Toward a General Solution Verification Method for Complex PDE problem with Hands off Coding

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I. Introduction and Motivation

This paper addresses the challenge of Solution Verification (SV) and accuracy assessment for computing solutions of complex Partial Differential Equation (PDE) model. Our main target applications are bio-heat transfer and blood flow simulation problems. However, our long term goal is to provide a postprocessing package that can be associated with any existing numerical simulation package, for example, widely used commercial codes such as ADINA, Ansys, Fluent, Numeca, Star-CD etc... and provide an a posteriori error estimate to their simulation. Important design decision are based on simulation done with these software. Unfortunately, we know that to verify a numerical solution and provide a quantitative assessment on the numerical accuracy of the solution is difficult.

The problem of accuracy assessment is a necessary step that comes after the code verification step and before the code validation step to complete the global task of providing a reliable virtual experiment tool [37], [38].

Our major goal in this paper is to pursue our work on the design of a new method that offer a general framework to do solution verification efficiently. The standard approach in applied mathematics to handle the problem of solution verification is to work on the approximation theory of the PDE. For each specific PDE problem, the right Finite Element (FE) approximation may provide the correct a posteriori error estimate. Unfortunately this approach may require a complete rewriting of an existing CFD code based on Finite Volume (FV) for example and lack generality. Usually a posteriori estimators fails if the PDE solution is stiff or if the grid resolution is not adequate. Since grid refinement itself is based on a posteriori estimator, this pose an obvious problem. Large Reynolds number flow are common in many applications, not to mention turbulence problems. For those applications, rigorous solution verification may not be achievable by the current state of the art of numerical analysis. If the PDE problem gets complicated, for example, in the presence of fluid-structure interaction, or other complex multi-physic coupling, one faces the same problem. The general practice in scientific computing is to simulate PDEs for which applied mathematics neither numerical analysis guaranty the result. For a number of fluid dynamic methods used in bioengineering such as the immersed boundary technique, or the chimera technique, there is no solid theoretical framework that can provides such rigorous a posteriori estimators. For complex bioengineering problems, the fact that there exist a functional space framework to derive a posteriori estimate is more the exception than the generality.

Because of the time lag between the development of rigorous mathematical tools and the common scientific computing practice, our goal is to improve existing SV tools such as the convergence index of Roache et Al, and the Richardson Extrapolation (RE) technique [51], [50], that are used daily by practitioner, by something more elaborate and reliable that can take advantage of both existing a posteriori estimators when they are available, and new distributed computing tools since SV is computer intensive.

Our method relies on four main ideas that are (1) embedding of the problem of error estimation into an optimum design framework that can extract the best information from a set of two or three existing numerical results, (2) solve the problem as a (non)linear set of discrete equations to produce a general tool, and renounce on using the specific approximation theory used to compute the PDE solution since we usually have no access to the detailed knowledge of the internal structure of the code that produces the numerical solution, (3) provide a framework that can reuse any a posteriori estimator if they are available (4) take advantage of distributed computing (or grid computing) to get a cost effective SV.

While it is impossible to review the work done in the theory of a posteriori estimators in a very limited amount of space, we will try to give a brief overview of this work in Section 2. Section 3 will describe our new method, while Section 4 will present some numerical results for two benchmark problems. Section 5 is our conclusion.

II. Background and Context

From the applied mathematics point of view, a posteriori estimates have been studied for many years [5], [29], [54], [58], [49]. There is a vast literature on this subject. Most of this work has been done in the framework of finite element analysis, but there are some works on finite volume as well as finite differences available [55], [34]. A posteriori estimates are mainly used to drive adaptive mesh refinement and there is, however, still a lack of a reliable error indicator for Euler and Navier-Stokes calculations that gives estimates on given meshes. This is especially apparent when the grid resolution is inadequate. Grid refinement in these studies are, in fact, supposed to increase the reliability of the a posteriori estimate computation. As a matter of fact, we recall that straightforward local refinement of grids based on the dominant flow features such as shock waves, slip lines or stagnation points, by using indicators based on large gradients, may even lead to incorrect results [59].

A standard and well known procedure to establish a posteriori estimates is to solve local residual problems. The so-called "equilibrated residual method" is one of the most reliable and accurate techniques [4], [5], [31]. However, theory for this method has been essentially limited to linear PDEs. Examples of application in the literature include the Poisson problem, linear singular perturbation problem with simple boundary layers, the Stokes and Oseen equation and linear elasticity theory. This family of techniques, that has the advantage of being mathematically rigorous in the finite element framework, can be generalized, up to a certain point and on a case per case basis, to non-linear elliptic problems or hyperbolic equations [12], [1], [2], [3], [9], [8], [11], [13], [43], [48]. But it may not apply to finite volumes computation, where there is no equivalence theorem with finite element formulations. We also notice that the estimates given by the theory does not provide quantitative free bound on the error. In practice, the unknown constants in the error estimates obtained by this theory have naturally a bad asymptotic behavior for boundary layer problems as the disparity of scales increases, unless one refines the grid. Also, the method has to be significantly modified depending on the existence of additional length scales, such as boundary layers or not, or to take into account the influence of the error in discretization of non-homogeneous boundary conditions[5].

More recently, a general framework for finite element a posteriori error control that can be applied to linear and non-linear elliptic problems has been introduced by Patera et al.[33], [44], [45], [53]. This new theory focuses on the fact that one is not necessarily interested in the solution u itself, but rather in a linear (local) functional output $Q(u)$ or a stress $\sigma(u)$. A posteriori Finite-Element free constant output bounds can be constructed for the Incompressible Navier-Stokes equation. The procedure to construct the a posteriori estimate on a given triangular mesh of step 'H' uses a fine grid solution 'h' as a reference solution. The efficiency of this method is demonstrated in the construction of upper and lower bounds of linear-functional outputs of the PDE solution. Examples considered are the Helmholtz problem and the viscous Burgers equation in one space dimension, 1^{st} order linear convection in two dimensional spaces, and Navier-Stokes equation written in Boussinesq approximations in a highly convective flow regime.

This procedure uses the concept of duality, in which an equivalent dual adjoin formulation of the primal is exploited. The error in the functional can then be related to the local residual errors of the primal solution through the adjoin variable. This method is promising and technically impressive. Its implementation seems, at first sight, complex, and is restricted to the finite element framework with appropriate so-called "broken space" to relate the coarse mesh space with the fine mesh approximation. The results on the bounds hold for $H \to h$ but for all $H < H^*$ where H^* is generally an unknown threshold discretization parameter [33]. A traditional drawback is the fact that the coarse mesh solution H may not be good enough to provide useful information, and that is why the method does use refinement to build confidence in the output results. Therefore, a large Reynolds number is an issue in such a computation. Furthermore, a reliable error estimate on the fine grid solution of step 'h' does not necessarily lead on a reliable error estimate for the true solution, unless one has external knowledge to guess what will be a good, fine grid solution. Let us noticed that [56] and [57] present

a practical solution to the error estimate of functional outputs that is used to drive grid refinement on complex CFD problems. It is also an adjoin-based error correction procedure that is very close to the engineering point of view.

It seems that there is a lot of new activity on a posteriori estimates based on residual methods for the problem and/or its adjoin in the variational theory framework, but recovery methods that rely on building a better solution to derive an a posteriori estimate are still very much in use in the engineering world. Among these methods are the Richardson extrapolation technique [50] and the so-called ZZ Super Convergence Patch Recovery Method [39], [40] . These methods are applicable to linear as well as nonlinear problems [6], [7] [7]. As explained in [5] Sect 4.7 p82, this technique may require that the grid resolves the smallest scale.

A third stream of work strongly related to a posteriori estimates [24], [26], [27], [25], [30], [32], concern real complex phenomena that are nonlinear, stochastic and multiscale with no clear cut between the different scaling. Problems with turbulent flows, flow in porous media, or weather prediction are classical examples. The stochastic method of Glimm et al. for the prediction of complex phenomena divides them into two components. One is the forward problem, starting from the governing equation and initial data; the other is the inverse problem to minimize the uncertainties in the model from given observations of the system. The main thrust of the method is to predict functional output of the solution with coarse grid solution only based on a probability error model that includes error in the numerical computation, error in the observation or experiments used to calibrate the model, and error in the data used in the model. Randomness occurs at several levels such as the specification of the model or the solution process itself. The solution process and the model must support a probabilistic framework; that is why stochastic PDEs is a natural application field. [32] gives a detailed description of the method applied to scale up flow in porous media. To focus on the role of this method for a posteriori estimate of the numerical error in direct computation of PDEs, the assumption is that the numerical error can be divided into two components: first, a highly variable component with sensitive dependence on data, and second, a systematic component with smooth but also unknown dependence on the data. Both components are present in the random process which models the error. Sensitive dependence of the error on the data must be subsumed within the randomness of the error process. Further, the output functional that is the objective to be predicted must not be sensitive to the global error. The error estimate depends on the specific choice of the error statistic model. It seems that this approach is very useful when uncertainties in the model are dominant versus the numerical error, and when a fine grid solution is never accessible. In complex numerical methods such as PDF/Monte Carlo for turbulent flows [60], the construction of the real model for numerical accuracy is a difficult task. It is remarkable, that once it is done, one may minimize the main component of the error that is random via time-averaging. Then Richardson extrapolation can provide a significant improvement on the accuracy of the solution. Nevertheless, the theory developed by J.Glimm and co-workers is a giant step toward the understanding of the effect of combined observation errors, model error, and numerical simulation error in effective prediction.

All the works described above are extremely important contributions to the understanding of a posteriori estimates for PDEs. There are advantages and disadvantages for each of them. Method of choice depends very much on the complexity of the application and the amount of CPU time that is available. The main challenge is still *to estimate numerical accuracy on under-resolved grids* [38]. As a matter of fact, in complex modeling best grid solutions provided by our best computing resources can be fairly under-resolved at least locally [10], [28]. A posteriori estimates should be redesigned to provide solution verification assessment in this context [41], [42].

Furthermore there are still some basic physical mechanisms on propagation of the error that must be reintroduced in the analysis of the error such as, following [47] (1) diffusion terms cause slow isotropic error decay, but global error pollution may occur from local irregularities, (2) advection terms propagate local errors in the transport direction, but errors decay exponentially in the cross wind direction, and (3) reaction terms cause isotropic exponential error decay, but stiff behavior may occur in the coupling error components. These ideas have been used extensively to build efficient, parallel iterative domain decomposition solvers that optimize fast decay of numerical error introduced at artificial interfaces- for example see [14], [16], [21], [22] and its references. Therefore, it might be useful to keep track of specific error propagation even if for models in which (1) to (3) are present, it is

impossible to account for all error interactions by analytical means. Finally, there are still possibilities of catastrophic failures of a posteriori estimates in case of meta-stabilities or bifurcations [15] . A good method for complex modeling should therefore provide numerical indicators of ill-conditioning problems and potential catastrophic errors.

We present in this paper an entirely different framework with a similar goal that is to construct reliable a posteriori estimates for *general* PDEs or system of PDEs.

III. METHOD

Let us first describe the general concept of our method. This work is a generalization of the LSE method discussed in [17], [18], [19].

A. General Concept

We consider a boundary value problem (Ω is a polygonal domain and $n = 2$ or 3) :

$$N[u(x)] = f(x),\ x \in \Omega \subset \mathbb{R}^n,\ u = g \text{ on } \partial\Omega. \tag{1}$$

We assume that the PDE problem is well posed and has a unique smooth solution. We consider a finite volume approximation of (1) on a family of meshes $M(h)$ parametrized by $h > 0$ a small parameter. The smaller h, the finer should be the discretization. We denote symbolically the corresponding family of systems of discrete equations:

$$N_h U_h = F_h. \tag{2}$$

Let p_h denotes the projection of the continuous solution u onto the mesh $M(h)$. We assume a priori that ($||.||$ is a given discrete norm):

$$||U_h - p_h(u)|| \ \rightarrow\ 0, \text{ as } h \ \rightarrow\ 0, \tag{3}$$

Let $M(h_1)$ and $M(h_2)$ be two different meshes used to build two approximations U_1 and U_2 of the PDE problem (1). A consistent linear extrapolation formula should have the form

$$\alpha U_1 + (1-\alpha)U_2,$$

where α is a weight function. In classical RE the α function is a constant. In our optimized extrapolation method α is an unknown space dependent function solution of the following optimization problem, where G is an objective function to be defined:

$$P_\alpha\text{: } \textit{Find } \alpha \in \Lambda(\Omega)\ \subset\ L_\infty \textit{ such that } G(\alpha\ U_1\ +\ (1-\alpha)\ U_2) \textit{ is minimum.}$$

The Optimized Extrapolated Solution (OES) if it exists, is denoted $V_e = \alpha U_1 + (1-\alpha)U_2$. For computational efficiency, $\Lambda(\Omega)$ should be a finite vector space of very small dimension compared to the size of the system (2). The objective function G might be derived from any existing a posteriori error estimators if possible. Our ambition is to provide a numerical estimate on $||U_j - U_\infty||,\ j = 1, 2$, without computing U_∞ explicitly. The solution U_j can be verified then assuming (3). The fine mesh $M(h_\infty)$ should be set such that it captures all the scales of the continuous solution with the level of accuracy required by the application. We have a priori $h_\infty << h_1,\ h_2$. Both coarse grid solutions U_1 and U_2 must be projected onto $M(h_\infty)$. We will denote $\tilde{U}_1$ and $\tilde{U}_2$ the corresponding functions. We choose then to minimize the consistency error for the numerical approximation of (1) on a fine mesh $M(h_\infty)$. The objective function is then

$$G(U^\alpha) = ||N_{h_\infty}\ U^\alpha - F_{h_\infty}||, \text{ where } U^\alpha = \alpha\ \tilde{U}_1\ +\ (1-\alpha)\ \tilde{U}_2. \tag{4}$$

The choice of the discrete norm should depend on the property of the solution. In the LSE method [18], [19] we chose the discrete L_2 norm. The choice of the L_1 or the L_∞ norm has been tested for stiff elliptic problems [20].

One of the difficulties encountered with a two-level extrapolation method is the existence of subsets of $M(h_\infty)$ where $\tilde{U}_1$ and $\tilde{U}_2$ are much closer to each other than what the expected order of accuracy based on local error analysis should provide. In such areas the sensitivity of the extrapolation to the variation of α is very weak and the problem is ill posed. These subsets should be treated as outliers of the optimization computation procedure. A potentially more robust optimization procedure consists of using three levels of grid solution. The optimization problem writes then

$P_{\alpha,\beta}$: *Find* $\alpha, \beta \in \Lambda(\Omega) \subset L_\infty$ *such that* $G(\alpha\, U_1 + \beta\, U_2 + (1-\alpha-\beta)\, U_3)$ *is minimum.*

We notice that if all $U_j,\ j = 1..3$, coincide at the same space location there is either no local convergence or all solutions U_j are exact. In such a situation, one cannot expect improved local accuracy from any extrapolation technique. The robustness of the OES method is that no a priori assumption on the asymptotic formula on the convergence rate of the numerical method is made, as opposed to RE.

Let us assume that the optimization problem P_α or $P_{\alpha,\beta}$ has been solved and that the OES solution, V_e, has been obtained. We are going to discuss now its application to provide a posteriori error estimators.

Let us denote U_j to be one of the coarse grid approximations at our disposal. A global a posteriori estimate of the error $||U_j - p_h(u)||$ may come in two different ways. For the sake of simplicity, we will assume that L_2 is the discrete norm used to compute (4).

• First is the recovery method based on the idea that the optimized extrapolated solution is more accurate than the coarse grid solution: let us denote $\tilde{U}_j$ the coarse grid solution projected onto the fine grid $M(h_\infty)$ via a suitable interpolation procedure. Let us assume that the extrapolated solution is decisively more accurate than the one based on interpolation from the coarse grid solution, namely,

$$||V_e - p_h(u)||_2 \ll ||\tilde{U}_j - p_h(u)||_2. \tag{5}$$

Then $||\tilde{U}_j - V_e||_2 \sim ||\tilde{U}_j - p_h(u)||_2$ and $||V_e - \tilde{U}_2||_2$ is a good error indicator to assess the accuracy on U_2.

We have seen in our numerical experiments with steady incompressible Navier-Stokes (NS) solutions that this method may give a good *lower* bound error estimate. But we do not know in general if the hypothesis (5) is correct. There is no guarantee that a smaller residual for V_e than for U_2 on the fine grid $M(h_\infty)$ leads to a smaller error.

• Second is a global *upper* bound that follows from a stability estimate with the discrete operator. Let us assume that system (2) as well as the PDE problem (1) is linear. We denote A_h the matrix of the linear system. We have

$$||V_e - U^0||_2 \ < \ \mu\, G(V_e), \text{ where } \mu \geq ||(A_{h_\infty})^{-1}||_2$$

We conclude then

$$||\tilde{U}_2 - U^0||_2 \ < \ \mu\, G(V_e) + ||V_e - \tilde{U}_2||_2. \tag{6}$$

The procedure to derive an estimate for μ uses a combination of standard eigenvalue computation procedures applied to $A_{h_j},\ j = 1 \ldots 3$ and some extrapolation technique designed for scalar functions. (6) is a good global a posteriori error estimator provided that

$$||U^0 - p_h(u)||_2 \ << \ ||U^0 - \tilde{U}_2||_2. \tag{7}$$

One way to test the hypothesis (7) is to measure the sensitivity of the upper bound (6) with respect to the choice of the fine grid $M(h_\infty)$. This is a feasible test because the fine grid solution is never computed in OES. Our verification procedure checks that $||U^0 - \tilde{U}_2||_2$ increases toward an asymptotic limit as $M(h_\infty)$ gets finer.

The algorithm procedure to construct V_e solution of P_α or $P_{\alpha,\beta}$ is straightforward when the operator is linear and the objective function is the discrete L_2 norm of the residual. Let $e_i,\ i = 1 \ldots m$ be a set of basis function of $\Lambda(\Omega)$. The solution process can be decomposed into three steps.

1) Interpolation of the coarse grid solution from $M(h_j),\ j = 1 \ldots p$ to $M(h_\infty)$, with $p = 2$ for the two level method, respectively 3 for the three level method.
2) Evaluate the residual $R[e_i\ (\tilde{U}_j - \tilde{U}_{j+1})],\ i = 1 \ldots m,\ j = 1 \ldots p-1,$ and $R[\tilde{U}_p]$ on the fine grid $M(h_\infty)$.
3) The solution of the least square linear algebra problem that has m unknowns for each weight coefficient α and β used in the extrapolation procedure. In practice, m is much lower than the number of grid points on any coarse grid used.

We have generalized the LSE method to non-linear elliptic problems via a Newton like loop [18], [19]. We have also obtained preliminary results for unsteady parabolic problems [17]. Most of this work has been done on solutions produced by our own finite volume code on a fairly large variety of

linear and nonlinear PDE problems on structured grids. To apply these techniques on solution produced by commercial code that have thousands of lines, and work with unstructured grids requires a more general and abstract approach, that we present in the next section.

B. Solution Verification of off the shelf CFD code

We propose to generalize our method to time independent, i.e steady, CFD solutions produced by existing code. The challenge is that in most commercial codes, one cannot rely on the exact knowledge of the discretization method, neither have access to information on the internal structure of the code. What we propose is fundamentally different than existing methods. We describe in the following the main ideas without seeking an exact formal mathematical description of a given specific PDE problem.

Let $(E, ||.||_E)$ and $(F, ||.||_F)$ be two normed linear space, $N_h : E \rightarrow F$ be the operator corresponding to the CFD problem. Further let us denote $S \in F$ the input data of the CFD code and $U \in E$ the solution we are looking for.

In practice we look for an approximation of the accuracy of the solution U_h on the mesh $M(h)$ produced by the code $\mathcal{C}$ that operates on the data S_h:

$$\mathcal{C} : \ S_h \ \rightarrow U_h$$

The objective is still to get an error estimate versus a very fine grid solution U_∞ that is never computed, because the cost is prohibitive. We will skip the index h when it is not essential. The space $E,\ F$ have in practice (very large) finite dimensions since they are for the discrete solutions on $M(h_\infty)$, and discrete data S_{h_∞}.

We assume that the code $\mathcal{C}$ has a procedure that provides the residual, i.e $V \rightarrow \rho = N(U_h) - N(V)$, where $V \in E,\ \rho \in F$. We note that this hypothesis is realistic, since most of the commercial code offer this feature or either provides a (first order explicit) time stepping procedure:

$$\frac{U_h^{n+1} - U_h^n}{dt} = N(U_h^n) - S, \tag{8}$$

The residual is then $\rho = \frac{U_h^1 - U_h}{dt}$. We assume that the following problem

$$N(u) \ = \ s, \ \forall s \in B(S, d).$$

is well posed for $s \in B(S, r)$, where B is a ball of center S and radius r in $(F, ||.||_F)$. There should exist a unique solution for all data in $B(S, r)$ and the dependency of the solution on these data is supposed to be smooth enough to use a second order Taylor expansion.

Let us suppose that $N(U_h) \in B(S, r)$, that is

$$||\rho||_F = ||N(U_h) - S||_F < d. \tag{9}$$

We would like to get an error estimate on $e = U_h - U_\infty = \mathcal{C}(S + \rho) - \mathcal{C}(S)$. A Taylor expansion writes

$$\mathcal{C}(S) \ = \ \mathcal{C}(S + \rho) - (\rho \cdot \nabla_s)\mathcal{C}(S + \rho) + \frac{1}{2}\rho \cdot [\rho \cdot R(S)] \tag{10}$$

$$\text{where } ||R(S)||_E \ \leq \ K = \sup_{s \in B(S,d)} ||\nabla_s^2 \mathcal{C}(s)||_E. \tag{11}$$

Therefore

$$||e||_E \ \leq \ ||\rho||_F \ (||\nabla_s \mathcal{C}(S + \rho)||_E \ + \ \frac{K}{2}||\rho||_F). \tag{12}$$

This completely general error estimate point out to two different tasks:

• Task 1: compute an accurate upper bound on $||\nabla_S \mathcal{C}(S + \rho)||$

• Task 2: obtain a solution $U_\infty + e$ that gives a residual $||\rho||$ small enough to make the estimate useful, i.e compatible with (9).

Task 2 is the purpose of the OES method, while Task 1 can be achieve by a perturbation method that can use of $\mathcal{C}$.

C. Task 1: Stability Estimate

Let $\{b_i^E,\ i=1..N\}$, (respt. $\{b_i^F,\ i=1..N\}$) be a basis of E_h, (respt. F_h) and $\varepsilon \in I\!R$ such that $\varepsilon = o(1)$. Let $(V_i^{\mp})_{i=1..N}$, be the family of solutions of the following problems:

$$N(U_h \mp \varepsilon V_i) = S + \rho \mp \varepsilon b_i$$

We get from finite differences the approximation

$$C_{h_\infty} = ||\nabla_S \mathcal{C}(S+\rho)|| \approx ||(\frac{1}{2}(V_j^+ - V_j^-))_{j=1..N}|| + O(\varepsilon^2). \tag{13}$$

We can get in as similar manner an approximation of the norm of the Hessian $\nabla_s^2 \mathcal{C}(S+\rho)$. For ρ small enough, we can verify that the upper bound is given at first approximation by:

$$||e||_E \ \leq\ C_{h_\infty} ||\rho||_F. \tag{14}$$

The column vectors $V_j^{\mp}$ can be computed with embarrassing parallelism. It is, however, unrealistic to compute these solutions that lies on the fine grid $M(h_\infty)$.

To make this task manageable, we have to reduce the dimension of the problem. We use the following two observations. While the solution of the CFD problem can be very much grid dependent, the conditioning number of the problem is in general much less sensitive to the grid. The idea is then to compute an approximation of C_{h_∞} by extrapolation from an estimate of two or three coarse grid computation of C_{h_j}. Further, let us assume that the fine grid $M(h_\infty)$ is a regular Cartesian grid. The number of terms to represent accurately the projected solution $\tilde{U}_j,\ j=1..3$ with a spectral expansion or a wavelet approximation at a given accuracy is much less than the dimension of the coarse grid used in a Finite Element/Finite Volume computation. We propose to use preferably a grid $M_{h\infty}$ that has enough regularity to allow a representation of the solution U_∞ with some form of reduced representation, using either trigonometric expansion or wavelets.

The grid M_{h_∞} may have many more grid points than necessary, and therefore it might not be computationally efficient to perform a true fine grid computation. But we do not have to do this computation anyway.

Further, regular grids are far more easy to construct. If for some reasons $M_{h\infty}$ has to be unstructured, we can also use spectral elements. An ideal method might be to use a proper orthogonal decomposition that captures the main feature of the solution [23].

Let us denote $\hat{E}$ and $\hat{F}$ the spaces corresponding to one of these compact representation of the solution and residual. Let $(\hat{b}_j^{E/F}, j=1..\hat{N})$, be the corresponding base with $\hat{N} \ll N$. Let $\hat{q}_{E/F}$ be a mapping $E/F \to \hat{E}/\hat{F}$, respectively $q_{\hat{E}/\hat{F}}$ be a mapping $\hat{E}/\hat{F} \to E/F$, and let $\hat{\mathcal{C}}:\ \hat{S}_h \to \hat{U}_h$ be the code that uses this postprocessing of the residual and solution.

Figure 1 illustrates the relation between the different discrete spaces, along with the corresponding mappings. The mapping $\hat{q}_E$ can be a least square approximation of the solution u into $\hat{E}$, acting as a filter on the solution, while $q_{\hat{E}}$ is a projection onto E.

The construction of q_E and $\hat{q}_E$, respectively q_F and $\hat{q}_F$ does not consider the nature of the true approximation space used in the code $\mathcal{C}$ since the implementation details are most of the time unavailable: the mappings involve only the discrete representations of the functions.

To summarize the procedure for Task 1, The estimate on $\mathcal{C}_{h_\infty}$ will be applied to verify the code $\hat{\mathcal{C}}$ based on the computation of $(\hat{V}_j^{\mp}, j=1..\hat{N})$ vectors on the coarse grids $M(h_j),\ j=1..3$ done by the code $\hat{\mathcal{C}}$. We notice that the computation of the vector $\hat{V}_j^{\mp}$ can be done with embarrassing parallelism. Further, because ε is small the code $\hat{\mathcal{C}}$ can use as an initial guess in its iterative process the solution U_h that is hopefully very close to the unknown $\hat{U}_h \pm \epsilon \hat{V}_j^{\mp}$.

There are several issues that needs to be carefully investigated when applying this procedure. Let us mention two of them. First, our method assumes that the spectral properties of the operator follows some fairly regular asymptotic properties for the coarse meshes under consideration as $h \to 0$. This hypothesis should be verified. Second, ε must be chosen carefully as a function of the mesh size, in order to avoid a dramatic inaccuracy on the stability estimate of C_{h_∞}. We recall however that we are only looking for an order of magnitude of C_{h_∞} and not its true value.

Let us discuss our second task that is to compute a solution on the fine grid that is good enough to recover an error estimate.

D. Task 2: Optimized Extrapolation

We use here an optimized extrapolation method. To reduce the dimension of this problem we search for the unknown weight functions in a small space that can be described either by trigonometric expansion, or wavelet expansion, or possibly spectral elements. If Ω is the physical domain for the CFD solution, the unknown weight function can be extended to a square domain $(0,1)^n$, such that $\Omega \subset (0,1)^n$. As a matter of fact no boundary conditions are required on the unknown weight functions. Let $\{\theta_j,\ j = 1..m\}$ be the set of basis function of $\Lambda(\Omega)$. We look for the solution of the optimization problem in the two level case

$$\text{Find } (\alpha_j) \in I\!R^m, \text{such that } ||G([\sum_{j=1..m} \alpha_j \Theta_j]\tilde{U}_1 + [1 - \sum_{j=1..m} \alpha_j \Theta_j]\tilde{U}_2)||_F \text{ is minimum.} \quad (15)$$

We have a similar formulation for the three level OES. Following the same argument as before, we will rather look for this minimum in $\hat{F}$. As shown in [18], [19], we need a filtering process of the solution to have this minimization procedure numerically efficient. In practice the interpolation of the coarse grid solution to the fine grid M_{h_∞} introduces spurious high order oscillations that may make the optimization process (15) unreliable. The postprocessing $\hat{q}_F$ regularized the problem. We can obtain easily the result when the weight function is a scalar function. To make this computation robust we use a surface response methodology [36] that is rather trivial in the scalar case. This procedure consist to compute a lower order polynomial best fit of the function $||G(\alpha\tilde{U}_1 + (1-\alpha)\tilde{U}_2)||$ by sampling α according to the expected convergence order range of the code. The minimization on α is then done on this polynomial approximation by a standard method. The sampling process is a cumbersome embarrassing parallel process that can take advantage of a computational grid [46].

It is, however, impractical when the dimension of the problem for the α search is more than few units. One may use a combination of genetic algorithm and local optimization search to solve the nonlinear optimization problem (15). There is an extensive knowledge and set of available optimization software [35] that we can reuse indeed. We have observed in our experiments that if the solution provided by the CFD code is very coarse, the use of space dependent weight functions might not be required. Similarly, if the solution is very accurate and the code has uniform convergence, we do not need either space dependent weight function. But, in the case of stiff problems we would like to get some adaptivity on the construction of the weight function. This is an open problem that we are currently working on.

Let us describe now the design of the software that we are developing as a solution verification system independent of the CFD code.

IV. Software and Numerical Examples

Let us present first the algorithm.

A. Algorithm

The algorithm of our method writes in its simplest version

1) *Call coarse Mesh* : generate the (coarse) meshes $M(h_1)$ and $M(h_2)$. If h_i is the average space step for the grid $M(h_i)$ we should have $h_2 < h_1$ but this is not necessary.
2) *Call fine Mesh* : generate a fine mesh $M(h_\infty)$ that is supposed to solve all the scales of the problem. $M(h_\infty)$ is preferably a structured mesh. We must have $h_\infty << h_1, h_2$.
3) *Call Solver* : solve the problem on $M(h_1)$ and $M(h_2)$, possibly in parallel.
4) *Call Projection* : project the coarse solutions U_1 and U_2 onto $M(h_\infty)$ and post-process them to avoid spurious oscillations due to the interpolant.
5) *Solve minimization problem* (15) : we can create, for example, sample solutions $U_\alpha = [\alpha\tilde{U}_1 + (1-\alpha)\tilde{U}_2]$ and/or use an off the shelf optimization package.
6) *Get Stability Constant*: compute in parallel an increasing set of perturbated solutions $U_h \pm \varepsilon V_i^{\pm}$ until eventually convergence of the stability estimate.

Step 5 and Step 6 of our algorithm are source of large set of cumbersome computations that can be done in parallel with a minimum of synchronization. This is a key feature to make our solution verification cost effective. We have designed our verification software to run on the web.

We use a standard client/server/slave architecture. The a posteriori estimate takes then a very small fraction of the time that it would take to compute the fine grid solution. More details on this parallel implementation can be found in [46].

We are now going to illustrate the numerical performance of our a posteriori estimator. We have selected two test cases, one in fluid dynamic and one in heat transfer. We will carry out our method starting from two coarse grid calculations only. Both test cases are solved with the finite element commercial package named ADINA . ADINA is a comprehensive finite element software that enable analysis of structures, fluid simulations , and fluid flow simulations with structural interactions. More information can be found at http://www.adina.com/. We are going to consider first a steady incompressible viscous flow in the backward facing step configuration.

B. Backward-facing step flow test case

The incompressible flow is governed by the following equations

$$\left\{ \begin{array}{rcl} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} + \nabla \mathbf{p} & = & \nu \Delta \mathbf{u} \\ \nabla \mathbf{u} & = & 0 \end{array} \right. \tag{16}$$

where ν is the viscosity of the fluid.

Figures 2 and 4 show some example of an unstructured mesh and results of the calculation of the backward-facing step flow at Reynolds number 500. The length of the cavity is $L_x = 10$, the width is $L_y = 2$, and the size of the step is 1. The inflow is set to be a Poiseuille flow with maximum velocity 1 *U.I.* The outflow is left free. We assume no slip boundary conditions on the walls. To provide a rigorous measurement of the error we proceed as follow.

- we compute a solution (U_{FE}, p_{FE}) on a relatively fine grid, with 10347 elements, denoted $M(h_\infty)$.
- we construct a near by manufactured solution [52]
- we construct a modified Navier Stokes problem that has the manufactured solution as an exact analytical solution. The flow speed is then prescribed on all boundaries.
- we apply the algorithm to compute an a posteriori estimate to this modified NS flow problem as described in Section IV-A.

Thanks to the manufactured solution, we can verify with no ambiguity the upper-bound obtained with our a posteriori error estimator.

The steady numerical solutions are obtained using a transient scheme for the incompressible Navier-Stokes equation with ADINA.

Let us describe the process to construct our near by manufactured solution that differs in the details from[52]. The general principle of this method is to generate a couple of continuous function (u^{ms}, p^{ms}) that approximates the FE numerical solution of the PDE. This solution is then used into the NS set of equations (16). We obtain an analytical formula for the residual denoted $\mathbf{F}_s$. $\mathbf{F}_s$ is added as a source term in (16). (u^{ms}, p^{ms}) is then an exact analytical solution of the following problem:

$$\left\{ \begin{array}{rcl} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} + \nabla p & = & \nu \Delta \mathbf{u} + \mathbf{F_s} \\ \nabla \mathbf{u} & = & 0 \\ u & = & \Phi \text{ on } \partial\Omega. \end{array} \right. \tag{17}$$

We look here for a trigonometric approximation of the numerical solution in the form:

$$\left\{ \begin{array}{rcl} u^{ms} & = & \sum_{i=0}^{N_1} \alpha_i \mathbf{u}_i \\ p^{ms} & = & \sum_{i=0}^{N_2} \beta_i p_i \\ \nabla \mathbf{u}_i & = & 0 \end{array} \right. \tag{18}$$

where $\alpha_i, \beta_i \in I\!R$.

We choose the following set of basis functions

$$\mathbf{u_{k_1}} \quad = \quad \left\{ \begin{array}{c} cos(k_1 \pi \frac{x}{L_x}) sin(k_1 \pi \frac{y}{L_y}) \\ -sin(k_1 \pi \frac{x}{L_x}) cos(k_1 \pi \frac{y}{L_y}) \end{array} \right. \tag{19}$$

that satisfies the divergence free condition and

$$p_{k_2} \;=\; cos(k_2\pi\frac{x}{L_x})cos(k_2\pi\frac{y}{L_y}). \tag{20}$$

We proceed then in two steps. First, we interpolate the flow field variable of the fine mesh FE solution onto a regular grid of Ω that is globally about the same mesh size to ease the fitting with the trigonometric functions of (19). We note $(\tilde{U}_{FE}, \tilde{p}_{FE})$ the discrete functions obtained on this grid. Second, we use a least square interpolation on this regular grid function to calculate the weight coefficients α_i and β_i. Ω is a subset of the rectangle $(0, L_x)\times(0, L_y)$ and the numerical convergence is not granted: we have to verify a priori if this interpolated solution is giving an accurate approximation of the numerical solution.

The discrete error for the backward-facing step flow between the numerical solution and its trigonometric approximation is shown in figure 5. Let $M(h_j), j = 1 \ldots N$ be the nodes of the mesh. We use the standard discrete vector norm $||\;||_1,\; ||\;||_2,\; ||\;||_\infty$, to measure the numerical error on the trace of the numerical solution at those points. To be more specific

$$||(f_j)_{j=1..M}||_1 \;=\; \frac{1}{M}\;\sum_j |f_j|, ||(f_j)_{j=1..M}||_2 \;=\; \sqrt{\frac{1}{M}\;\sum_j (f_j)^2}, ||(f_j)_{j=1..M}||_\infty \;=\; \max_j |f_j|.$$

These norms are unrelated to the FE space used in ADINA calculation. Our verification procedure follows the spirit of an experimental measurement where one may get information at specific locations. Figure 3 shows the evolution of the error in L_2 norm as a function of the number N_1 and N_2 of basis functions. We choose $N_1 = 38$ for the velocity and $N_2 = 33$ for the pressure that gives some small differences with $(\tilde{U}_{FE}, \tilde{p}_{FE})$. Specifically we have

$$||\tilde{U}_{FE} - u^{ms}||_2 = 1.76e^{-04},\; ||\tilde{U}_{FE} - U^{ms}||_\infty = 2.12e^{-03}$$

and

$$\frac{||\tilde{p}_{FE} - p^{ms}||_2}{||\tilde{p}_{FE}||_2} = 2.25e^{-03}.$$

The expression of F_s is computed symbolically using Maple for each element of the basis and its expression is injected into ADINA as a forcing term in the momentum equation. We have now a manufactured solution for the back step that is near by the numerical solution obtained with the FE code on a very fine mesh. Let us compute the coarse grid solution of the corresponding new NS problem with RHS F_s, and construct the a posteriori error estimate. The only motivation to work with this new NS problem is to provide the exact numerical accuracy.

We proceed with the a posteriori estimate calculation using the algorithm of Section IV-A: we compute two coarse mesh solutions of problem (17) with respectively 1506 and 2630 elements. Both meshes are generated automatically by ADINA and are not coincident. Both solution are interpolated to the fine mesh $M(h_\infty)$ with a P1 method. We postprocess each solutions on the fine grid with few explicit time stepping reusing ADINA on the fine grid. This explicit time stepping is used as a relaxation procedure to damp all the spurious high frequencies of the solution introduced by the P1 interpolation. Its efficiency decreases as the Reynolds number increases. Practically, the residual decreases extremely fast during the first relaxations that get rids of the perturbation introduced by the P1 interpolation. After this initial stage the convergence speed slow down significantly. The turning point in the convergence behavior set up our stop criterion. A small number of explicit time steps provides by no means a convergence to the fine mesh solution. In our test case we have 10 to 30 time steps. To avoid this time stepping, one might use a high order interpolation procedure on the coarse meshes solutions, such as Bspline, to impose the smoothness of the solution on the fine mesh. This is in practice far more expensive than the relaxation steps and increase the complexity of the code.

Based on these two coarse grid solutions that we have postprocess on $M(h_\infty)$, we construct now the OES $\alpha\tilde{U}_1 + (1-\alpha)\tilde{U}_2$. We restrict ourselves to scalar weight function. The expected order of convergence is between 0 and 3, and we look for $\alpha \in \Lambda = (-0.2\;,\; 2.2)$.

Figure 6 shows the evolution of the true error as a function of the residual in the L_2 norm for α varying in Λ. One can observe that there is no need to postprocess every linear combination $\alpha\tilde{U}_1 + (1-\alpha)\tilde{U}_2$ with further explicit time stepping. Figure 7 shows that minimizing the error is

reasonably achieved by minimizing the residual for both L_1 and L_2 norm. This correspondence is less well satisfied in the L_∞ norm as shown by Figure 8. We speculate that this discrepancy comes from the fact that the NS solution is not smooth along the back step wall at the corners. It also appears that the optimal combination α is weakly dependent on the choice of the discrete norm used.

The computation of the stability estimate described in Section III-C uses the following set of basis function $b_i(x/L_x)b_j(y/L_y)$, where

$$b_0(z) = 1,\ b_1(z) = cos(\pi z),\ b_2(z) = sin(\pi z),\ b_3(z) = sin(2\pi z), \ldots b_j(z) = sin((j-1)\pi z), \ldots$$

We compute the stability estimates or condition numbers with both coarse grids. The amplitude of the perturbation ϵ is set to 0.01. We observe that larger values such as $\epsilon = 0.1$ are leading to oscillation in the computation of the stability constant. Smaller values of ϵ than 0.01 do not provide a sufficient amplitude for the perturbation to have a significant impact on the computed solution.

Because the sensitivity analysis is done with the initial two coarse grid calculations, we use very few time steps starting from the unperturbed coarse grid calculation. From the two condition numbers corresponding to each coarse grid solution we extrapolate the stability constant for the NS calculation on the fine grid mesh M_∞. We apply the same optimal α linear combination than the one obtain in the OES. The number of basis function to get the stability estimate is of order 600.

Figure 9 shows the convergence behavior for the L_2 norm of the computation of the conditioning number. The estimate growths as the number of basis function growths until it reach a plateau. The dimension of the reduced space of approximation $\hat{E}_h$ of the solution growths accordingly.

Using this stability estimate, one can derive the a posteriori error for the OES.

Figure 10 gives the error for the OES computed in the reduced space to the fine grid solution in L_2 norm. The low horizontal line is the true error in L_2 norm for the OES obtained previously. One observe that the extrapolation on the stability constant that combines the calculation of the estimate on both coarse grids with the best α obtained in the OES process does improve the accuracy of the error estimate. The fundamental result in this figure is that we can provide an upper bound on the numerical error of the simulation for each coarse grid calculation within 10% of the 'true' error. In particular, it was not necessary here to use three coarse grid solutions to deal with the cancelation problem.

Similar results are obtained for other norms, as shown in Figures 11 and 12. As observed previously the quality of the estimate is less satisfying for the L_∞ norm, but it still provides an error estimate for the coarse grid calculation with 10%. These results show that our procedure is independent of the norm used to do the calculations, and more specifically, it does not required to know the norm of the finite element used by ADINA in that case. We observe also that the error estimate in any of these discrete norm requires the same calculation. Once we have generated the surface response for α and the sensitivity calculation for NS, one can provide the a posteriori error estimate for all three discrete norms $L_1,\ L_2,\ L_\infty$ at once.

Based on the OES we can proceed by giving error estimate on each coarse grid solution. This would not have been possible with the Richardson extrapolation that has inaccuracy. We observe also that the OES can be used as an initial guess for a new NS calculation on a refined grid if the a posteriori error estimate does not fit the goal of the simulation.

One of the drawback of our method is that one has to compute numerous residual to get the stability estimate. The computation load is then fairly high. For instance, for a perturbation space containing 600 basis functions, we need to run 2400 times the NS code with different small perturbation source terms. As a matter of fact, we have two coarse grid calculations and the Jacobian in (13) is computed with central finite differences.

However, each of these computations can be done with embarrassing parallelism on a grid of computers. We have developed a software system to launch the simulation trough a web interface on an heterogeneous network of PCs, servers and Beowulf clusters. The description of this computer software is beyond the scope of this paper. The parallel efficiency of this system has been reported in [46].

While we have applied our a posteriori calculation on a modified NS problem that has an exact solution, we could also have done the same work on the initial NS problem. We did this simulation and we obtain then very similar results. Next, we present results for a heat transfer problem that is a more stiff problem to solve.

C. Heat Transfer test case

The model, for our second example is governed by the energy equation:

$$\frac{\partial}{\partial x_i}\left(k_{ij}(T)\frac{\partial T}{\partial x_j}\right) + Q(T) = \rho c_p(T)\frac{\partial T}{\partial t} \text{ on } \Omega \times (0,t)$$

with i, j running from 1 to 2 for this model. This two dimensional problem is solved in a square domain $\Omega = (0, L_x) \times (0, L-y)$. T is the temperature, t is time, ρ is the material density, c_p is the specific heat as a function of T, Q is the volumetric heat source as a function of t, and k_{ij} is the thermal conductivity tensor as a function of T.

The boundary conditions are :

- $-\left(k_{ij}\frac{\partial T}{\partial x_j}\right)\cdot n = h_1(T)(T-T_\infty) + \sigma\varepsilon_1\left(T^4 - T_\infty^4\right)$, on Γ_{N_1} (radiation, convection);
- $-\left(k_{ij}\frac{\partial T}{\partial x_j}\right)\cdot n = h_2(T)(T-T_\infty) + \sigma\varepsilon_2\left(T^4 - T_\infty^4\right)$,on Γ_{N_2} (radiation, convection);
- $-\left(k_{ij}\frac{\partial T}{\partial x_j}\right)\cdot n = h_3(T)(T-T_\infty)$, on Γ_{N_3} (convection),

where $T_\infty = 313.0K$, $\varepsilon_1 = \varepsilon_2 = 0.25$, $\sigma = 5.670 \times 10^{-8}$ is the Stefan-Boltzmann constant $(W \cdot m^{-2} \cdot K^{-1})$, and $h_3 = 1.0$ $(W \cdot m^{-2} \cdot K^{-1})$. The functions $h_1(T)$, $h_2(T)$, and c_p are given by tables.

The temperature is initialized to $T_0 = 313.0K$ in the structure. The difficulty of this study is due to the fact that the structure is compounded of different materials, for which coefficients might depend on space and temperature, and finally, two regions will undergo a phase change. The problem is therefore very stiff and the solution is almost discontinuous near the wall as shown in Figure 13.

We are only interested here in the steady state solution. Our result for this test case will be very similar to the previous one with NS. We will give then the main results with less details.

To simulate the heat transfer in this structure, we have used quad elements in each physical subdomain. The total number of elements in the grids $M(h_j),\ j = 1..2$, are respectively 8767 and 21072. To validate our result we compute a fine grid solution with a mesh of 57258 elements for $M(h_\infty)$. All meshes here are unstructured and present large aspect ratio. The fine grid computation is 10 times more expensive to compute than the two coarse grid solution process. This fine grid computation becomes intractable in 3D unless one use a large parallel system. In this case, also, we have shown that OES outperforms RE (Figure 14 and 15). We have been able to get a robust upper bound on the error following the strategy described in Section III-C that is reliable even with relatively coarse grids $M(h_i),\ i = 1, 2$.

In order to evaluate the stability constant for this problem, perturbation source terms need to be added to the different equations of the model, including boundary conditions. The modified equations are given by

$$\frac{\partial}{\partial x_i}\left(k_{ij}(T)\frac{\partial T}{\partial x_j}\right) + Q(T) = \rho c_p(T)\frac{\partial T}{\partial t} + F_s \text{ on } \Omega \times (0,t)$$

The boundary conditions are :

- $-\left(k_{ij}\frac{\partial T}{\partial x_j}\right)\cdot n = h_1(T)(T-T_\infty) + \sigma\varepsilon_1\left(T^4 - T_\infty^4\right) + F_{s_1}$, on Γ_{N_1} (radiation, convection);
- $-\left(k_{ij}\frac{\partial T}{\partial x_j}\right)\cdot n = h_2(T)(T-T_\infty) + \sigma\varepsilon_2\left(T^4 - T_\infty^4\right) + F_{s_2}$,on Γ_{N_2} (radiation, convection);
- $-\left(k_{ij}\frac{\partial T}{\partial x_j}\right)\cdot n = h_3(T)(T-T_\infty) + F_{s_3}$, on Γ_{N_3} (convection),

The perturbation functions chosen are, as in the previous test case, trigonometric functions.

Figures 9 and 10 gives respectively the stability estimate as the function of the size of the number of basis functions used and the error estimate in L_2 norm. We need of the order of 400 basis functions to reach a plateau in the error estimate. Once again we obtain an a posteriori error estimate on the coarse grid solution within 10% of the 'true' error versus the reference solution on $M(h_\infty)$. To be more specific, we are able to estimate a solution with an absolute error of 0.2 Kelvin, using coarse grid solutions with absolute errors of 10 Kelvin.

Moreover, this evaluation can be done efficiently since the estimations are done on a vector space of dimension 400 compare to the size of the FE fine mesh (57258 elements). Each coarse grid

computation for a set of perturbation functions was done in an average time of 2 minutes, for the finest of the two coarse grids, on a standard PC, leading approximately to a total elapse time of 20 hours for the error estimate. The fine grid computation, that is our reference solution, is also taking on its own 20 hours. The a posteriori estimate procedure seems, at first site, not so efficient. However, as mentioned earlier, our a posteriori error estimate procedure has embarrassing parallelism and can be distributed easily on a network of PCs. The overall elapse time for the solution verification is then considerably less than the elapse time to compute a fine grid solution.

The interesting point in the numerical results of Section 4.2 and 4.3 is that the same framework works for two completely different set of PDEs and lead to similar results. It seems to show that our a posteriori error estimate method is a fairly robust procedure. But we need indeed to test our algorithm further on many more benchmark problems.

V. Conclusion

In this paper, we have explored a new framework to provide a posteriori estimation for CFD simulations produced with a commercial package. The challenge comes from the fact we do not have access to the source code, neither we may not know precisely what FE approximation is used. In such context the only method that is used by practitioner is mesh refinement and eventually a Richardson extrapolation procedure. This process is time consuming and may not even be possible for complex PDE problems. We have proposed an alternative solution that still use a fine mesh as a reference, but do not require the calculation of the CFD solution on this fine mesh.

Our method starts from the calculation of two or three solutions and search for an optimized extrapolation on a fine mesh that is assumed to resolve all scales of interest. With this process we can retrieve the best information from coarse grid solutions. Because we solve the problem as a (non)linear set of discrete equations and ignore the FE framework, that may not be at disposal for our specific application, our method is extremely general and can use all kind of discrete norm to measure the error. A sensitivity analysis is necessary to complete the work by providing the error estimate for the best consistent extrapolation solution we found.

The drawback of our method is that it is computationally intensive and requires hundred of evaluation of the residual. On the other hand this process has embarrassing parallelism and constitute an effective way of using a grid of computers. Both OES and sensitivity analysis presented in this paper can be speedup by using state of the art optimizers. This new development is part of our ongoing research. We are also currently developing the software that can process our a posteriori estimate with web computing. This will allow to test our technique on a larger number of test cases with increasing complexity.

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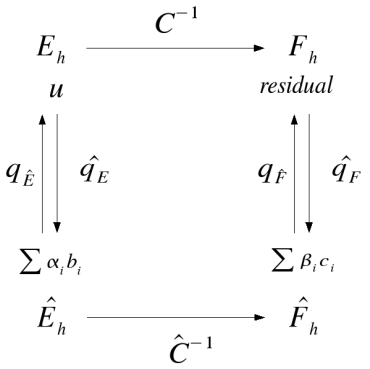

Fig. 1. Mappings and vector spaces.

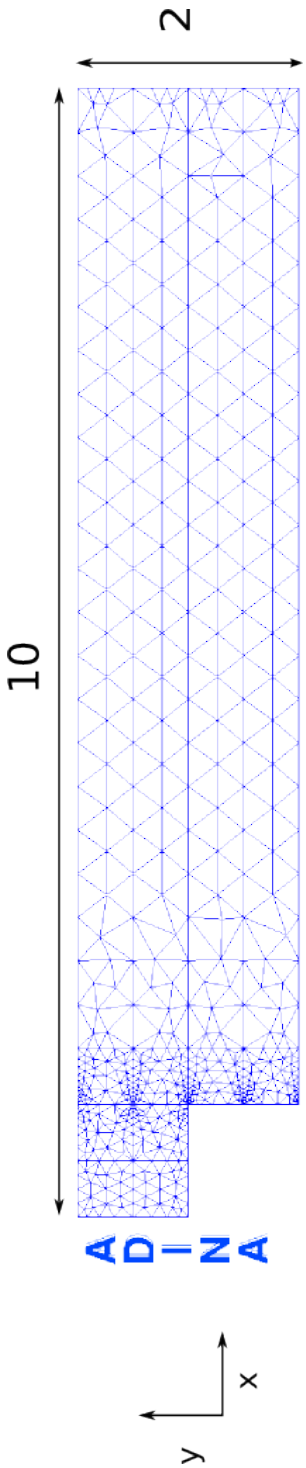

Fig. 2. Coarse unstructured mesh for the backstep test case generated by Adina.

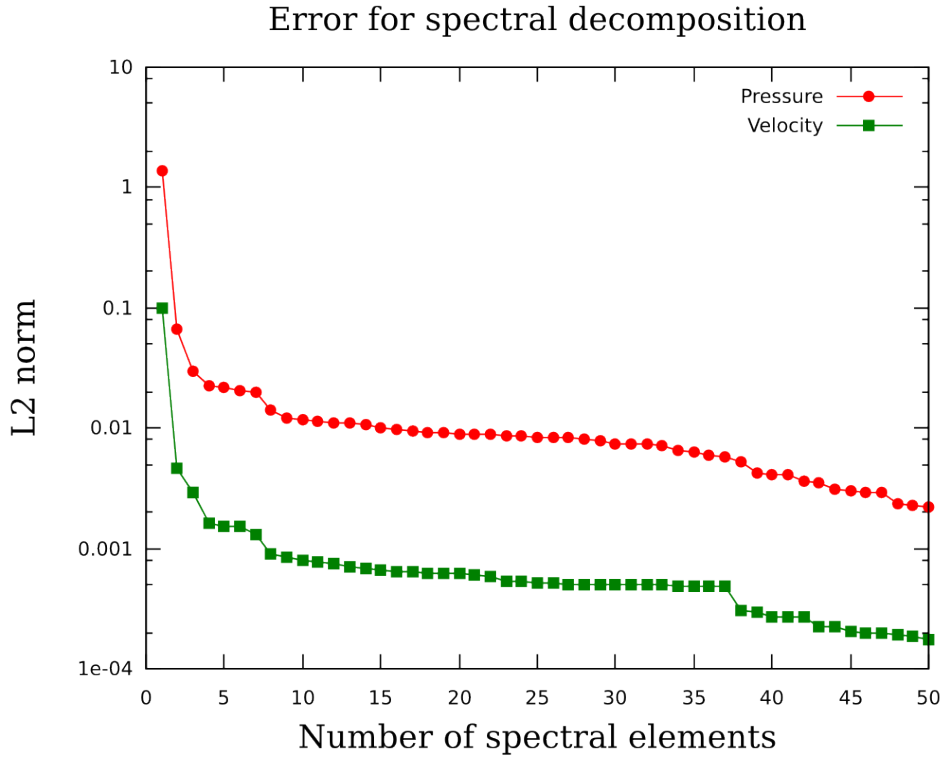

Fig. 3. Numerical error in L_2 norm as a function of the number of trigonometric basis function of the near by manufactured solution.

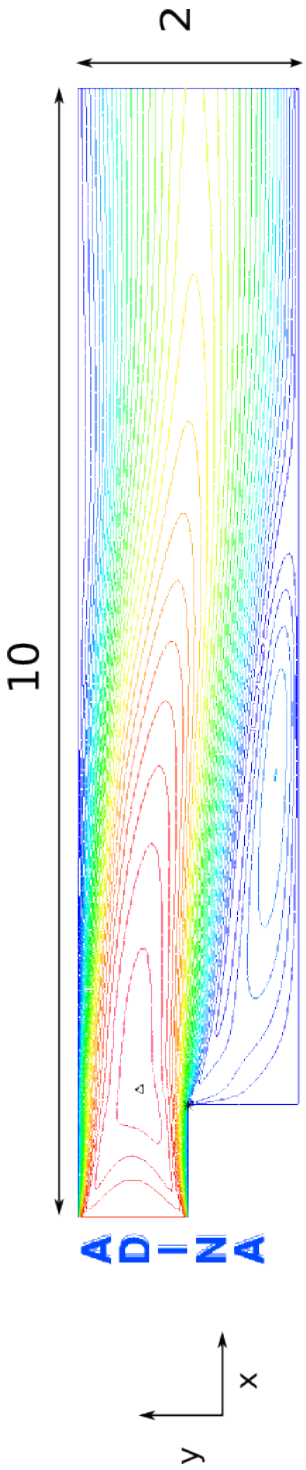

Fig. 4. Contour lines of the amplitude of the velocity field.

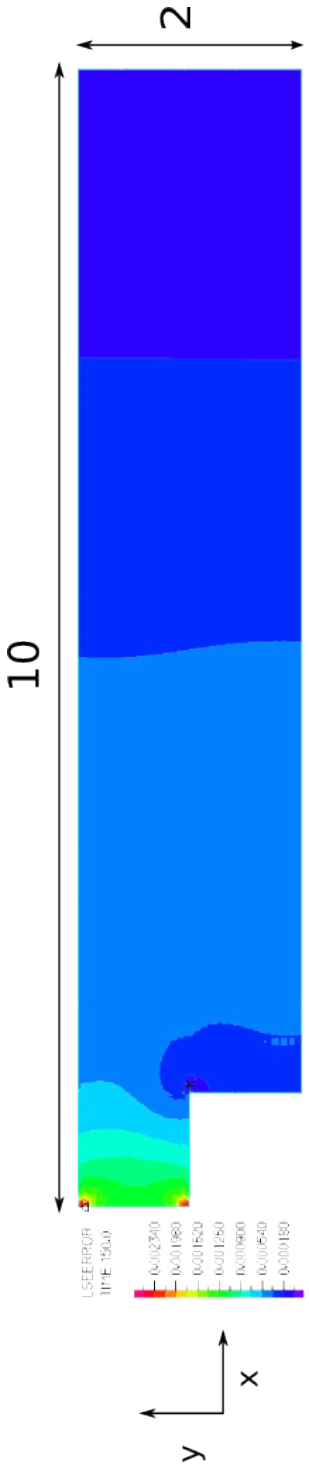

Fig. 5. Difference between the NS FE solution and the near by manufactured solution.

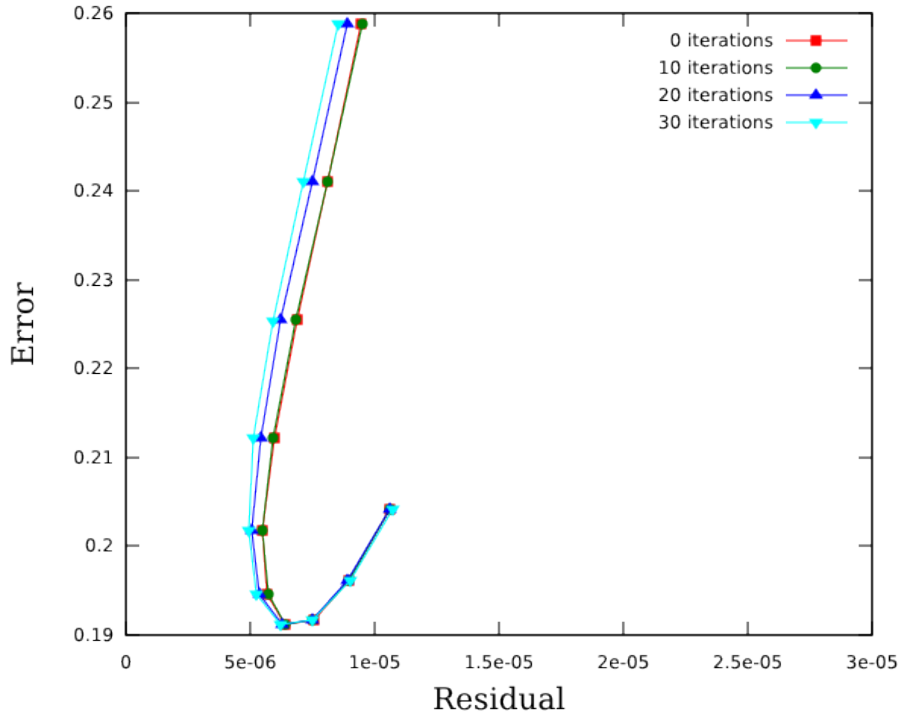

Fig. 6. Dependence of the Error and Residual in L_2 norm on the number of relaxation used in postprocessing $\alpha\tilde{U}_1 + (1-\alpha)\tilde{U}_2$.

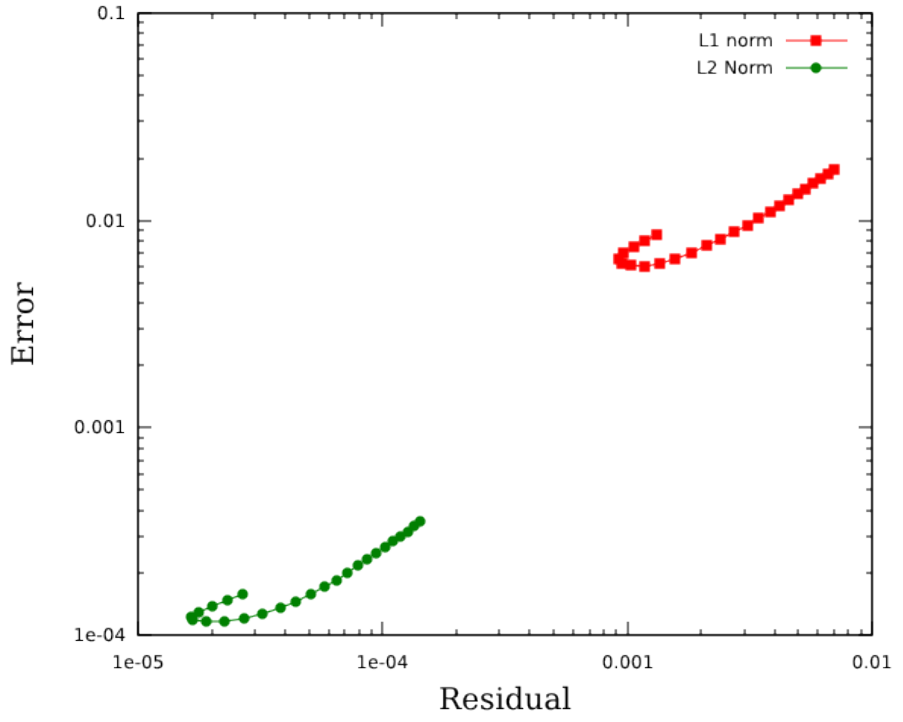

Fig. 7. Comparison of OES solutions for different discrete norms

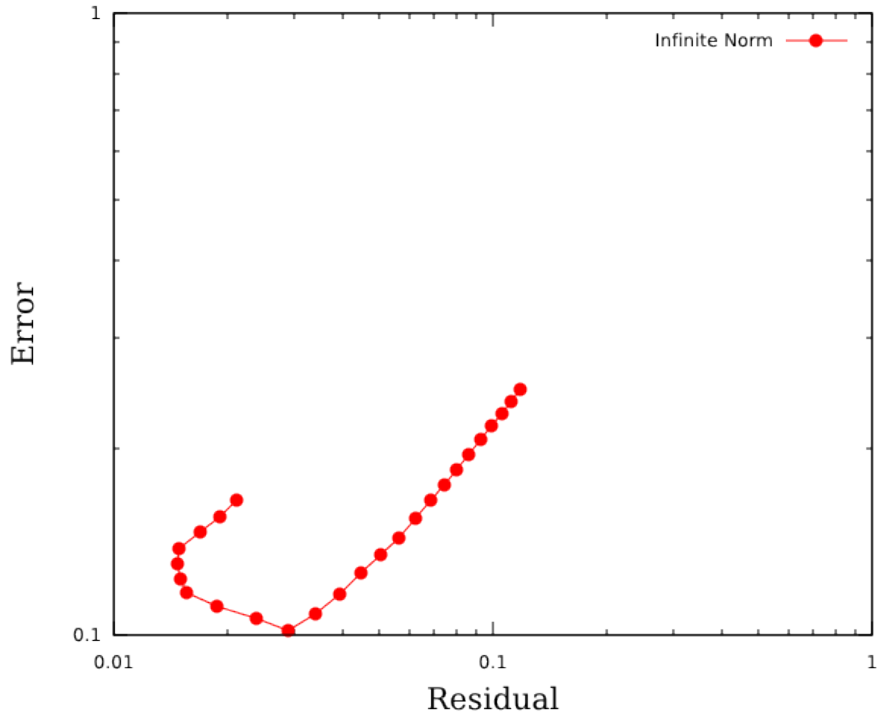

Fig. 8. OES solution with the L_∞ norm.

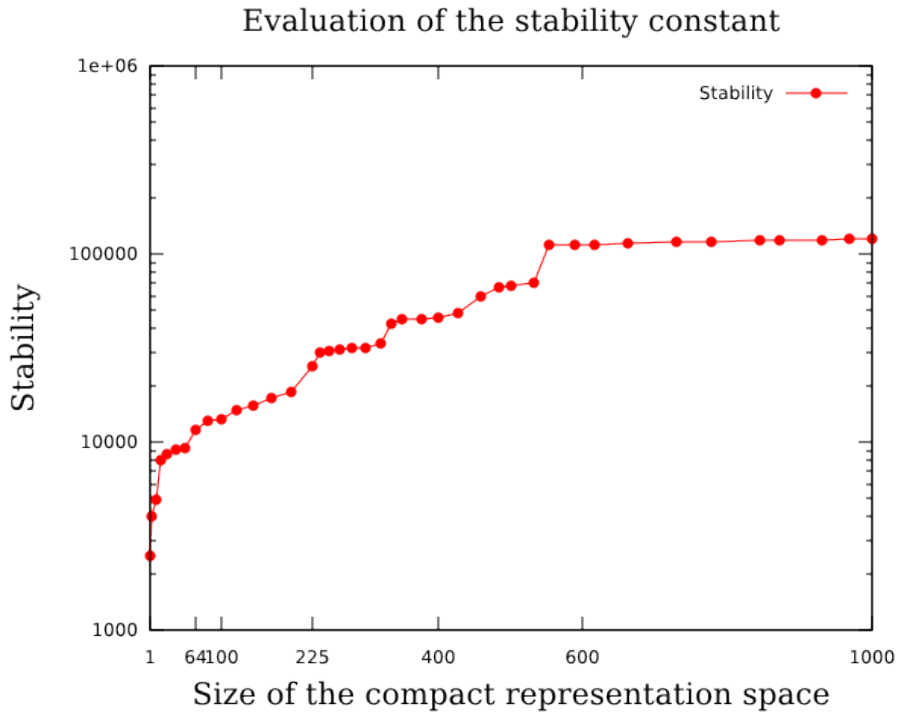

Fig. 9. Evaluation of the stability constant for the modified NS problem in L_2 norm.

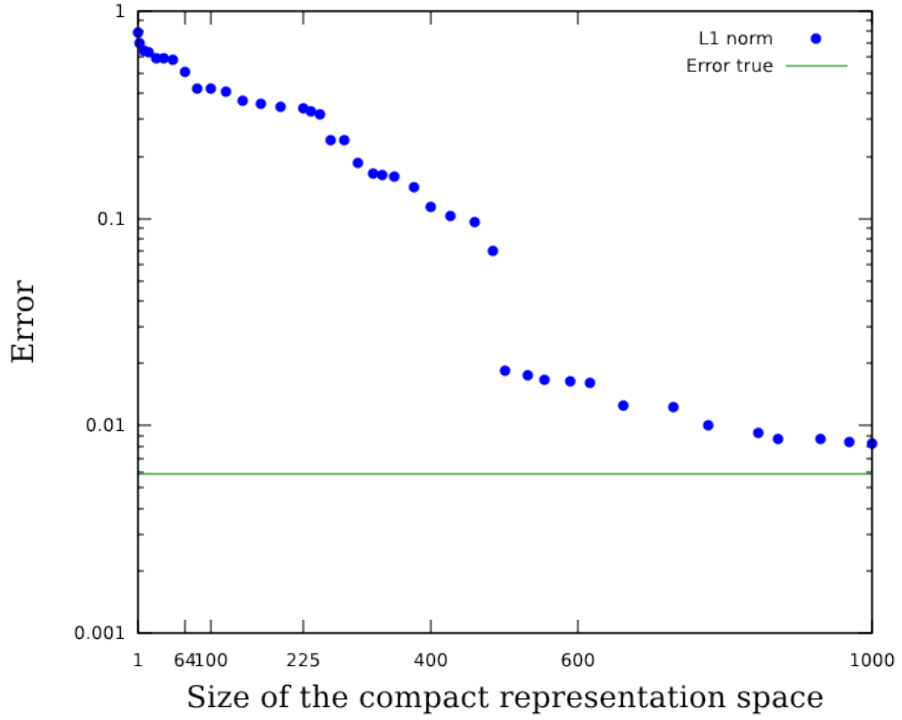

Fig. 10. Evolution of the error versus the manufactured solution for the L_1 norm.

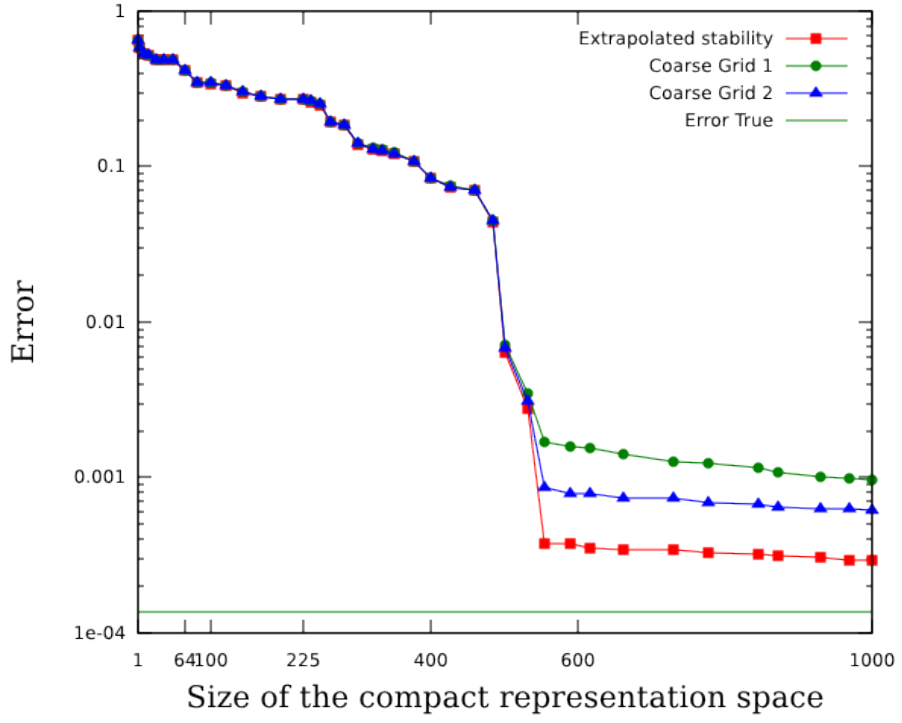

Fig. 11. Evolution of the error versus the manufactured solution for the L_2 norm.

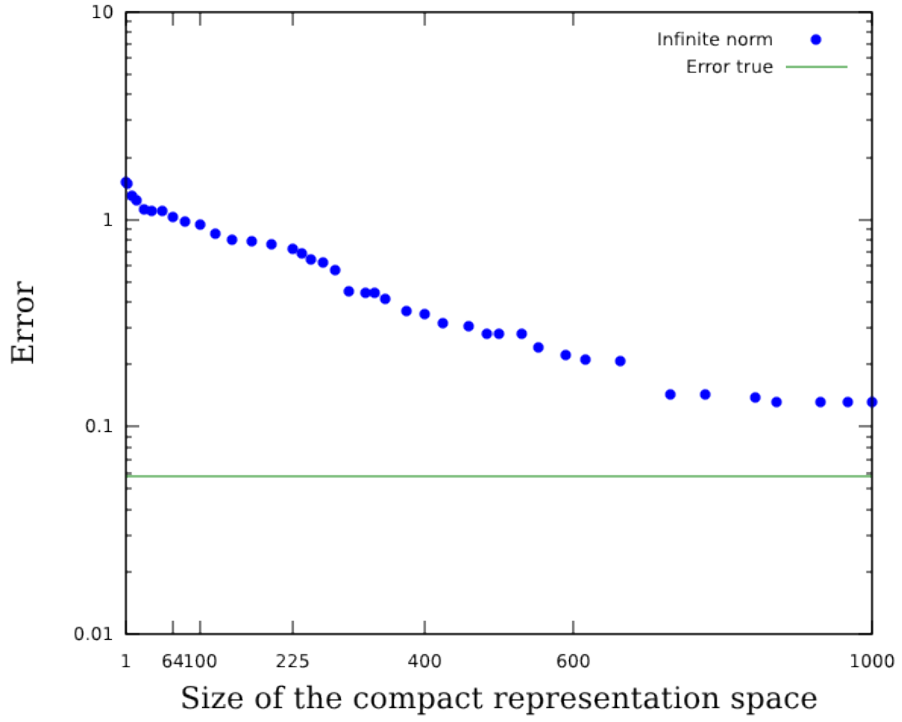

Fig. 12. Evolution of the error versus the manufactured solution for the L_∞ norm.

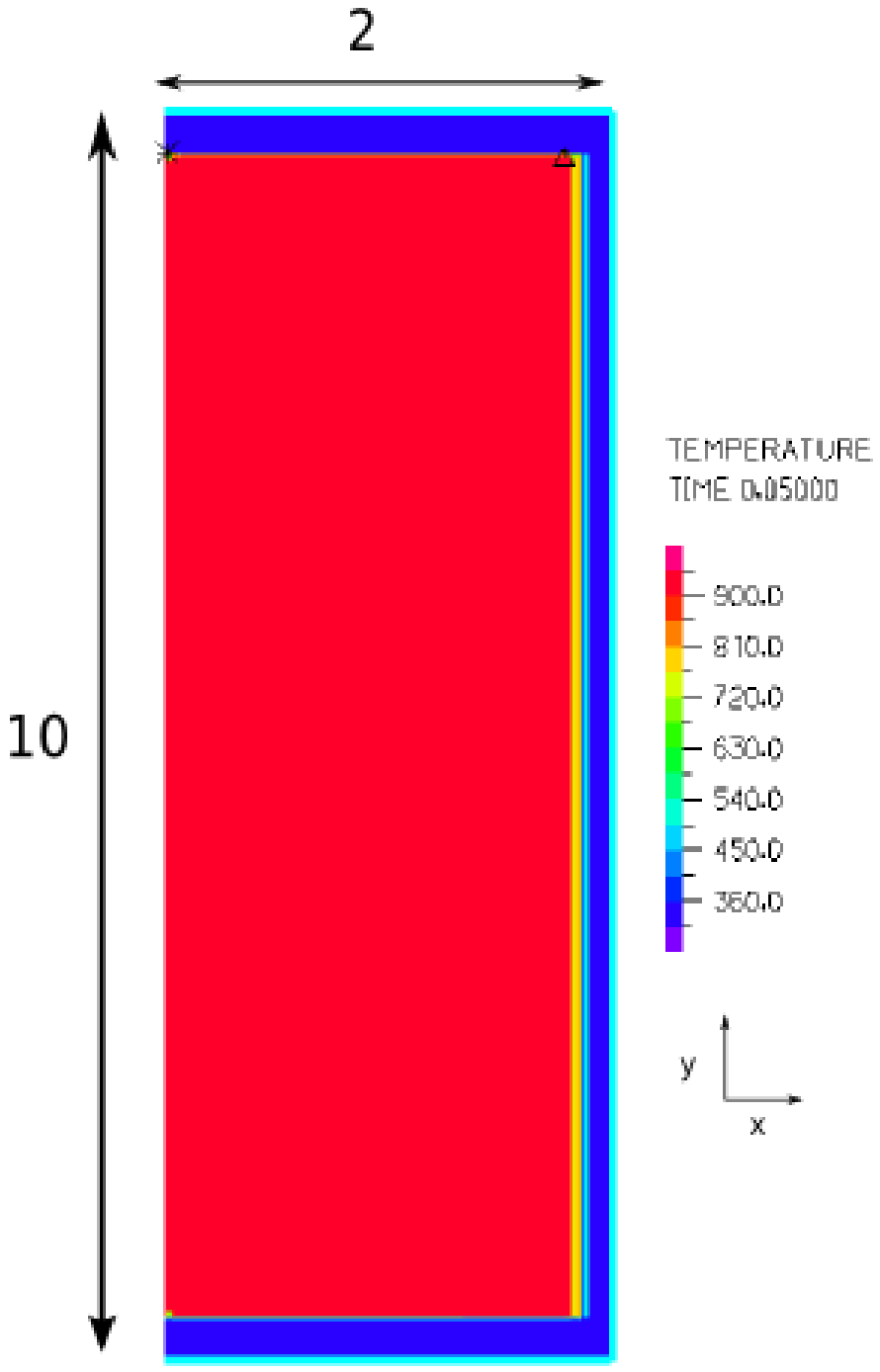

Fig. 13. Steady state solution of the heat transfer problem

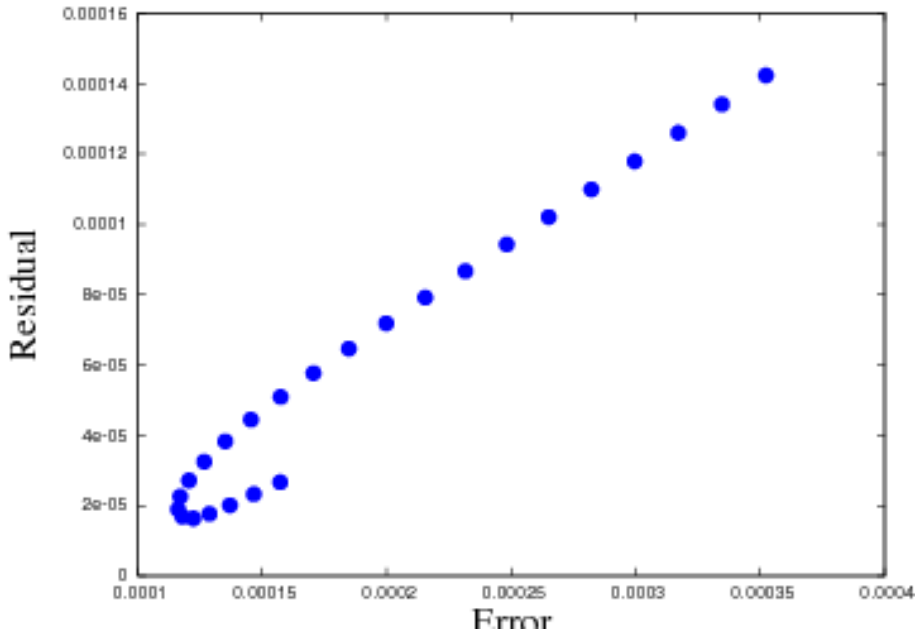

Fig. 14. Residual versus the error in the L_2 norm for a sequence of α values.

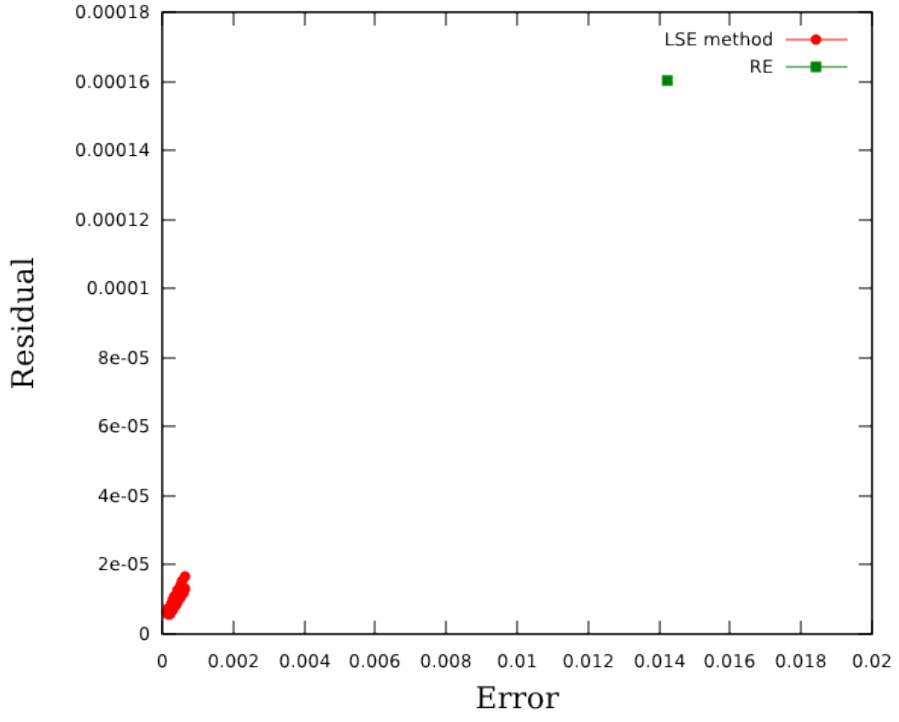

Fig. 15. Performance of OES versus Richardson Extrapolation in the L_2 norm.

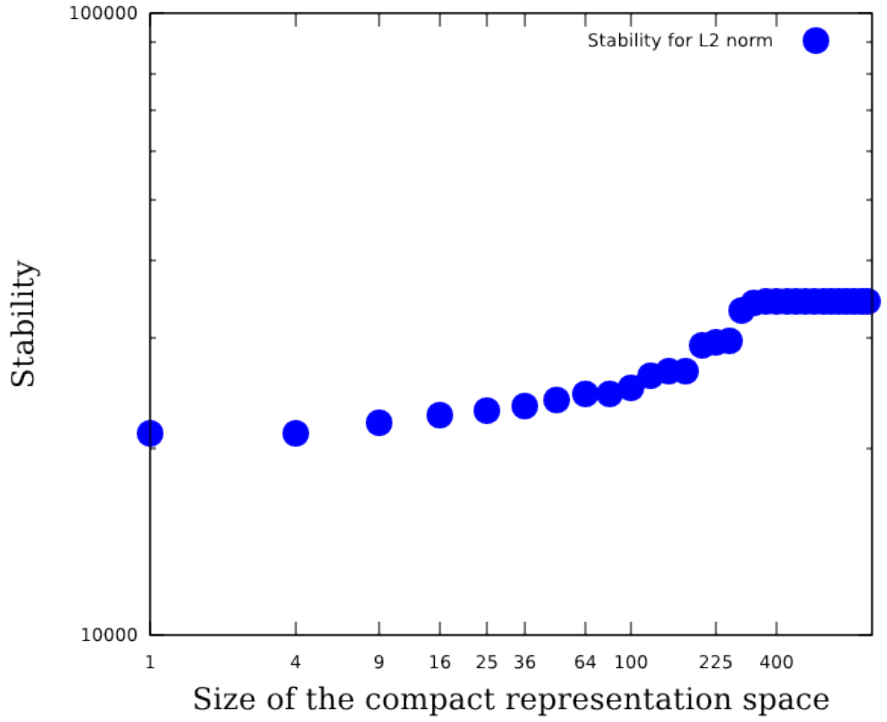

Fig. 16. Evaluation of the stability constant for the heat transfer problem.

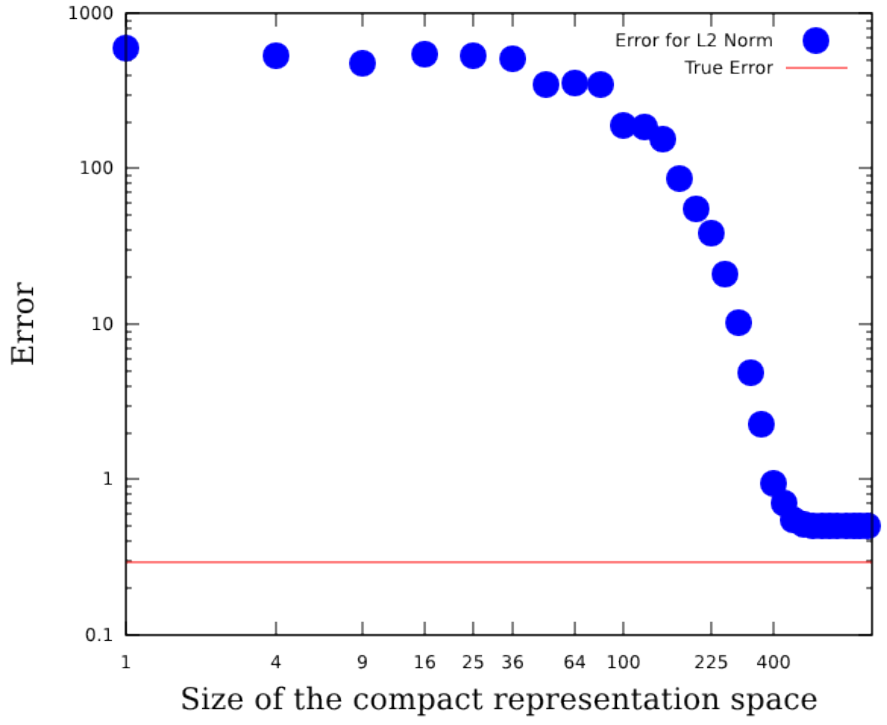

Fig. 17. Evolution of the error versus the fine grid solution for the heat transfer problem.