\perp \mathfrak{C} p \mathfrak{C} \mathcal{H}] = [p^\perp \mathfrak{C} \mathcal{H}] = p^\perp \mathcal{H}$$

by Lemma 4.3.6, and the other equation is proved similarly. The desired result then follows readily from straightforward calculations and Lemma 2.2.8. □

Note 4.6.3 (Rieffel subequivalence for Σ^*-algebras)**.** We will not provide the details, but by following [9, 8.2.24], replacing norm-closures with σ-closures, one may prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are strongly Σ^*-Morita equivalent via a Σ^*-imprivitivity bimodule $\mathfrak{X}$, then there are lattice isomorphisms between the following: (1) the σ-closed two-sided ideals of $\mathfrak{A}$, (2) the σ-closed two-sided ideals of $\mathfrak{B}$, (3) the ${}_{\mathfrak{A}}\sigma$-closed (equivalently, $\sigma_{\mathfrak{B}}$-closed) $\mathfrak{A} - \mathfrak{B}$ submodules of $\mathfrak{X}$, and (4) the σ-closed two-sided ideals of $\mathcal{L}^{\mathscr{B}}(\mathfrak{X})$.

Note 4.6.4 (The TRO picture)**.** Recall that a *ternary ring of operators* (*TRO* for short) is a closed subspace $X \subseteq B(\mathcal{H}, \mathcal{K})$ for Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, such that $xy^*z \in X$ for all $x, y, z \in X$. Just as Theorem 4.6.2 is a "corners picture" of strong Σ^*-Morita equivalence, there is also a "TRO picture" of the same. Namely, two Σ^*-algebra $\mathfrak{A}$ and $\mathfrak{B}$ are strongly Σ^*-Morita equivalent if and only if there is a WOT sequentially closed TRO $\mathfrak{X} \subseteq B(\mathcal{H}, \mathcal{K})$ for some $\mathcal{H}, \mathcal{K}$, such that $\mathscr{B}(\mathfrak{X}\mathfrak{X}^\star) \cong \mathfrak{A}$ and $\mathscr{B}(\mathfrak{X}^\star\mathfrak{X}) \cong \mathfrak{B}$ Σ^*-isomorphically. (This is similar to the "corner picture" and "TRO picture" of Σ^*-modules presented in Theorem 3.1.10.)

We will omit most of the details of the proof since they are direct and much in the same line as other proofs already given here. For the forward direction, one fixes a faithful Σ^*-representation $\mathfrak{B} \hookrightarrow B(\mathcal{H})$ and takes $\mathfrak{X}$ to be the image of a Σ^*-imprivitivity bimodule in $B(\mathcal{H}, \mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})$. The converse follows from the fact that if $\mathfrak{X}$ is a WOT sequentially closed TRO, then $\mathfrak{X}$ is a Σ^*-imprivitivity bimodule between $\mathscr{B}(\mathfrak{X}\mathfrak{X}^\star)$ and $\mathscr{B}(\mathfrak{X}^\star\mathfrak{X})$.

Let $\mathfrak{B} \subseteq B(\mathcal{H})$ be a Σ^*-algebra, let I be a cardinal number, and denote by $\mathbb{K}_I$ the C^*-algebra of compact operators on a Hilbert space of dimension I. Recalling that the C^*-algebra tensor product $\mathbb{K}_I \otimes \mathfrak{B}$ may be viewed as a certain space of infinite matrices over $\mathfrak{B}$ (see, e.g., [9, 1.2.26, 1.5.2]), we prefer to denote this space as $\mathbb{K}_I(\mathfrak{B})$, and we view it canonically as a concrete C^*-algebra in $B(\mathcal{H}^{(I)})$. In the corollary below, we take $\mathscr{B}(\mathbb{K}_I(\mathfrak{B}))$ to mean the WOT sequential closure of $\mathbb{K}_I(\mathfrak{B})$ in $B(\mathcal{H}^{(I)})$. (This is a special case of the obvious Σ^*-analogue of the spatial C^*-algebra tensor product, which is used below in the construction of the Σ^*-module exterior tensor product.)

Let $C_I(\mathfrak{B})$ denote the collection of operators in $B(\mathcal{H}, \mathcal{H}^{(I)})$ of the form $\zeta \mapsto (b_i(\zeta))_{i \in I}$ for $b_i \in \mathfrak{B}$, $\zeta \in \mathcal{H}$. This coincides with the notation in [9, 1.2.26], and it is shown there that $C_I(\mathfrak{B})$ is a norm-closed TRO in $B(\mathcal{H}, \mathcal{H}^{(I)})$. Denote by $C_I^\sigma(\mathfrak{B})$ the WOT sequential closure of $C_I(\mathfrak{B})$ in $B(\mathcal{H}, \mathcal{H}^{(I)})$.

Corollary 4.6.5. *The space $C_I^\sigma(\mathfrak{B})$ is a $\mathscr{B}(\mathbb{K}_I(\mathfrak{B})) - \mathfrak{B}$ Σ^*-imprivitivity bimodule.*

Proof. This follows immediately from Theorem 4.6.2 by noting that

$$\begin{bmatrix} \mathscr{B}(\mathbb{K}_I(\mathfrak{B})) & C_I^\sigma(\mathfrak{B}) \\ \overline{C_I^\sigma(\mathfrak{B})} & \mathfrak{B} \end{bmatrix}$$

is a Σ^*-algebra in $B(\mathcal{H}^{(I)} \oplus \mathcal{H})$ (indeed, it is the WOT sequential closure of the $\mathbb{K}_I(\mathfrak{B}) - \mathfrak{B}$ linking algebra of $C_I(\mathfrak{B})$ in $B(\mathcal{H}^{(I)} \oplus \mathcal{H})$) with complementary σ-full corners $\mathscr{B}(\mathbb{K}_I(\mathfrak{B}))$ and $\mathfrak{B}$. $\square$

The proposition below gives a generalization and simpler proof of Theorem 3.1.12, which gives existence of the Σ^*-module completion of a C^*-module X over a Σ^*-algebra (but it is not clear how one could deduce the facts about $\mathscr{B}(X)$ in Proposition 3.1.14 from this shorter route). For a right C^*-module X over a C^*-algebra B, denote by $\mathcal{L}$ the linking algebra of X, i.e. $\mathcal{L} := \begin{bmatrix} \mathbb{K}_B(X) & X \\ \overline{X} & B \end{bmatrix}$. It is well known that this is a C^*-algebra with the canonical product and involution coming from the module action and inner product on X, and that if B is represented faithfully and nondegenerately on a Hilbert space $\mathcal{H}$, then there is a canonical faithful corner-preserving representation of $\mathcal{L}$ on $B((X \otimes_B \mathcal{H}) \oplus \mathcal{H})$.

Proposition 4.6.6. *Let X be a right C^*-module over a nondegenerate C^*-subalgebra $B \subseteq B(\mathcal{H})$, and denote by $\mathscr{B}(X)$ the WOT sequential closure of X in $B(\mathcal{H}, X \otimes_B \mathcal{H})$. Then $\mathscr{B}(X)$ is a Σ^*-module over $\mathscr{B}(B)$ with inner product and module action extending those of X. Additionally, if S is a $\sigma_{\mathscr{B}(B)}$-closed subset of $\mathscr{B}(X)$ with $X \subseteq S$, then $S = \mathscr{B}(X)$.*

Proof. As mentioned above, $\mathcal{L}$ may be viewed as a C^*-subalgebra of $B((X \otimes_B \mathcal{H}) \oplus \mathcal{H})$. Taking the WOT sequential closure there, we obtain a concrete Σ^*-algebra $\mathscr{B}(\mathcal{L})$, and it is elementary to argue that $\mathscr{B}(\mathcal{L})$ may be identified with the space of 2×2

matrices

$$\begin{bmatrix} \mathscr{B}(\mathbb{K}_B(X)) & \mathscr{B}(X) \\ \mathscr{B}(\overline{X}) & \mathscr{B}(B) \end{bmatrix},$$

where all these WOT sequential closures are taken with respect to the appropriate Hilbert space representations (for example, $\mathscr{B}(\overline{X})$ is the WOT sequential closure of $\overline{X}$ in $B(X \otimes_B \mathcal{H}, \mathcal{H})$). By Theorem 3.1.10 (2), $\mathscr{B}(X)$ is a Σ^*-module over $\mathscr{B}(B)$ with inner product and module action derived from the multiplication in $\mathscr{B}(\mathcal{L})$.

An easy argument using the algebra structure of $\mathscr{B}(\mathcal{L})$ shows that a subset $S \subseteq \mathscr{B}(X)$ is $\sigma_{\mathscr{B}(B)}$-closed in $\mathscr{B}(X)$ if and only if S is WOT sequentially closed as a subset of $B(\mathcal{H}, X \otimes_B \mathcal{H})$, which proves the "additionally" claim. $\square$

Now let X be a right C^*-module over a C^*-algebra B. As shown in [9, Proposition 8.5.17], X^{**} admits a canonical left B^{**}-action and B^{**}-valued inner product under which it becomes a right W^*-module over the W^*-algebra B^{**}, and if additionally X is an $A-B$ C^*-imprivitivity bimodule, then X^{**} is an $A^{**} - B^{**}$ W^*-imprivitivity bimodule.

Extending the definition of the Davies-Baire envelope of a C^*-algebra (see above Proposition 4.5.4), denote by $\Sigma(X)$ the weak* sequential closure of X in X^{**}. As another application of Theorem 4.6.2, we prove in the proposition below a Σ^*-analogue of the result in the last paragraph, which essentially just notes that the structure X^{**} has as a W^*-module over B^{**} "restricts" to $\Sigma(X)$ to make it a Σ^*-module over $\Sigma(B)$.

Proposition 4.6.7. *If X is a right C^*-module over a C^*-algebra B, then $\Sigma(X)$*

is canonically a right Σ^*-module over $\Sigma(B)$ whose action is the restriction of the canonical one of B^{**} on X^{**}. If X is an $A-B$ C^*-imprivitivity bimodule between A and B, then $\Sigma(X)$ is a $\Sigma(A)-\Sigma(B)$ Σ^*-imprivitivity bimodule.

Proof. For the first statement, let $\mathcal{L} := \begin{bmatrix} \mathbb{K}_B(X) & X \\ \overline{X} & B \end{bmatrix}$. By basic functional analysis, $\mathcal{L}^{**} = \begin{bmatrix} \mathbb{K}_B(X)^{**} & X^{**} \\ \overline{X}^{**} & B^{**} \end{bmatrix}$, and by the proof of [9, 8.5.17], the action of B^{**} on X^{**} induced by $\mathcal{L}^{**}$ coincides with the canonical action. Routine arguments show that $\Sigma(\mathcal{L}) = \begin{bmatrix} \Sigma(\mathbb{K}_B(X)) & \Sigma(X) \\ \Sigma(\overline{X}) & \Sigma(B) \end{bmatrix}$, so the result follows from Theorem 3.1.10 (2).

The final statement in the proposition follows quickly from a similar argument and Theorem 4.6.2. Indeed, if $\mathcal{L} := \begin{bmatrix} A & X \\ \overline{X} & B \end{bmatrix}$, then $\Sigma(\mathcal{L}) = \begin{bmatrix} \Sigma(A) & \Sigma(X) \\ \Sigma(\overline{X}) & \Sigma(B) \end{bmatrix}$, and σ-fullness of the 1-1 and 2-2 corners holds since $\mathscr{B}({}_{\Sigma(A)}\langle \Sigma(X)|\Sigma(X)\rangle) \supseteq \overline{{}_A\langle X|X\rangle} = A$ and $\mathscr{B}(\langle \Sigma(X)|\Sigma(X)\rangle_{\Sigma(B)}) \supseteq \overline{\langle X|X\rangle_B} = B$. $\square$

4.7 The Σ^*-module exterior tensor product

For a right C^*-module X over a C^*-algebra $B \subseteq B(\mathcal{H})$, let $\mathscr{B}(X)$ be the WOT sequential closure of X in $B(\mathcal{H}, X \otimes_B \mathcal{H})$, so that by Proposition 4.6.6, $\mathscr{B}(X)$ is a Σ^*-algebra over $\mathscr{B}(B)$. Hence we may also view $\mathscr{B}(X)$ in $B(\mathcal{H}, \mathscr{B}(X) \otimes_{\mathscr{B}(B)} \mathcal{H})$, but as the next lemma shows, this is really no different from viewing $\mathscr{B}(X)$ in $B(\mathcal{H}, X \otimes_B \mathcal{H})$.

Lemma 4.7.1. *If X is a right C^*-module over a C^*-algebra $B \subseteq B(\mathcal{H})$, then there is a canonical unitary $U : X \otimes_B \mathcal{H} \to \mathscr{B}(X) \otimes_{\mathscr{B}(B)} \mathcal{H}$. The spatial isomorphism between $B(\mathcal{H}, X \otimes_B \mathcal{H})$ and $B(\mathcal{H}, \mathscr{B}(X) \otimes_{\mathscr{B}(B)} \mathcal{H})$ induced by U restricts to the identity between the canonical copies of $\mathscr{B}(X)$.*

Proof. Standard arguments show that there is an isometry $X \otimes_B \mathcal{H} \to \mathscr{B}(X) \otimes_{\mathscr{B}(B)} \mathcal{H}$ determined by the rule $x \otimes \zeta \mapsto x \otimes \zeta$ on simple tensors. A calculation using the WOT sequential closure of the linking algebra of X shows that

$$\langle \xi_1(h_1), \xi_2(h_2) \rangle_{X \otimes_B \mathcal{H}} = \langle h_1, \langle \xi_1 | \xi_2 \rangle_{\mathscr{B}(B)} h_2 \rangle_{\mathcal{H}}$$

for $\xi_1, \xi_2 \in \mathscr{B}(X)$ and $h_1, h_2 \in \mathcal{H}$. It follows that the rule $\xi \otimes \eta \mapsto \xi(\eta)$, for $\xi \in \mathscr{B}(X) \subseteq B(\mathcal{H}, X \otimes_B \mathcal{H})$ and $\eta \in \mathcal{H}$, determines an isometry $\mathscr{B}(X) \otimes_{\mathscr{B}(B)} \mathcal{H} \to X \otimes_B \mathcal{H}$, which is easily seen to be the inverse of U.

The last statement is easy to check. □

Corollary 4.7.2. *The set $\{x \otimes \zeta : x \in X, \zeta \in \mathcal{H}\}$ is total in $\mathscr{B}(X) \otimes_{\mathscr{B}(B)} \mathcal{H}$.*

We now work out some of the details of the Σ^*-analogue of the *exterior tensor product* of C^*-modules (see 4.2.3), which we will then apply to prove a fact needed in the proof of our Σ^*-analogue of the Brown-Green-Rieffel stable isomorphism theorem (mimicking the route taken to prove the stable isomorphism theorem in [25] and [36]). To this end, let $\mathfrak{A} \subseteq B(\mathcal{K})$ and $\mathfrak{B} \subseteq B(\mathcal{H})$ be concrete Σ^*-algebras (although the definition works fine if $\mathfrak{A}$ and $\mathfrak{B}$ are merely C^*-algebras); let $\mathfrak{A} \otimes \mathfrak{B} \subseteq B(\mathcal{K} \otimes^2 \mathcal{H})$ denote the spatial C^*-algebra tensor product of $\mathfrak{A}$ and $\mathfrak{B}$; and let $\mathfrak{A} \otimes^{\Sigma^*} \mathfrak{B}$ denote the σ-closure of $\mathfrak{A} \otimes \mathfrak{B}$ in $B(\mathcal{K} \otimes^2 \mathcal{H})$, that is,

$$\mathfrak{A} \otimes^{\Sigma^*} \mathfrak{B} := \mathscr{B}_{\mathcal{K} \otimes^2 \mathcal{H}}(\mathfrak{A} \otimes \mathfrak{B}).$$

Call $\mathfrak{A} \otimes^{\Sigma^*} \mathfrak{B}$ the Σ^*-*spatial tensor product of* $\mathfrak{A}$ *and* $\mathfrak{B}$. (This is different from Dang's Σ^*-tensor product from [15, B.III], which is the Σ^*-analogue of the maximal C^*-algebra tensor product.)

For the remainder of this section, take $\mathfrak{X}$ and $\mathfrak{Y}$ to be right Σ^*-modules over concrete Σ^*-algebras $\mathfrak{A} \subseteq B(\mathcal{K})$ and $\mathfrak{B} \subseteq B(\mathcal{H})$ respectively. Let $\mathfrak{X} \otimes \mathfrak{Y}$ denote the C^*-module exterior tensor product of $\mathfrak{X}$ and $\mathfrak{Y}$ (see 4.2.3), which is a right C^*-module over $\mathfrak{A} \otimes \mathfrak{B} \subseteq B(\mathcal{K} \otimes^2 \mathcal{H})$. Denote by $\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}$ the Σ^*-module completion of $\mathfrak{X} \otimes \mathfrak{Y}$ from 3.1.11–3.1.14, i.e.

$$\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y} := \mathscr{B}(\mathfrak{X} \otimes \mathfrak{Y}),$$

where the σ-closure is taken in $B(\mathcal{K} \otimes^2 \mathcal{H}, (\mathfrak{X} \otimes \mathfrak{Y}) \otimes_{\mathfrak{A} \otimes \mathfrak{B}} (\mathcal{K} \otimes^2 \mathcal{H}))$. This is called the Σ^*-*module exterior tensor product of* $\mathfrak{X}$ *and* $\mathfrak{Y}$.

Lemma 4.7.3. (1) *If* $(\xi_n), \xi \in \mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}$ *are such that* $\xi_n \xrightarrow{\sigma_{\mathfrak{A} \otimes^{\Sigma^*} \mathfrak{B}}} \xi$*, then* $|\xi_n\rangle\langle\gamma| \xrightarrow{\sigma} |\xi\rangle\langle\gamma|$ *in* $\mathscr{B}(\mathbb{K}_{\mathfrak{A} \otimes^{\Sigma^*} \mathfrak{B}}(\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}))$ *for all* $\gamma \in \mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}$.

(2) *If* $S \in \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y}))$ *and* $T_n \xrightarrow{\sigma} T$ *in* $\mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X}))$*, then* $T_n \otimes S \xrightarrow{\sigma} T \otimes S$ *in* $\mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X})) \otimes^{\Sigma^*} \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y}))$.

Proof. (1) follows by a direct application of Lemma 3.1.7.

For (2), recall that $\mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X}))$ and $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y}))$ can be viewed as concrete Σ^*-algebras in $B(\mathfrak{X} \otimes_{\mathfrak{A}} \mathcal{H})$ and $B(\mathfrak{Y} \otimes_{\mathfrak{B}} \mathcal{K})$, so that $\mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X})) \otimes^{\Sigma^*} \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y}))$ is a concrete Σ^*-algebra in $B((\mathfrak{X}\otimes_{\mathfrak{A}}\mathcal{K})\otimes^2(\mathfrak{Y}\otimes_{\mathfrak{B}}\mathcal{H}))$. It thus suffices to check convergence on tensors of the form $(x \otimes \zeta) \otimes (y \otimes \eta)$ since $(T_n \otimes S)$ is bounded and tensors of this form are total in $(\mathfrak{X} \otimes_{\mathfrak{A}} \mathcal{K}) \otimes^2 (\mathfrak{Y} \otimes_{\mathfrak{B}} \mathcal{H})$ by elementary arguments. The required calculation is straightforward. □

Lemma 4.7.4. *The subspaces*

$$\operatorname{span}\{|x_1 \otimes y_1\rangle\langle x_2 \otimes y_2| : x_1, x_2 \in \mathfrak{X};\ y_1, y_2 \in \mathfrak{Y}\}$$

and

$$\operatorname{span}\{|x_1\rangle\langle x_2| \otimes |y_1\rangle\langle y_2| : x_1, x_2 \in \mathfrak{X};\ y_1, y_2 \in \mathfrak{Y}\}$$

are σ-dense in $\mathscr{B}(\mathbb{K}_{\mathfrak{A}\otimes^{\Sigma^*}\mathfrak{B}}(\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}))$ and $\mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X})) \otimes^{\Sigma^*} \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y}))$ respectively.

Proof. Call the displayed sets V_1 and V_2 respectively. These are $*$-algebras, so that $\mathscr{B}(V_1)$ and $\mathscr{B}(V_2)$ are Σ^*-algebras. We have the evident inclusions

$$\mathscr{B}(V_1) \subseteq \mathscr{B}(\mathbb{K}_{\mathfrak{A}\otimes^{\Sigma^*}\mathfrak{B}}(\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}))$$

$$\mathscr{B}(V_2) \subseteq \mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X})) \otimes^{\Sigma^*} \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y})).$$

So to prove the lemma, it suffices to show $\mathbb{K}_{\mathfrak{A}\otimes^{\Sigma^*}\mathfrak{B}}(\mathfrak{X}\otimes^{\Sigma^*}\mathfrak{Y}) \subseteq \mathscr{B}(V_1)$ and $\mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X}))\otimes \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y})) \subseteq \mathscr{B}(V_2)$, which follow if we can show

$$|\xi\rangle\langle\gamma| \in \mathscr{B}(V_1) \text{ for all } \xi, \gamma \in \mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y} \tag{4.1}$$

and

$$T \otimes S \in \mathscr{B}(V_2) \text{ for all } S \in \mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X})) \text{ and } T \in \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y})). \tag{4.2}$$

For (1), fix $\gamma \in \mathfrak{X} \otimes \mathfrak{Y}$, and let $\mathscr{S} = \{\xi \in \mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y} : |\xi\rangle\langle\gamma| \in \mathscr{B}(V_1)\}$. Clearly $\mathfrak{X} \otimes \mathfrak{Y} \subseteq \mathscr{S}$, and it follows by Lemma 4.7.3 (1) that $\mathscr{S}$ is $\sigma_{\mathfrak{A}\otimes^{\Sigma^*}\mathfrak{B}}$-closed in $\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}$. By the "additionally" statement in Proposition 4.6.6, we get that $\mathscr{S} = \mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}$. A similar argument using this proves (1).

A similar argument to that in the first paragraph, using the second part of Lemma 4.7.3, gives (2) here. $\square$

Theorem 4.7.5 (cf. [25], end of page 37)**.** *There is a canonical Σ^*-isomorphism*

$$\mathscr{B}(\mathbb{K}_{\mathfrak{A}\otimes^{\Sigma^*}\mathfrak{B}}(\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y})) \cong \mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X})) \otimes^{\Sigma^*} \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y})).$$

Proof. Recall as above that the two Σ^*-algebras in the conclusion of the claim can be viewed as concrete Σ^*-algebras as follows:

$$\mathscr{B}(\mathbb{K}_{\mathfrak{A}\otimes^{\Sigma^*}\mathfrak{B}}(\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y})) \subseteq B((\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}) \otimes_{\mathfrak{A}\otimes^{\Sigma^*}\mathfrak{B}} (\mathcal{K} \otimes^2 \mathcal{H})),$$
$$\mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X})) \otimes^{\Sigma^*} \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y})) \subseteq B((\mathfrak{X} \otimes_{\mathfrak{A}} \mathcal{K}) \otimes^2 (\mathfrak{Y} \otimes_{\mathfrak{B}} \mathcal{H})).$$

For convenience, label the Hilbert spaces $\mathcal{M} := (\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}) \otimes_{\mathfrak{A}\otimes^{\Sigma^*}\mathfrak{B}} (\mathcal{K} \otimes^2 \mathcal{H})$ and $\mathcal{N} := (\mathfrak{X} \otimes_{\mathfrak{A}} \mathcal{K}) \otimes^2 (\mathfrak{Y} \otimes_{\mathfrak{B}} \mathcal{H})$, and label the Σ^*-modules $\mathfrak{M} := \mathscr{B}(\mathbb{K}_{\mathfrak{A}\otimes^{\Sigma^*}\mathfrak{B}}(\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}))$ and $\mathfrak{N} := \mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X})) \otimes^{\Sigma^*} \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y}))$. We will show that there is a canonical unitary $U : \mathcal{M} \to \mathcal{N}$ which implements a spatial isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$. To see the existence of such a unitary, first define U on simple tensors of simple tensors in $\mathcal{M}$ by

$$U((x \otimes y) \otimes (\zeta \otimes \eta)) = (x \otimes \zeta) \otimes (y \otimes \eta)$$

for $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$, $\zeta \in \mathcal{K}$, and $\eta \in \mathcal{H}$. It is then straightforward using the definitions of the inner products in these various C^*-modules and Hilbert spaces to see that U

extends to an isometry on the span of elements of the form $(x \otimes y) \otimes (\zeta \otimes \eta)$ in $\mathcal{M}$. Since the set of elements of the form $(x \otimes y) \otimes (\zeta \otimes \eta)$ (resp. $(x \otimes \zeta) \otimes (y \otimes \eta)$) is total in $\mathcal{M}$ (resp. $\mathcal{N}$) by Corollary 4.7.2 (resp. by elementary principles), it follows that U extends to a unitary $U : \mathcal{M} \to \mathcal{N}$.

Let $\varphi : B(\mathcal{M}) \to B(\mathcal{N})$ denote the isomorphism $m \mapsto UmU^*$. A straightforward calculation gives the formula

$$\varphi(|x_1 \otimes y_1\rangle\langle x_2 \otimes y_2|) = |x_1\rangle\langle x_2| \otimes |y_1\rangle\langle y_2|$$

for all $x_1, x_2 \in \mathfrak{X}$ and $y_1, y_2 \in \mathfrak{Y}$, showing that φ restricts to an isomorphism from

$$\operatorname{span}\{|x_1 \otimes y_1\rangle\langle x_2 \otimes y_2| : x_1, x_2 \in \mathfrak{X};\ y_1, y_2 \in \mathfrak{Y}\}$$

onto

$$\operatorname{span}\{|x_1\rangle\langle x_2| \otimes |y_1\rangle\langle y_2| : x_1, x_2 \in \mathfrak{X};\ y_1, y_2 \in \mathfrak{Y}\}.$$

It is easy to check that if C is a C^*-subalgebra of $B(\mathcal{H}_1)$ and $V : \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary implementing an isomorphism $\psi : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$, then $\psi(\mathscr{B}(C)) = \mathscr{B}(\psi(C))$. Applying this fact to φ and using Lemma 4.7.4, we get that φ restricts to a Σ^*-isomorphism from $\mathscr{B}(\mathbb{K}_{\mathfrak{A} \otimes^{\Sigma^*} \mathfrak{B}}(\mathfrak{X} \otimes^{\Sigma^*} \mathfrak{Y}))$ onto $\mathscr{B}(\mathbb{K}_{\mathfrak{A}}(\mathfrak{X})) \otimes^{\Sigma^*} \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{Y}))$. $\square$

4.8 A Brown-Green-Rieffel stable isomorphism theorem for Σ^*-modules

We conclude by proving a Σ^*-analogue of the Brown-Green-Rieffel stable isomorphism theorem, which asserts that σ-unital C^*-algebras are strongly C^*-Morita equivalent if and only if they are stably isomorphic. (See any of the texts mentioned in Section 2 for a proof, and see [9, 8.5.31] for the W^*-version.) Unfortunately, we have not been able to find a condition on the algebras in our setting as elegant as the condition of being σ-unital in the C^*-setting (the basic problem is that we need a condition on the algebras guaranteeing that some Σ^*-imprivitivity bimodule is Σ^*-countably generated on both sides). Our method of proof is a translation to the Σ^*-setting of the proofs given in [9, 8.2.7] (for C^*-algebras) and [9, 8.5.31] (for W^*-algebras). We first recall and rephrase some definitions and results from Section 3.2 that we will need.

Definition 4.8.1 (cf. Definition 3.2.1)**.** A right (resp. left) Σ^*-module $\mathfrak{X}$ over a C^*-algebra $\mathfrak{A} \subseteq B(\mathcal{H})$ is $\Sigma^*_{\mathfrak{A}}$-*countably generated* (resp. ${}_{\mathfrak{A}}\Sigma^*$-countably generated) if there is a countable set $\{x_i\}_{i=1}^{\infty} \subseteq \mathfrak{X}$ such that $\{\sum_{i=1}^{N} x_i a_i : a_i \in \mathfrak{A}, N \in \mathbb{N}\}$ (resp. $\{\sum_{i=1}^{N} a_i x_i : a_i \in \mathfrak{A}, N \in \mathbb{N}\}$) is WOT sequentially dense in $\mathfrak{X} \subseteq B(\mathcal{H}, \mathfrak{X} \otimes_A \mathcal{H})$ (resp. in $\mathfrak{X} \subseteq B(\overline{\mathfrak{X}} \otimes_A \mathcal{H}, \mathcal{H})$).

Definition 4.8.2 (cf. Lemma 3.2.14 and above)**.** If $\mathfrak{X}$ is a Σ^*-module over $\mathfrak{B}$, denote by $C^w(\mathfrak{X})$ the Σ^*-module with underlying vector space

$$\{(x_n) \in \prod_{n \in \mathbb{N}} \mathfrak{X} : \sum_n \langle x_n | x_n \rangle \text{ is } \sigma\text{-convergent in } \mathfrak{B}\},$$

inner product $\langle (x_n)|(y_n)\rangle := \sum_n \langle x_n|y_n\rangle$, and entrywise $\mathfrak{B}$-module action. It is not hard to see from the definitions that if $\mathfrak{X} = \mathfrak{B}$, this coincides with $C^\sigma_{\mathbb{N}}(\mathfrak{B})$ from above Corollary 4.6.5 (more precisely, there is a canonical unitary between $C^w(\mathfrak{B})$ and $C^\sigma_{\mathbb{N}}(\mathfrak{B})$ induced by the canonical unitary between $\mathcal{H}^{(\mathbb{N})}$ and $C^w(\mathfrak{B}) \otimes_{\mathfrak{B}} \mathcal{H}$).

Definition 4.8.3 (cf. Definition 3.2.23)**.** For a left Σ^*-module $\mathfrak{X}$ over a Σ^*-algebra $\mathfrak{C}$, a countable subset $\{x_k\}$ of $\mathfrak{X}$ is a called a *left weak quasibasis* for $\mathfrak{X}$ if for any $x \in \mathfrak{X}$, the sequence of finite sums $(\sum_{k=1}^n \langle x|x_k\rangle x_k)_n$ ${}_{\mathfrak{C}}\sigma$-converges to x. A similar definition holds for *right weak quasibases.*

Proposition 4.8.4 (cf. Proposition 3.2.7)**.** *Let $\mathfrak{X}$ be a right Σ^*-module over a concrete Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$. Then $\mathfrak{X}$ is $\Sigma^*_{\mathfrak{B}}$-countably generated if and only if there is a sequence (e_n) in $\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})$ such that $e_n \xrightarrow{\sigma} I$ in $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ (recall the Σ^*-algebra structure on $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$ mentioned above Proposition 4.3.9).*

Proposition 4.8.5 (cf. Proposition 3.2.21)**.** *Let $\mathfrak{X}$ be a submodule of a Σ^*-module $\mathfrak{Y}$ over a Σ^*-algebra $\mathfrak{B}$. If $\mathfrak{X}$ meets the following requirements:*

(1) *$\mathfrak{X}$ is $\sigma_{\mathfrak{B}}$-closed in $\mathfrak{Y}$;*

(2) *$\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) = \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$;*

then $\mathfrak{X}$ is orthogonally complemented in $\mathfrak{Y}$.

Theorem 4.8.6 (cf. Theorem 3.2.26)**.** *Let $\mathfrak{C} \subseteq B(\mathcal{H})$ be a unital Σ^*-algebra, and let $\mathfrak{X}$ be a left Σ^*-module over $\mathfrak{C}$. Then $\mathfrak{X}$ is ${}_{\mathfrak{C}}\Sigma^*$-countably generated if and only if $\mathfrak{X}$ has a left weak quasibasis.*

Proposition 4.8.7. *If $\mathfrak{X}$ is a Σ^*-module over a Σ^*-algebra $\mathfrak{B}$, then there is a canonical unitary between $C^w(\mathfrak{X})$ and the Σ^*-module exterior tensor product $\ell^2 \otimes^{\Sigma^*} \mathfrak{X}$.*

Proof. Fix a faithful Σ^*-representation $\mathfrak{B} \hookrightarrow B(\mathcal{H})$. By the proof of Lemma 3.2.14, $C^w(\mathfrak{X})$ is unitarily equivalent to $\mathscr{C} := \{T \in B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})}) : P_n T \in \mathfrak{X}\}$, where $P_n : (\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})} \to \mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H}$ is the projection onto the n^{th} coordinate. This unitary restricts to a unitary between the closure of

$$\{T \in \mathscr{C} : P_n T = 0 \text{ for all but finitely many } n\}$$

and $C(\mathfrak{X})$ (this is the C^*-module direct sum of countably many copies of $\mathfrak{X}$). It is easy to argue from this that $C(\mathfrak{X})$ is WOT sequentially dense in $C^w(\mathfrak{X})$ when these are viewed in $B(\mathcal{H}, (\mathfrak{X} \otimes_B \mathcal{H})^{(\mathbb{N})})$. By basic properties of the interior tensor product of C^*-module (see [9, 8.2.12(3)]), the Hilbert spaces $(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})}$ and $C(X) \otimes_{\mathfrak{B}} \mathcal{H}$ are unitarily equivalent, and it is easy to check that the induced isomorphism between $B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})})$ and $B(\mathcal{H}, C(\mathfrak{X}) \otimes_{\mathfrak{B}} \mathcal{H})$ restricts to the identity on the copies of $C(\mathfrak{X})$ in each of these. Additionally, it is a well-known C^*-module fact (see [25, pg. 35]) that $C(\mathfrak{X})$ is unitarily equivalent the exterior tensor product $\ell^2 \otimes \mathfrak{X}$. Putting all these together yields the following chain of canonical unitaries:

$$\begin{aligned} C^w(\mathfrak{X}) &\cong \mathscr{B}_{B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})})}(C(X)) \\ &\cong \mathscr{B}_{B(\mathcal{H}, C(X) \otimes_{\mathfrak{B}} \mathcal{H})}(C(X)) \\ &\cong \mathscr{B}_{B(\mathcal{H}, (\ell^2 \otimes \mathfrak{X}) \otimes_{\mathfrak{B}} \mathcal{H})}(\ell^2 \otimes \mathfrak{X}) \\ &= \ell^2 \otimes^{\Sigma^*} \mathfrak{X}. \end{aligned}$$

□

Lemma 4.8.8. *If $\mathfrak{X}$ and $\mathfrak{Y}$ are Σ^*-modules over $\mathfrak{B}$, then $C^w(C^w(\mathfrak{X})) \cong C^w(\mathfrak{X})$ and $C^w(\mathfrak{X} \oplus \mathfrak{Y}) \cong C^w(\mathfrak{X}) \oplus C^w(\mathfrak{X} \oplus \mathfrak{Y})$ unitarily.*

Proof. By the proof of Lemma 3.2.14, if $\mathfrak{B} \subseteq B(\mathcal{H})$ is a faithful Σ^*-representation, then

$$C^w(\mathfrak{X}) \cong \{T \in B(\mathcal{H}, (\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})}) : P_n T \in \mathfrak{X} \text{ for all } n \in \mathbb{N}\}$$

and

$$C^w(C^w(\mathfrak{X})) \cong \{T \in B(\mathcal{H}, (C^w(\mathfrak{X}) \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})}) : P_n T \in C^w(\mathfrak{X}) \text{ for all } n \in \mathbb{N}\}$$

unitarily. Identifying $C^w(\mathfrak{X}) \otimes_{\mathfrak{B}} \mathcal{H}$ with $(\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})}$ via the canonical unitary, it follows that $(C^w(\mathfrak{X}) \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})} \cong ((\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})})^{(\mathbb{N})}$. By properties of Hilbert space direct sums, $((\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})})^{(\mathbb{N})} \cong (\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N} \times \mathbb{N})} \cong (\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})}$. The latter unitary is not canonical, but for any choice of such a unitary, it readily follows that the unitary $(C^w(\mathfrak{X}) \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})} \cong (\mathfrak{X} \otimes_{\mathfrak{B}} \mathcal{H})^{(\mathbb{N})}$ induces a C^*-module unitary between the sets displayed above.

For the second claim, one can easily check that $C^w(\mathfrak{X} \oplus \mathfrak{Y}) \cong C^w(\mathfrak{X}) \oplus C^w(\mathfrak{Y})$ via the unitary $(x_n, y_n) \mapsto ((x_n), (y_n))$, and that $C^w(\mathfrak{X}) \cong C^w(\mathfrak{X}) \oplus C^w(\mathfrak{X})$ via the unitary $(x_n) \mapsto (x_{2n-1}, x_{2n})$. So

$$\begin{aligned} C^w(\mathfrak{X} \oplus \mathfrak{Y}) &\cong C^w(\mathfrak{X}) \oplus C^w(\mathfrak{Y}) \\ &\cong C^w(\mathfrak{X}) \oplus C^w(\mathfrak{X}) \oplus C^w(\mathfrak{Y}) \\ &\cong C^w(\mathfrak{X}) \oplus C^w(\mathfrak{X} \oplus \mathfrak{Y}). \end{aligned}$$

□

We now present a nice consequence of the Σ^*-version of Kasparov's stabilization theorem just mentioned that will allow us to prove the Σ^*-stable isomorphism theorem. This result is analogous to the C^*-result in [9, 8.2.6] and W^*-result in [9, 8.5.28].

Proposition 4.8.9. *If $\mathfrak{X}$ is a σ-full right Σ^*-module over $\mathfrak{B} \subseteq B(\mathcal{H})$ such that $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))$ is unital (i.e. $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) = \mathbb{B}_{\mathfrak{B}}(\mathfrak{X})$) and $\mathfrak{X}$ is ${}_{\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))}\Sigma^*$-countably generated, then $C^w(\mathfrak{X}) \oplus C^w(\mathfrak{B}) \cong C^w(\mathfrak{X})$ unitarily.*

Proof. (cf. [9, proof of Corollary 8.2.6].) The hypotheses allow us to invoke Theorem 4.8.6 on the left Σ^*-module $\mathfrak{X}$ over $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))$ to obtain a left weak quasibasis $\{x_k\}$. For any $x \in \mathfrak{X}$, we have

$$x \sum_{k=1}^{n} \langle x_k | x_k \rangle = \sum_{k=1}^{n} |x\rangle\langle x_k| x_k \xrightarrow{\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))^{\sigma}} x,$$

so that $x \sum_{k=1}^{n} \langle x_k | x_k \rangle \xrightarrow{\sigma_{\mathfrak{B}}} x$ by Lemma 3.1.7. By the "right version" of Lemma 4.3.8 in the current paper, $\sum_{k=1}^{n} \langle x_k | x_k \rangle \xrightarrow{\sigma} 1_{M(\mathfrak{B})}$. In particular, $\mathfrak{B}$ is unital and $(x_k) \in C^w(\mathfrak{X})$.

Now define a map $\varphi : \mathfrak{B} \to C^w(\mathfrak{X})$ by $\varphi(b) = (x_k b)$ for $b \in \mathfrak{B}$. By the calculation $\sum_{k=1}^{n} \langle x_k b | x_k b \rangle = b^* \sum_{k=1}^{n} \langle x_k | x_k \rangle b \leq b^* b$, we have that φ does indeed map into $C^w(\mathfrak{B})$, and it is easy to see that φ is a $\mathfrak{B}$-module map, so that $\mathrm{Ran}(\varphi)$ is a $\mathfrak{B}$-submodule of $C^w(\mathfrak{B})$. We now check the conditions in Proposition 4.8.5 for the submodule $\mathrm{Ran}(\varphi) \subseteq C^w(\mathfrak{B})$. For condition (1), we need to show that if we have a sequence (b_n) from $\mathfrak{B}$ such that $\langle \varphi(b_n) | (y_k) \rangle$ is σ-convergent for all $(y_k) \in C^w(\mathfrak{B})$, then there is a $b \in \mathfrak{B}$ such that $\langle \varphi(b_n) | (y_k) \rangle \xrightarrow{\sigma} \langle \varphi(b) | (y_k) \rangle$ for all $(y_k) \in C^w(\mathfrak{B})$.

Suppose then that we have such a sequence (b_n) as in the former. Then the sequence with terms

$$b_n^* = b_n^* \sum_k \langle x_k | x_k \rangle = \langle (x_k b_n) | (x_k) \rangle = \langle \varphi(b_n) | \varphi(1) \rangle$$

σ-converges in $\mathfrak{B}$, so that $b_n \xrightarrow{\sigma} b$ for some $b \in \mathfrak{B}$. Then for any $(y_k) \in C^w(\mathfrak{X})$,

$$\langle \varphi(b_n) | (y_k) \rangle = \langle (x_k b_n) | (y_k) \rangle = b_n^* \sum_k \langle x_k | y_k \rangle \xrightarrow{\sigma} b^* \sum_k \langle x_k | y_k \rangle = \langle \varphi(b) | (y_k) \rangle.$$

To see that $\varphi(\mathfrak{B})$ meets condition (2) from Proposition 4.8.5, i.e., that the identity map $I_{\varphi(\mathfrak{B})} \in \mathbb{B}_{\mathfrak{B}}(\varphi(\mathfrak{B}))$ is in $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\varphi(\mathfrak{B})))$, let $b \in \mathfrak{B}$ and compute

$$|\varphi(1)\rangle\langle\varphi(1)|(\varphi(b)) = \varphi(1)\langle (x_k) | (x_k b) \rangle = \varphi(1) \sum_k \langle x_k | x_k \rangle b = \varphi(1) b = \varphi(b).$$

Hence $I_{\varphi(\mathfrak{B})} = |\varphi(1)\rangle\langle\varphi(1)| \in \mathbb{K}_{\mathfrak{B}}(\varphi(\mathfrak{B}))$. So we may apply Proposition 4.8.5 to conclude that $\varphi(\mathfrak{B})$ is orthogonally complemented in $C^w(\mathfrak{B})$.

The calculation

$$\langle \varphi(b) | \varphi(c) \rangle = \langle (b x_k) | (c x_k) \rangle = b^* \left(\sum_k \langle x_k | x_k \rangle \right) c = b^* c$$

shows that $\varphi : \mathfrak{B} \to \varphi(\mathfrak{B})$ is a unitary map, so $C^w(\mathfrak{X}) \cong \mathfrak{B} \oplus \mathfrak{W}$ for some Σ^*-module $\mathfrak{W}$ ($\mathfrak{W}$ is a Σ^*-module by Proposition 3.2.21). By Lemma 4.8.8, we have

$$\begin{aligned} C^w(\mathfrak{X}) \cong C^w(C^w(\mathfrak{X})) \cong C^w(\mathfrak{B} \oplus \mathfrak{W}) &\cong C^w(\mathfrak{B}) \oplus C^w(\mathfrak{B} \oplus \mathfrak{W}) \\ &\cong C^w(\mathfrak{B}) \oplus C^w(C^w(\mathfrak{X})) \cong C^w(\mathfrak{B}) \oplus C^w(\mathfrak{X}). \end{aligned}$$

□

Definition 4.8.10. Say that a unital Σ^*-algebra $\mathfrak{B}$ has *Property (D)* if for every closed ideal I in $\mathfrak{B}$ such that $\mathscr{B}(I) = \mathfrak{B} = M(I)$, there is a sequence (e_n) in I with $e_n \xrightarrow{\sigma} 1$.

Note 4.8.11. Clearly every simple unital Σ^*-algebra (in particular, every von Neumann algebra factor that is type I_n, type II_1, or countably decomposable and type III) has Property (D). Every infinite type I von Neumann algebra factor also has Property (D). Indeed, if $\mathcal{H}$ is infinite dimensional and nonseparable, then $B(\mathcal{H})$ has no WOT sequentially dense ideals; if $\mathcal{H}$ is infinite dimensional and separable, then the unique closed ideal $\mathbb{K} \subseteq B(\mathcal{H})$ meets the conditions and conclusion in the definition.

For a Σ^*-algebra $\mathfrak{B} \subseteq B(\mathcal{H})$, note that the WOT sequential closure of $\mathbb{K} \otimes \mathfrak{B}$ in $B(\ell^2 \otimes^2 \mathcal{H})$, is all of $\mathbb{M}(\mathfrak{B})$ (where the latter is the space of infinite matrices over $\mathfrak{B}$ indexed by $\mathbb{N}$ with uniformly bounded finite submatrices—see [9, 1.2.26]). Indeed, take $A \in \mathbb{M}(\mathfrak{B})$ viewed as a matrix with entries in $\mathfrak{B}$, and let $A_n \in \mathbb{K} \otimes \mathfrak{B}$ be the finite submatrix of A supported in the entries $1, \ldots, n$. An easy argument (using finitely supported columns in $\mathcal{H}^{(\mathbb{N})}$ and Lemma 2.2.8 if you like) shows that $A_n \xrightarrow{WOT} A$. Since $\mathbb{K} \otimes \mathfrak{B} \subseteq \mathscr{B}(\mathbb{K}) \otimes \mathfrak{B} \subseteq B(\ell^2 \otimes^2 \mathcal{H})$, it follows that the WOT sequential closure of $\mathscr{B}(\mathbb{K}) \otimes \mathfrak{B}$, i.e. $\mathscr{B}(\mathbb{K}) \otimes^{\Sigma^*} \mathfrak{B}$ in the notation introduced above Lemma 4.7.3, coincides with $\mathbb{M}(\mathfrak{B})$.

Theorem 4.8.12. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two unital Σ^*-algebras with Property (D). Then $\mathfrak{A}$ and $\mathfrak{B}$ are Σ^*-Morita equivalent if and only if $\mathbb{M}(\mathfrak{A}) \cong \mathbb{M}(\mathfrak{B})$ Σ^*-isomorphically.*

Proof. ($\Longrightarrow$) Let $\mathfrak{X}$ be an $\mathfrak{A} - \mathfrak{B}$ Σ^*-imprivitivity bimodule. Then $\mathfrak{A} \cong \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))$ Σ^*-isomorphically by Theorem 4.3.11, so by Property (D), there is a sequence (e_n) in $\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})$ with $e_n \xrightarrow{\sigma} I$ in $\mathbb{B}_{\mathfrak{B}}(\mathfrak{X}) = \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X}))$. By Proposition 4.8.4, $\mathfrak{X}$ is $\Sigma^*_{\mathfrak{B}}$-countably generated. It is similarly shown that $\mathfrak{X}$ is $\Sigma^*_{\mathfrak{A}}$-countably generated. By

Theorem 3.2.19, $C^w(\mathfrak{B}) \cong \mathfrak{X} \oplus C^w(\mathfrak{B})$, so that by Lemma 4.8.8,

$$C^w(\mathfrak{B}) \cong C^w(\mathfrak{X} \oplus C^w(\mathfrak{B})) \cong C^w(\mathfrak{X}) \oplus C^w(\mathfrak{B}).$$

By Proposition 4.8.9, we have $C^w(\mathfrak{X}) \oplus C^w(\mathfrak{B}) \cong C^w(\mathfrak{X})$. Hence $C^w(\mathfrak{X}) \cong C^w(\mathfrak{B})$ unitarily. By symmetry and the other-handed versions of everything above, also $C^w(\mathfrak{X}) \cong C^w(\mathfrak{A})$ unitarily. Hence

$$\begin{aligned}\mathbb{M}(\mathfrak{B}) &\cong \mathscr{B}(\mathbb{K}) \otimes^{\Sigma^*} \mathfrak{B} \cong \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\ell^2 \otimes^{\Sigma^*} \mathfrak{B})) \cong \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(C^w(\mathfrak{B}))) \cong \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(C^w(\mathfrak{X}))) \\ &\cong \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\ell^2 \otimes^{\Sigma^*} \mathfrak{X})) \cong \mathscr{B}(\mathbb{K}) \otimes^{\Sigma^*} \mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{X})) \cong \mathscr{B}(\mathbb{K}) \otimes^{\Sigma^*} \mathfrak{A} \cong \mathbb{M}(\mathfrak{A}),\end{aligned}$$

where we have used the observation above the statement of the current theorem for the first and last isomorphisms, Theorem 4.7.5 for the second and sixth isomorphisms (using the Σ^*-module ℓ^2 over $\mathscr{B}(\mathbb{K})$ and the fact that $\mathscr{B}(\mathbb{K}_{\mathfrak{B}}(\mathfrak{B})) \cong \mathfrak{B}$ Σ^*-isomorphically) and Proposition 4.8.7 for the third and fifth isomorphisms. Since these are all Σ^*-isomorphisms, the claim is proved.

($\Longleftarrow$) This direction follows immediately from our results that Σ^*-Morita equivalence is an equivalence relation coarser than Σ^*-isomorphism (Theorem 4.4.6), that $\mathfrak{A}$ is Σ^*-Morita equivalent to $\mathscr{B}(\mathbb{K}_I(\mathfrak{A}))$ for all I (Corollary 4.6.5), and the fact mentioned above the theorem that $\mathscr{B}(\mathbb{K}(\mathfrak{A})) = \mathbb{M}(\mathfrak{A})$ (and similarly for $\mathfrak{B}$). $\square$

Chapter 5

Weak* Sequentially Closed Banach Spaces and Operator Spaces

5.1 Introduction

There are many interesting and fruitful connections between C^*-module theory and operator space theory. For example, D. Blecher in [6] proved that the interior tensor product of C^*-modules coincides with their module Haagerup tensor product when one views them as operator modules. (The latter tensor product is a module version of the Haagerup tensor product for operator spaces, a natural and remarkable construction that only really appears in the context of operator space theory.) Blecher then used this observation in [8] to prove a satisfying version of Morita's fundamental theorem for (strong) C^*-algebraic Morita equivalence.

Our goal in this chapter is to start the process of connecting our theory of Σ^*-modules with operator space theory. We begin in Section 5.2 by giving a fast overview of the necessary background on operator spaces and TROs. In Section 5.3, we define "abstract" weak* sequential analogues of both dual Banach spaces and dual operator spaces, and prove representation theorems for both of these. In Section 5.4, we prove a TRO result analogous to the Zettl-Effros-Ozawa-Ruan result that a TRO with a Banach space predual is a W^*-TRO. We then apply this to prove a characterization of Σ^*-modules among C^*-modules over Σ^*-algebras.

5.2 Background

5.2.1 Operator spaces

There are four excellent textbooks on operator spaces, [9, 19, 32, 35], each of which emphasizes different aspects of the theory. The following outline of basic operator space theory is pretty bare-bones, so see one of these references for more information. Our notation probably aligns most with that of [9].

A *concrete operator space* is a closed linear subspace X of $B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ (beware: some authors do not require their operator spaces to be closed). Given such an X, there is for each $n \in \mathbb{N}$ a canonical norm $\|\cdot\|_n$ on the vector space $M_n(X)$ gotten by viewing $M_n(X)$ as a subspace of $M_n(B(\mathcal{H}))$ and identifying the latter with $B(\mathcal{H}^{(n)})$ (where $\mathcal{H}^{(n)}$ denotes the Hilbert space direct sum of n copies of $\mathcal{H}$).

It turns out that to view operator spaces abstractly, the appropriate data to consider are precisely these "matrix norms." An *abstract operator algebra* is a vector space X equipped with a sequence of norms $\|\cdot\|_n : M_n(X) \to [0, \infty)$ such that X embeds as a concrete operator space in some $B(\mathcal{H})$ in such a way that the given matrix norms coincide with those induced by the embedding into $B(\mathcal{H})$. Research in operator space theory took off in the late 80s as a result of Z-J. Ruan's abstract characterization of operator spaces in [40], which gives simple criteria for when a vector space X equipped with a sequence of matrix norms is an abstract operator space. When we say "operator space" in the following, we usually mean an abstract operator space with an implicit sequence of matrix norms.

Let X and Y be two operator spaces, and let $\varphi : X \to Y$ be a linear map. The n^{th} *amplification of* φ is the map $\varphi_n : M_n(X) \to M_n(Y)$ defined by the rule $[x_{ij}] \mapsto [\varphi(x_{ij})]$. Since $M_n(X)$ and $M_n(Y)$ are normed spaces, φ_n has an operator norm. We say that φ is *completely bounded* if $\sup_n \|\varphi_n\| < \infty$. In the case that φ is completely bounded, the *cb-norm* of φ is the number $\|\varphi\|_{cb} := \sup_n \|\varphi_n\|$. We say that φ is a *completely isometry* if each φ_n is isometric, and similar definitions apply to "completely contractive," "complete quotient map," etc.

Denote by $CB(X, Y)$ the space of all completely bounded linear maps $\varphi : X \to Y$. For each $n \in \mathbb{N}$, $M_n(CB(X, Y))$ can be canonically identified as a vector space with $CB(X, M_n(Y))$, and it turns out that the matrix norms induced on $CB(X, Y)$ by these identifications makes $CB(X, Y)$ into an operator space. By a fundamental fact about operator spaces (see [9, 1.2.6]), each functional $\varphi \in X^*$ is completely bounded with $\|\varphi\| = \|\varphi\|_{cb}$. So $CB(X, \mathbb{C}) = X^*$ as sets of functions on X, and we can thus endow X^* with an operator space structure in which the 1-norm coincides with the usual norm. In this case, $X^* = CB(X, \mathbb{C})$ is called the *operator space dual* of X. We repeat for emphasis that the matrix norms on X^* come from the canonical linear isomorphism $M_n(X^*) \cong CB(X, M_n)$.

Finally, we remark that a C^*-module X over a C^*-algebra A has a canonical operator space structure arising from the identification of X with a TRO in its linking algebra (see below). The matrix norms in this case also coincide with those coming from the canonical structure $M_n(X)$ has as a C^*-module over $M_n(A)$. See [9, 8.2] for a lot more on the perspective of C^*-modules as operator spaces.

5.2.2 Ternary rings of operators

We have already seen TROs a little in Chapters 3 and 4, but in this chapter, we will need to be more systematic in our dealing with them; hence we now give a short rundown of their basic theory. Our main references in preparing this work were [9, Sections 4.4 and 8.3] and [18].

Recall from the discussion two paragraphs above Proposition 3.1.9 that a *ternary ring of operators*, or *TRO* for short, is a norm-closed subspace $Z \subseteq B(\mathcal{H}, \mathcal{K})$ for Hilbert spaces $\mathcal{H}, \mathcal{K}$, such that $xy^*z \in Z$ for all $x, y, z \in Z$. The map $Z \times Z \times Z \to Z$, $(x, y, z) \mapsto xy^*z$ is called the *triple product* on Z. If Z is a TRO, we denote by $Z^\star$ the set $\{z^* \in B(\mathcal{K}, \mathcal{H}) : z \in Z\}$; this is also a TRO. (Note the slight distinction in notation between $Z^\star$ and the dual Z^*.) It is elementary to see that the closed linear span of the set $\{y^*z : y, z \in Z\}$, denoted $Z^\star Z$, is a C^*-algebra in $B(\mathcal{H})$, and similarly

$ZZ^\star$, the closed linear span of $\{xy^* : x, y \in Z\}$, is a C^*-algebra in $B(\mathcal{K})$.

For two TROs Y and Z, a linear map $\varphi : Y \to Z$ is a *TRO-homomorphism* if $\varphi(xy^*z) = \varphi(x)\varphi(y)^*\varphi(z)$ for all $x, y, z \in Y$. A *TRO-isomorphism* is an injective and surjective TRO-homomorphism. A *TRO-representation* of a TRO X is a TRO-isomorphism $\varphi : X \to B(\mathcal{H}, \mathcal{K})$ for Hilbert spaces $\mathcal{H}, \mathcal{K}$.

If Y and Z are TROs and $\varphi : Y \to Z$ is a TRO-isomorphism, there is a $*$-isomorphism $\tilde{\varphi} : Y^\star Y \to Z^\star Z$ determined by the rule $x^*y \mapsto \varphi(x)^*\varphi(y)$. Indeed, one way to prove this uses the fact that for $\eta \in Z^\star Z$, $\|\eta\| = \sup\{\|z\eta\| : z \in \mathrm{Ball}(X)\}$ (see [24, Lemma 4.2(iv)] for the latter). Similarly, there is a canonical $*$-isomorphism $YY^\star \cong ZZ^\star$.

A TRO $Z \subseteq B(\mathcal{H}, \mathcal{K})$ is called *nondegenerate* if $[Z\mathcal{H}] = \mathcal{K}$ and $[Z^\star\mathcal{K}] = \mathcal{H}$. As sketched above Proposition 3.1.9, one may always replace a TRO with a TRO-isomorphic copy that is nondegenerate, so just as with C^*-algebras, one may usually assume without loss of generality that a given TRO is nondegenerate.

If Z is a TRO, a *sub-TRO* of Z is a closed subspace X of Z that is closed under the triple product. A *TRO-ideal* in Z is a norm-closed subspace J of Z such that $JZ^\star Z \subseteq J$ and $ZZ^\star J \subseteq J$ (it follows automatically that also $ZJ^\star Z \subseteq J$—see [18, above Proposition 2.2]). It is easy to check that the kernel of a TRO-homomorphism is a TRO-ideal. Conversely, a quotient of a TRO by a TRO-ideal is a TRO in the canonically induced triple product (see [18, Proposition 2.2]), so every TRO-ideal is the kernel of some TRO-homomorphism.

A TRO $Z \subseteq B(\mathcal{H}, \mathcal{K})$ admits a canonical structure as an operator space by viewing Z canonically as a subspace of $B(\mathcal{H} \oplus \mathcal{K})$; the matrix norms induced on Z coincide with those gotten by viewing $M_n(Z) \subseteq M_n(B(\mathcal{H}, \mathcal{K}))$ and identifying the latter with $B(\mathcal{H}^{(n)}, \mathcal{K}^{(n)})$.

A result due in various parts to Harris, Kaup, Hamana, Kirchberg, and Ruan (see notes to [9, 4.4.6 and 8.3.2] for the history) states that for a surjective linear map φ from a TRO onto another TRO, φ is a TRO-isomorphism if and only if it is completely isometric (and interestingly, Hamana proved that these hold if and only φ is 2-isometric—see [24, Proposition 4.1]). Thus the triple product on a TRO is determined by its operator space structure, and conversely, any two TRO-representations of a TRO give rise to the same operator space structure.

A *W^*-TRO* is a weak*-closed TRO $Z \subseteq B(\mathcal{H}, \mathcal{K})$. If Z is a W^*-TRO and J is a weak*-closed TRO-ideal, then Z/J is also a W^*-TRO in the canonically induced triple product (see [18, Proposition 2.2]). Also, if Z is a W^*-TRO, the weak*-closure of $Z^\star Z$ turns out to coincide with $M(Z^* Z)$ (see [18, Appendix]). Thus, if Y and Z are TRO-isomorphic W^*-TROs, there is a canonical $*$-isomorphism between the weak*-closure of $Y^\star Y$ and the weak*-closure of $Z^\star Z$.

Finally, if Z is a TRO, the second dual Z^{**} is canonically a W^*-TRO containing Z as a weak*-dense sub-TRO (see [9, proof of Proposition 8.5.17]). We will need the following facts about how triple products in Z^{**} act on functionals in Z^*:

$$\langle \eta x^* y, \varphi \rangle = \langle \eta, \varphi(\,\cdot\, x^* y) \rangle \text{ and } \langle x y^* \eta, \varphi \rangle = \langle \eta, \varphi(x y^* \,\cdot\,) \rangle$$

for all $\eta \in Z^{**}$, $x, y \in Z$, and $\varphi \in Z^*$. This is a special case of a triple product version of the Arens product on the second dual of an Arens regular Banach algebra.

We could not find a reference for such an "Arens triple product," but undoubtedly this is at least known to TRO experts. At any rate, the centered equations above are straightforward to prove using the facts that Z is weak*-dense in Z^{**} and that the triple product on the latter is separately weak*-continuous in each variable.

5.3 Definitions and representation theorems

In this chapter, we deal abstractly with weak* sequentially closed subspaces of dual Banach spaces, which we call Σ-Banach spaces, and their operator space analogue, Σ-operator spaces. Our abstract characterization for Σ-operator spaces is similar to Davies' abstract characterization for Σ^*-algebras from [16] (see 2.2.3), and part of the proof is modeled after some pieces in the proof of the representation theorem for operator space as presented in [19, Section 2.3]. The abstract characterization we give for Σ-Banach spaces is the obvious variant.

Note 5.3.1. To be clear, when we say "dual Banach space," we mean a Banach space E *together with* a weak*-topology on E coming from an isometric isomorphism of E with the dual Y^* of some other Banach space Y.

Before we formally define Σ-Banach spaces and Σ-operator spaces, we set some notation and terminology. Recall from Definition 2.2.1 that a *σ-convergence system* for a Banach space X is a set $\mathscr{S}$ whose elements are pairs $((x_n), x)$ consisting of a sequence $(x_n) \subset X$ and an element $x \in X$.

For a Banach space X with a σ-convergence system $\mathscr{S}$, let $\mathscr{S}^\sigma$ be the set of

"$\mathscr{S}$-continuous" functionals on X, that is,

$$\mathscr{S}^\sigma := \{\psi \in X^* : \psi(x_n) \to \psi(x) \text{ for all } ((x_n), x) \in \mathscr{S}\}.$$

Now suppose that X is an operator space with a σ-convergence system $\mathscr{S}$. For each $k \in \mathbb{N}$, let

$$\mathscr{S}_k := \{((u_n), u) : (u_n) \text{ is a sequence in } M_k(X),\ u \in M_k(X),$$
$$\text{and } (((u_n)_{ij}), u_{ij}) \in \mathscr{S} \text{ for all } i, j = 1, \dots, k\}.$$

Roughly speaking, $\mathscr{S}_k$ is the collection of pairs consisting of a sequence and element in $M_k(X)$ such that looking at each pair entrywise gives pairs in $\mathscr{S}$. For each $k \in \mathbb{N}$, define

$$\mathscr{S}_k^\sigma := \{\psi \in M_k(X)^* : \psi(u_n) \to \psi(u) \text{ for all } ((u_n), u) \in \mathscr{S}_k\}.$$

In particular, $\mathscr{S}_1 = \mathscr{S}$ and $\mathscr{S}_1^\sigma = \mathscr{S}^\sigma$.

Definition 5.3.2. A *concrete* Σ*-Banach space* is a weak* sequentially closed subspace X of a dual Banach space E.

An *abstract* Σ*-Banach space* is a Banach space X together with a σ-convergence system $\mathscr{S}$ on X such that the following conditions hold:

(1) If $((x_n), x) \in \mathscr{S}$, then (x_n) is a bounded sequence.

(2) If (x_n) is a sequence in X such that the sequence $(\varphi(x_n))$ converges for all $\varphi \in \mathscr{S}^\sigma$, then there is an $x \in X$ such that $((x_n), x) \in \mathscr{S}$.

(3) For any $x \in X$ and $\epsilon > 0$, there is a $\psi \in \mathrm{Ball}(\mathscr{S}^\sigma)$ such that $|\psi(x)| > \|x\| - \epsilon$.

Definition 5.3.3. A *concrete* Σ*-operator space* is a weak* sequentially closed subspace X of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

An *abstract* Σ*-operator space* is a pair $(X, \mathscr{S})$ consisting of an (abstract) operator space X and a σ-convergence system $\mathscr{S}$ such that the following conditions hold:

(1) If $((x_n), x) \in \mathscr{S}$, then (x_n) is bounded.

(2) If (x_n) is a sequence in X such that $(\varphi(x_n))$ converges for all $\varphi \in \mathscr{S}^\sigma$, then there is a $x \in X$ such that $((x_n), x) \in \mathscr{S}$.

(3) For any $k \in \mathbb{N}$, $\epsilon > 0$, and $u \in M_k(X)$, there is a $\psi \in \mathrm{Ball}(\mathscr{S}_k^\sigma)$ such that $|\psi(u)| > \|u\| - \epsilon$.

Note 5.3.4. Evidently, every Σ-operator space has an underlying Σ-Banach space structure.

Observe that the ψ in condition (3) in both Definition 5.3.2 and Definition 5.3.3 may be taken to have norm exactly 1. Also note that condition (3) in Definition 5.3.2 (resp. Definition 5.3.3) implies that $\mathrm{Ball}(\mathscr{S}^\sigma)$ (resp. $\mathrm{Ball}(\mathscr{S}_k^\sigma)$) separates points in X (resp. $M_k(X)$). In particular, this shows that limits in $\mathscr{S}$ are unique in both cases, i.e. if (x_n) is a sequence in X and $x, x' \in X$ with $((x_n), x), ((x_n), x') \in \mathscr{S}$, then $x = x'$.

If $(X, \mathscr{S})$ is an abstract Σ-Banach space and E is a dual Banach space, an isometric map $\theta : X \to E$ is called a Σ*-representation* if: (1) $\theta(X)$ is weak* sequentially closed, and (2) for a sequence (x_n) and element x in X, $((x_n), x) \in \mathscr{S}$ if and only if $\theta(x_n) \xrightarrow{w^*} \theta(x)$.

Similarly, if X is an abstract Σ-operator space and $\mathcal{H}$ is a Hilbert space, a completely isometric linear map $\theta : X \to B(\mathcal{H})$ is called a Σ-*representation* if: (1) $\theta(X)$ is weak* sequentially closed, and (2) for a sequence (x_n) and element x in X, $((x_n), x) \in \mathscr{S}$ if and only if $\theta(x_n) \xrightarrow{w^*} \theta(x)$.

If $(X, \mathscr{S}_X)$ and $(Y, \mathscr{S}_Y)$ are two abstract Σ-Banach spaces, an isometric isomorphism $\phi : X \to Y$ is a Σ-*isomorphism* if ϕ and ϕ^{-1} are "continuous" with respect to $\mathscr{S}_X$ and $\mathscr{S}_Y$, that is, $((x_n), x) \in \mathscr{S}_X$ iff $((\phi(x_n)), \phi(x)) \in \mathscr{S}_Y$. Similar definitions hold for isometric isomorphisms between concrete Σ-Banach spaces and completely isometric isomorphisms between abstract and concrete Σ-operator spaces.

The proof of the "representation theorem" for Σ-Banach spaces is quite easy—it is essentially just a compilation of a few well-known facts and common arguments. For completeness, we include the proof below.

Theorem 5.3.5. *If $X \subseteq E$ is a concrete Σ-Banach space and $\mathscr{S}$ is the collection of pairs $((x_n), x)$ such that $x_n \xrightarrow{w^*} x$, then $(X, \mathscr{S})$ is an abstract Σ-Banach space.*

Conversely, every abstract Σ-Banach space admits a Σ-representation into a dual Banach space.

Proof. Suppose $X \subseteq E$ is a concrete Σ-Banach space, and let Z be the subspace of weak*-continuous functionals in E^*. If $x_n \xrightarrow{w^*} x$ in X, then $\langle x_n, \eta \rangle \to \langle x, \eta \rangle$ for all η in Z. It follows by the principle of uniform boundedness applied to the sets $\{\langle x_n, \eta \rangle : n \in \mathbb{N}\}$ over all $\eta \in Z$ that (x_n) is a bounded sequence. Now suppose that (y_n) is a sequence in X such that $(\varphi(y_n))$ converges for all $\varphi \in \mathscr{S}^\sigma$. Since $Z \subseteq \mathscr{S}^\sigma$, we have in particular that $\langle y_n, \eta \rangle$ converges for all $\eta \in Z$. By the same argument as above using the uniform boundedness principle, (y_n) is bounded. Thus (y_n) has a subnet (y_{n_λ}) that converges weak* to some $y \in E$. But then $\langle y_n, \eta\rangle \to \langle y, \eta\rangle$ for all $\eta \in Z$, so that $y_n \xrightarrow{w^*} y$. To finish the proof of the first statement, if $x \in X$ and $\epsilon > 0$, then we may pick $\eta \in Z \subseteq \mathscr{S}^\sigma$ such that $|\langle x, \eta\rangle| > \|x\| - \epsilon$.

For the second statement, let $(X, \mathscr{S})$ be an abstract Σ-Banach space. Referring to the definition of the latter, condition (1) and a standard triangle inequality argument shows that $\mathscr{S}^\sigma$ is closed in X^*, hence $\mathscr{S}^\sigma$ is a Banach space. By condition (3), X canonically isometrically embeds into $(\mathscr{S}^\sigma)^*$. It follows from condition (2) that the image of X under this embedding is weak* sequentially closed. This embedding evidently meets the other requirements for being a Σ-representation. $\square$

Now we turn to the (harder, but similar) proof of the representation theorem for Σ-operator spaces.

Note 5.3.6. Let $(X, \mathscr{S})$ be a Σ-operator space. A standard triangle inequality argument shows that $\mathscr{S}^\sigma$ is a closed subspace of X^*, so $\mathscr{S}^\sigma$ is an operator space with matrix norms determined by the canonical linear injection

$$M_k(\mathscr{S}^\sigma) \hookrightarrow CB(X, M_k).$$

Under the identification of $M_k(X^*)$ with $CB(X, M_k)$, for $\varphi \in M_k(X^*)$, we have $\varphi \in M_k(\mathscr{S}^\sigma)$ if and only if $\varphi(x_n) \to \varphi(x)$ for all $((x_n), x) \in \mathscr{S}$.

Lemma 5.3.7 (cf. [19] Lemmas 2.3.3 and 2.3.4)**.** *Let $(X, \mathscr{S})$ be an abstract Σ-operator space. For any $k \in \mathbb{N}$, $\epsilon > 0$, and $u \in M_k(X)$, there exists a $\rho \in$* Ball$(M_k(\mathscr{S}^\sigma))$ *such that $\|\rho_k(u)\| > \|u\| - \epsilon$.*

Proof. Given such k, ϵ, and u, let ψ be the functional in $\mathrm{Ball}(\mathscr{S}_k^\sigma)$ with $|\psi(u)| > \|u\| - \epsilon$ guaranteed by condition (3) in Definition 5.3.3, and assume without loss of generality that $\|\psi\| = 1$.

As shown in [19, proof of Lemma 2.3.3] one may construct the following: subspaces $K_0, H_0 \subseteq \mathbb{C}^k$, linear surjections $\tilde{\theta} : M_{1,k} \to K_0$, $\tilde{\pi} : M_{1,k} \to H_0$, and a bounded linear operator $\rho_0 \in B(K_0, H_0)$ such that:

- $\psi(\alpha^* x \beta) = \langle \rho_0(x)\tilde{\theta}(\beta), \tilde{\pi}(\alpha)\rangle$ for all $\alpha, \beta \in M_{1,k}$ and $x \in X$;
- the operator $\rho : X \to M_k$ defined $\rho(x) := \rho_0(x)P_{K_0}$ is a complete contraction with $|\psi(v)| \leq \|\rho_k(v)\|$ for all $v \in M_k(X)$.

(The latter follows from the displayed equation in the statement of Lemma 2.3.3 in [19]).

It remains only to check the "$\mathscr{S}$-continuity" condition. If $((x_n), x) \in \mathscr{S}$, then it follows from (2) in Definition 5.3.3 that $((\lambda x_n \mu), \lambda x \mu) \in \mathscr{S}$ for all $\lambda, \mu \in \mathbb{C}$. Hence if $\alpha, \beta \in M_{1,k}$, then $((\alpha^* x_n \beta), \alpha^* x \beta) \in \mathscr{S}_k$. Using the first bullet above,

$$\langle \rho_0(x_n)\tilde{\theta}(\beta), \tilde{\pi}(\alpha)\rangle = \psi(\alpha^* x_n \beta) \to \psi(\alpha^* x \beta) = \langle \rho_0(x)\tilde{\theta}(\beta), \tilde{\pi}(\alpha)\rangle.$$

Since $\tilde{\theta}$ and $\tilde{\pi}$ map onto K_0 and H_0 respectively, we have $\rho_0(x_n) \to \rho_0(x)$ in $B(K_0, H_0)$. It follows easily from this and the definition of ρ that $\rho(x_n) \to \rho(x)$ in M_k. $\square$

Proposition 5.3.8. *If $(X, \mathscr{S})$ is an abstract Σ-operator space, then $X \hookrightarrow (\mathscr{S}^\sigma)^*$ completely isometrically.*

Proof. We must show that for any $k \in \mathbb{N}$ and $x = [x_{ij}] \in M_k(X)$, the cb-norm of the map $\hat{x} : \mathscr{S}^\sigma \to M_k$, $\psi \mapsto [\psi(x_{ij})]$, coincides with $\|x\|$. By [19, Proposition 2.2.2], it suffices to show that for any $\epsilon > 0$, there is a $\varphi = [\varphi_{ab}] \in \mathrm{Ball}(M_k(\mathscr{S}^\sigma))$ such that

$$\|\hat{x}_k(\varphi)\| > \|x\| - \epsilon.$$

Since $\hat{x}_k(\varphi) = [\hat{x}(\varphi_{ab})] = [\varphi_{ab}(x_{ij})] = \varphi_k(x)$, this is provided by Lemma 5.3.7. $\square$

Theorem 5.3.9. *If $X \subseteq B(\mathcal{H})$ is a concrete Σ-operator space and $\mathscr{S}$ is the collection of pairs $((x_n), x)$ such that (x_n) is a weak* convergent sequence in X with limit x, then $(X, \mathscr{S})$ is an abstract Σ-operator space.*

Conversely, every abstract Σ-operator space admits a Σ-representation.

Proof. With $X \subseteq B(\mathcal{H})$ and $\mathscr{S}$ as in the first statement, properties (1) and (2) in Definition 5.3.3 are well-known properties of the weak* topology on $B(\mathcal{H})$, and property (3) follows easily by considering functionals on $M_k(X)$ of the form $u \mapsto \langle u((\zeta_i)), (\eta_i)\rangle$ for $(\zeta_i), (\eta_i) \in \mathrm{Ball}(\mathcal{H}^{(k)})$.

For the "conversely" statement, it is easily checked (cf. the Banach space case) that the completely isometric embedding $X \hookrightarrow (\mathscr{S}^\sigma)^*$ from Proposition 5.3.8 is a Σ-representation. $\square$

5.4 A characterization of Σ^*-modules

This section is mostly concerned with proving one result: Theorem 5.4.5, which says, loosely speaking, that if $\mathfrak{B}$ is a Σ^*-algebra and X is a C^*-module over $\mathfrak{B}$

possessing a Σ-Banach space structure meeting some compatibility requirements, then X is automatically a Σ^*-module over $\mathfrak{B}$. Our Theorem 5.4.5 is analogous to the remarkable result by Zettl and Effros-Ozawa-Ruan (see [44, 18]) that if X is a C^*-module over a W^*-algebra M such that X is a dual Banach space, then X is a W^*-module over M. However, our result is not quite as satisfying, as it requires some extra compatibility assumptions between the Σ-Banach space structure and module structure on X.

Following Effros-Ozawa-Ruan in [18, Theorem 2.6], to get to our desired Theorem 5.4.5, we take a route through a result that is most conveniently expressed in the language of TROs.

Definition 5.4.1. A Σ^*-*TRO* is weak* sequentially closed TRO.

One challenging thing about the theory of Σ^*-TROs as compared to that of general TROs and W^*-TROs is that we do not know if a quotient of a Σ^*-TRO by a weak* sequentially closed TRO-ideal is again a Σ^*-TRO. We are able to get around this issue in the following proposition because our purported Σ^*-TRO is contained as a weak* sequentially closed sub-TRO of a larger W^*-TRO (that is itself the quotient of a W^*-TRO by a weak*-closed TRO-ideal).

Proposition 5.4.2. *If X is a TRO and $\mathscr{S}$ is a σ-convergence system on X such that $(X, \mathscr{S})$ is a Σ-Banach space in which the triple product is separately $\mathscr{S}$-continuous in the first and third variables, then X is TRO-isomorphic (hence completely isometrically isomorphic) and Σ-isomorphic to a Σ^*-TRO. Thus $(X, \mathscr{S})$ is a Σ-operator space.*

Proof. The basic essence of the proof is to show that X is TRO-isomorphic to a weak* sequentially closed TRO in a W^*-TRO $X^{**}/\ker(\iota^*)$.

Let $\iota : \mathscr{S}^\sigma \hookrightarrow X^*$ be the inclusion, which is of course an isometry. As mentioned in Section 5.2.2, X^{**} is a W^*-TRO. We claim that the kernel of the quotient map $\iota^* : X^{**} \to (\mathscr{S}^\sigma)^*$ is a TRO ideal. Indeed, since X is weak*-dense in X^{**} and the triple product on X^{**} is separately weak*-continuous, it suffices to show that $\ker(\iota^*)X^\star X \subseteq \ker(\iota^*)$ and $XX^\star \ker(\iota^*) \subseteq \ker(\iota^*)$. Let $\eta \in \ker(\iota^*)$, $x, y \in X$, and $\varphi \in \mathscr{S}^\sigma$. Then by one of the centered equations in the last paragraph of Section 5.2.2,

$$\iota^*(\eta x^* y)(\varphi) = \langle \eta, \varphi(\,\cdot\, x^* y)\rangle = \iota^*(\eta)(\varphi(\,\cdot\, x^* y)) = 0$$

where the second to last equality makes sense since $\varphi(\,\cdot\, x^* y)$ is in $\mathscr{S}^\sigma$ by the assumption that the triple product on X is $\mathscr{S}$-continuous in the first variable. It follows similarly, using the assumption about continuity in the third variable, that $\iota^*(yx^*\eta) = 0$. So $\ker(\iota^*)$ is a weak*-closed TRO-ideal in X^{**}.

Thus $X^{**}/\ker(\iota^*)$ is a W^*-TRO that is weak*-homeomorphic and isometrically isomorphic to $(\mathscr{S}^\sigma)^*$ via the map $\rho : X^{**}/\ker(\iota^*) \to (\mathscr{S}^\sigma)^*$, $[\eta] \mapsto \iota^*(\eta)$. Let $\tilde{X} = \{[x] \in X^{**}/\ker(\iota^*) : x \in X\}$, and let $\hat{X}$ be the canonical isometric copy of X in $(\mathscr{S}^\sigma)^*$ (using the assumption that $(X, \mathscr{S})$ is a Σ-Banach space). Evidently ρ restricts to an isometric isomorphism $\tilde{X} \to \hat{X}$. Since ρ is a weak*-homeomorphism and $\hat{X}$ is weak* sequentially closed in $(\mathscr{S}^\sigma)^*$, we have that $\tilde{X}$ is weak* sequentially closed in $X^{**}/\ker(\iota^*)$. Evidently, the map $X \to \tilde{X}$, $x \mapsto [x]$, is a TRO-isomorphism (since X is a sub-TRO in X^{**} and the triple product on $X^{**}/\ker(\iota^*)$ is the canonical one induced from that of X^{**}). That this map is a Σ-isomorphism (of Σ-Banach spaces) follows again from the fact that ρ is a weak*-homeomorphism. $\square$

If Z is a TRO such that $Z(\overline{Z^\star Z}^{w^*}) \subseteq Z$, it turns out that $\overline{Z^\star Z}^{w^*}$ coincides with the multiplier algebra $M(Z^\star Z)$ (see [9, Proposition 8.5.3] for this statement in C^*-module terms). Thanks to this nice fact, the bridge between the W^*-TRO result [18, Theorem 2.6] and the W^*-module result mentioned above is short (see [9, Corollary 8.5.7]).

In our case however, things end up being slightly more complicated, and we will eventually have to add some extra assumptions to get our desired characterization of Σ^*-modules. The next lemma is a partial Σ^*-version of the fact mentioned in the previous paragraph.

Lemma 5.4.3. *If $X_0 \subseteq B(\mathcal{H}, \mathcal{K})$ is a nondegenerate Σ^*-TRO, then $M(X_0^\star X_0)$ is a concrete Σ^*-algebra in $B(\mathcal{H})$, and X_0 is a Σ^*-module over $M(X_0^\star X_0)$. Moreover, $\zeta_n \xrightarrow{WOT} \zeta$ in $M(X_0^\star X_0) \subseteq B(\mathcal{H})$ if and only if $x\zeta_n \xrightarrow{WOT} x\zeta$ in $X_0 \subseteq B(\mathcal{H}, \mathcal{K})$ for all $x \in X_0$.*

Proof. Since X_0 is an $X_0 X_0^\star - X_0^\star X_0$ C^*-imprivitivity bimodule, we have from a basic principle of strong C^*-Morita equivalence theory that there is a canonical $*$-isomorphism $X_0^\star X_0 \cong {}_{X_0 X_0^\star}\mathbb{K}(X_0)$. Thus we have canonical $*$-isomorphisms

$$M(X_0^\star X_0) \cong M({}_{X_0 X_0^\star}\mathbb{K}(X_0)) \cong {}_{X_0 X_0^\star}\mathbb{B}(X_0) \cong {}_{\mathscr{B}(X_0 X_0^\star)}\mathbb{B}(X_0).$$

It is straightforward to check that the composition of the above is the restriction of the canonical $*$-isomorphism $B(\mathcal{H}) \cong B(X_0^\star \otimes_{X_0 X_0^\star} \mathcal{K})$. By the other-handed version of Proposition 3.1.6, ${}_{\mathscr{B}(X_0 X_0^\star)}\mathbb{B}(X_0)$ is Σ^*-algebra in $B(X_0^\star \otimes_{X_0 X_0^\star} \mathcal{K})$. Hence

$M(X_0^\star X_0)$ is a Σ^*-algebra in $B(\mathcal{H})$. That X_0 is a Σ^*-module over $M(X_0^\star X_0)$ follows by the other-handed version of Theorem 3.1.8.

The final statement follows from nondegeneracy and the usual argument using Lemma 2.2.8. $\square$

Lemma 5.4.4. *Suppose $\mathfrak{B}$ is a σ-closed ideal in a Σ^*-algebra $\mathfrak{C}$. If $\mathfrak{X}$ is a Σ^*-module over $\mathfrak{B}$, then $\mathfrak{X}$ is a Σ^*-module over $\mathfrak{C}$.*

Proof. This follows easily from a basic C^*-module principle (see [9, 8.1.4 (4)]) and Proposition 3.1.4. $\square$

Theorem 5.4.5. *Let X be a right C^*-module over a Σ^*-algebra $(\mathfrak{B}, \mathscr{S}_{\mathfrak{B}})$, and let $\mathscr{S}_X$ be a σ-convergence system on X such that $(X, \mathscr{S}_X)$ is a Σ-Banach space. If the following continuity conditions hold:*

(1) *the map $X \to \mathfrak{B}$, $x \mapsto \langle x|y\rangle$, is $\mathscr{S}_X - \mathscr{S}_{\mathfrak{B}}$-continuous for all $y \in X$;*

(2) *the map $\mathfrak{B} \to X$, $b \mapsto xb$, is $\mathscr{S}_{\mathfrak{B}} - \mathscr{S}_X$-continuous for all $x \in X$;*

(3) *the map $X \to X$, $x \mapsto xb$, is $\mathscr{S}_X$-continuous for all $b \in \mathfrak{B}$;*

then X is a Σ^*-module over $\mathfrak{B}$ with $((x_n), x) \in \mathscr{S}_X$ if and only if $x_n \xrightarrow{\sigma_{\mathfrak{B}}} x$.

Conversely, if X is a right Σ^*-module over a Σ^*-algebra $(\mathfrak{B}, \mathscr{S}_{\mathfrak{B}})$, and we declare $((x_n), x) \in \mathscr{S}_X$ if and only if $x_n \xrightarrow{\sigma_{\mathfrak{B}}} x$, then $(X, \mathscr{S}_X)$ is a Σ-operator space for which the three conditions above hold.

Proof. For the forward direction, it follows from Lemma 5.4.4 that it suffices to prove the result in the case that X is σ-full over $\mathfrak{B}$.

Fix a faithful, nondegenerate Σ^*-representation $\mathfrak{B} \subseteq B(\mathcal{H})$. View X as a TRO in $B(\mathcal{H}, X \otimes_{\mathfrak{B}} \mathcal{H})$. If $x_n \xrightarrow{\mathscr{S}_X} x$, then $\langle y|x_n\rangle \xrightarrow{\mathscr{S}_{\mathfrak{B}}} \langle y|x\rangle$ for all $y \in X$ by condition (1) in the assumptions. So by condition (2),

$$zy^*x_n = z\langle y|x_n\rangle \xrightarrow{\mathscr{S}_X} z\langle y|x\rangle = zy^*x$$

for all $y, z \in X$. Also,

$$x_n y^* z = x_n\langle y|z\rangle \xrightarrow{\mathscr{S}_X} x\langle y|z\rangle = xy^*z$$

for all $y, z \in X$ by condition 3.

We then have by Proposition 5.4.2 that $(X, \mathscr{S}_X)$ is TRO-isomorphic and Σ-isomorphic to a Σ^*-TRO $X_0 \subseteq B(\mathcal{K}_1, \mathcal{K}_2)$. By Lemma 5.4.3, X_0 is a Σ^*-module over the concrete Σ^*-algebra $M(X_0^\star X_0) \subseteq B(\mathcal{K}_1)$. Since $M(\langle X|X\rangle) \cong M(X_0^\star X_0)$ $*$-isomorphically (this follows from a fact mentioned in the fourth paragraph of Section 5.2.2), we can transfer the Σ^*-algebra structure of $M(X_0^\star X_0)$ to $M(\langle X|X\rangle)$. Letting $\mathscr{R}$ be the induced σ-convergence system on $M(\langle X|X\rangle)$, we have that X is a right Σ^*-module over $(M(\langle X|X\rangle), \mathscr{R})$, and by Lemma 5.4.3, $\zeta_n \xrightarrow{\mathscr{R}} \zeta$ in $M(\langle X|X\rangle)$ if and only if $x\zeta_n \xrightarrow{\mathscr{S}_X} x\zeta$ for all $x \in X$.

Since $\mathfrak{B} \subseteq B(\mathcal{H})$ is nondegenerate, $\langle X|X\rangle \subseteq B(\mathcal{H})$ is also nondegenerate by Lemma 4.3.6 since X is σ-full over $\mathfrak{B}$. So we may identify $M(\langle X|X\rangle)$ with

$$\{T \in B(\mathcal{H}) : T\langle X|X\rangle \subseteq \langle X|X\rangle \text{ and } \langle X|X\rangle T \subseteq \langle X|X\rangle\}$$

$*$-isomorphically. Clearly $\mathfrak{B}$ is contained in the latter. Let $j : \mathfrak{B} \hookrightarrow M(\langle X|X\rangle)$ be the inclusion. Since $\mathfrak{B}$ is an ideal in $B(\mathcal{H})$, $j(\mathfrak{B})$ is an ideal in $M(\langle X|X\rangle)$. So if we can show that $j : (\mathfrak{B}, \mathscr{S}_{\mathfrak{B}}) \to (M(\langle X|X\rangle), \mathscr{R})$ is a Σ^*-embedding, the desired result follows from Lemma 5.4.4.

Suppose that $b_n \xrightarrow{\mathscr{S}_{\mathfrak{B}}} b$ in $\mathfrak{B}$. By continuity condition (2), $xb_n \xrightarrow{\mathscr{S}_X} xb$ for all $x \in X$. So $j(b_n) \xrightarrow{\mathscr{R}} j(b)$, and thus j is $\mathscr{S}_{\mathfrak{B}} - \mathscr{R}$-continuous. Now suppose $j(b_n)$ is a sequence in $j(\mathfrak{B})$ such that $j(b_n) \xrightarrow{\mathscr{R}} \eta$ for some $\eta \in M(\langle X|X\rangle)$. In particular, the sequence (xb_n) is $\mathscr{S}_X$-convergent for all $x \in X$, so that $(\langle y|x\rangle b_n)$ is $\mathscr{S}_{\mathfrak{B}}$-convergent for all $x, y \in X$ by condition (1). Since $[\langle X|X\rangle\mathcal{H}] = \mathcal{H}$ by Lemma 4.3.6, it follows from Lemma 2.2.8 that there is a $b \in \mathfrak{B}$ such that $b_n \xrightarrow{\mathscr{S}_{\mathfrak{B}}} b$. This is enough (by a slight modification of Lemma 2.2.2) to conclude that j is a Σ^*-embedding.

The converse follows from Theorem 3.1.10 (1) and the definition of $\sigma_{\mathfrak{B}}$-convergence.

□

The following is a different kind of solution to the problem of determining when a C^*-module is a Σ^*-module. In the last result, we had to add several continuity assumptions, but we were able to conclude that the module was a Σ^*-module over a pre-specified Σ^*-algebra. In this result, we do not have as many assumptions (although we do still need continuity assumptions), but we are not free to fix the coefficient Σ^*-algebra from the start. (Also, cf. the first part of and discussion above Lemma 4.3.7.)

Proposition 5.4.6. *Suppose X is a full right C^*-module over a C^*-algebra A, and that $\mathscr{S}$ is a σ-convergence system on X such that $(X, \mathscr{S})$ is a Σ-Banach space and* *the maps $x \mapsto x\langle y|z\rangle$ and $x \mapsto z\langle y|x\rangle$ are $\mathscr{S}$-continuous for all $y, z \in X$. Let $\mathscr{T}$ be the σ-convergence system on $M(A)$ defined by declaring $((\zeta_n), \zeta) \in \mathscr{T}$ if and only if $((x\zeta_n), x\zeta) \in \mathscr{S}$ for all $x \in X$. Then $(M(A), \mathscr{T})$ is a Σ^*-algebra, and X is a Σ^*-module over $(M(A), \mathscr{T})$.*

Proof. View X as a TRO in $B(X, X \otimes_A \mathcal{H})$ for some faithful, nondegenerate representation $A \subseteq B(\mathcal{H})$. By Proposition 5.4.2, $(X, \mathscr{S})$ is TRO-isomorphic and Σ-isomorphic to a Σ^*-TRO X_0. Then $M(A) = M(\langle X|X\rangle) \cong M(X_0^\star X_0)$. By Lemma 5.4.3, the latter is a Σ^*-algebra over which X_0 is a Σ^*-module, and the induced σ-convergence system on $M(A)$ coincides with $\mathscr{T}$. The result follows. $\square$

We conclude with a related result generalizing the last sentence in Lemma 4.3.7.

Proposition 5.4.7. *Suppose X is a σ-full right C^*-module over a Σ^*-algebra $\mathfrak{B}$, and that $\mathscr{S}$ is a σ-convergence system on X such that $(X, \mathscr{S})$ is a Σ-Banach space meeting the three continuity conditions in Theorem 5.4.5. The Σ^*-algebra structure on $M(\langle X|X\rangle)$ from Proposition 5.4.6 is the unique Σ^*-algebra structure on $M(\langle X|X\rangle)$ such that the latter contains $\mathfrak{B}$ as a Σ^*-subalgebra.*

Proof. Suppose that $\mathscr{R}$ is an arbitrary σ-convergence system on $M(\langle X|X\rangle)$ such that $(M(\langle X|X\rangle), \mathscr{R})$ is a Σ^*-algebra containing $\mathfrak{B}$ as a Σ^*-subalgebra. Fix a nondegenerate, faithful Σ^*-representation $(M(\langle X|X\rangle), \mathscr{S}) \subseteq B(\mathcal{H})$. Then the restriction $\mathfrak{B} \subseteq B(\mathcal{H})$ is a faithful Σ^*-representation, and the restriction $\langle X|X\rangle \subseteq B(\mathcal{H})$ is nondegenerate. (A nondegenerate representation of $M(A)$ always restricts to a nondegenerate representation of A.)

We then have $\xi_n \xrightarrow{\mathscr{R}} \xi$ in $M(\langle X|X\rangle)$ if and only if $\xi_n \xrightarrow{WOT} \xi$ in $B(\mathcal{H})$, which happens if and only if

$$\langle y|x\xi_n\rangle = \langle y|x\rangle\xi_n \xrightarrow{WOT} \langle y|x\rangle\xi = \langle y|x\xi\rangle \text{ in } B(\mathcal{H}) \text{ for all } x, y \in X$$

by Lemma 2.2.8. The latter is equivalent to saying $x\xi_n \xrightarrow{\sigma_{\mathfrak{B}}} x\xi$ for all $x \in X$, which by the final phrase in the forward direction of Theorem 5.4.5 is equivalent to $x\xi_n \xrightarrow{\mathscr{S}} x\xi$ for all $x \in X$.

Thus $\mathscr{R}$ coincides with the σ-convergence system on $M(\langle X|X\rangle)$ from Proposition 5.4.6. $\square$

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