# 2015 UH EITM Summer Institute

Reduced Form Calculation

Calculate the reduced form for $X_t$.

$$\begin{aligned}
Y_t^D &= \alpha + \beta X_t + \delta Z_t + \gamma Q_t + \eta_t \\
Y_t^S &= \varphi + \theta X_t + \phi X_{t-1} + \xi_t
\end{aligned}$$

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Reduced Form Calculation Answer Key

Calculate the reduced form for $X_t$.

$$
\begin{aligned}
Y_t^D &= \alpha + \beta X_t + \delta Z_t + \gamma Q_t + \eta_t \\
Y_t^S &= \varphi + \theta X_t + \phi X_{t-1} + \xi_t
\end{aligned}
$$

This can be accomplished by setting (1) equal to (2) and isolating $X_t$. The result is:

$$X_t = \Pi_1 + \Pi_2 Z_t + \Pi_3 Q_t + \Pi_4 X_{t-1} + \varepsilon_t,$$

where:

$$
\begin{aligned}
\Pi_1 &= \frac{\varphi - \alpha}{\beta - \theta} \\
\Pi_2 &= -\frac{\delta}{\beta - \theta} \\
\Pi_3 &= -\frac{\gamma}{\beta - \theta} \\
\Pi_4 &= \frac{\phi}{\beta - \theta} \\
\varepsilon_t &= \frac{\xi_t - \eta_t}{\beta - \theta}
\end{aligned}
$$

Cross sectional models are static; there is only a relation for one point in time. A dynamic model, on the other hand, involves relationships between variables that can be contemporaneous and over many periods. This prospect opens up many additional possibilities for resolution of theoretical issues. *It simply is a stronger empirical test to determine if a certain theory or hypothesis holds for not only one point in time but also over time.*
Below are the basics in persistence dynamics. They include:

- Lag Operators
- The Dynamic Multiplier
- Intervention Analysis (temporary and permanent)

**Lag Operators**

Lag operators are an often used notation. They serve important functions, most notably an important short-hand notation indicating how many lags of a particular variable are used in a model. The lag operator is usually represented by the letter "L," although the letter "B" has been used as well (Box and Jenkins 1976).

Consider now the variable $y_t$. We express the first, second, and third-order lags of this variable below:

$$\begin{aligned} Ly_t &= y_{t-1} && (1) \\ L^2 y_t &= L\left(Ly_t\right) = Ly_{t-1} = y_{t-2} && (2) \\ L^3 y_t &= L(L^2 y_t) = Ly_{t-2} = y_{t-3}. && (3) \end{aligned}$$

This generalizes to:

$$L^j = y_{t-j}. \quad (4)$$

Finally, to assess the accumulation of shocks to a series sum the lag operators. Here infinity is used as the end point. Note that the effect of shocks gets less pronounced the further into the past one goes. That is due to the $\phi^j$ "weight" that is attached to each lag:

$$\begin{aligned} \sum_{j=0}^{\infty} \phi^j L^j &= \phi^j \left(1 + L + L^2 + L^3 + ... + L^{\infty}\right) \\ \text{where } \phi^0 &= 1. && (5) \end{aligned}$$

### Dynamic Multiplier

One of the attributes of time series analysis is that point estimates for the immediate effect can be adjusted to *determine the long-run or steady state effect*. The identity for the dynamic mutlitplier can only be derived when there are lags in the dependent variable and this is why a lag operator provides the basis for the result. This provides the "persistence" and eventual decay of the initial effect from a given variable or variables. To show this take (5) above and multiply it by $(1-\phi L)$. The result is:

$$(1-\phi L)\left(1+\phi^j L+\phi^j L^2+\phi^j L^3+\ldots+\phi^j L^\infty\right). \tag{6}$$

Since we are dealing with a finite sample, say we adjust the infinite series to one that is $j$ periods long. Rewrite (6) as:

$$(1-\phi L)\left(1+\phi L+\phi^2 L^2+\phi^3 L^3+\ldots+\phi^j L^j\right), \tag{7}$$

simplify:

$$\begin{aligned} & \left(1+\phi L+\phi^2 L^2+\phi^3 L^3+\ldots+\phi^j L^j\right) \\ & -\left(\phi L+\phi^2 L^2+\phi^3 L^3+\phi^4 L^4+\ldots+\phi^{j+1} L^{j+1}\right) \\ = & \left(1-\phi^{j+1} L^{j+1}\right). \end{aligned} \tag{8}$$

As the sample size gets progressively larger, or as $j \to \infty$, the second term in $\left(1-\phi^{j+1}L^{j+1}\right)$ approaches 0. Therefore, we have the following result:

$$(1-\phi L)\left(1+\phi L+\phi^2 L^2+\phi^3 L^3+\ldots+\phi^j L^j\right)=1 \tag{9}$$

and

$$\left(1+\phi L+\phi^2 L^2+\phi^3 L^3+\ldots+\phi^j L^j\right)=\frac{1}{(1-\phi L)}. \tag{10}$$

The identity, $\frac{1}{(1-\phi L)}$, which makes use of the lag operator, tells us the accumulated effect. While this example is just for one lag, this identity could include many lags. It is sometimes the case that analysts ignore the cumulative effect and focus on the point estimates instead. This is a mistake as the point estimate effect can be dwarfed by the long-run cumulative effect. The derivation of this type of result can alter the *substantive significance* where the point estimate is eclipsed by the sheer magnitude of the cumulative effect.

## Intervention Analysis

A useful way to test hypotheses is to examine the effect of a specific policy change. The possibilities are great here since many subjects in the social sciences are influenced by changes in regime or policy. The interventions can be characterized in many ways, but they generally can be categorized as either *temporary* or *permanent*.

### Temporary Interventions

Temporary interventions follow the sequence in (11):

$$\left[\begin{array}{cccccccccc} 0 & 0 & 0 & 1 & 0 & 0 & . & . & . & 0 \end{array}\right], \tag{11}$$

which could be written as a column vector and take on the identity of a dummy variable:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}. \tag{12}$$

The intervention is characterized by the number "1" and it is called *temporary* since it is a one time only occurrence. What we want to know is not only the point estimate, the initial effect, but also how the effect accumulated. Consider, for example, the following AR(1) model (the constant and additional independent variables are dropped for convenience):

$$y_t = \rho y_{t-1} + \beta I_t + \varepsilon_t, \tag{13}$$

where $I_t$ is the intervention and $|\rho| < 1$. $\beta$ gives the point estimate or initial effect of the intervention, but the accumulation of the intervention's effect is based on :

$$y_t = \rho y_{t-1} + \beta I_t = \beta. \tag{14}$$

This dynamic effect can be shown by extending (14) for successive periods:

$$y_{t+1} = \rho y_t + \beta I_{t+1} = \rho\left(\beta\right), \tag{15}$$

and for $y_{t+2:}$

$$y_{t+2} = \rho y_{t+1} + \beta I_{t+2} = \rho^2\left(\beta\right). \tag{16}$$

From these substitutions we can come up with the simple rule:

$$\begin{aligned} y_{t+n} &= \rho y_{t+n-1} + \beta I_{t+n} && (17) \\ &= \rho\left(\rho^{n-1}\beta\right) + \beta\left(0\right) && (18) \\ &= \rho^n\beta. && (19) \end{aligned}$$

### A Temporary Intervention: An Example

Consider the following estimated model:

$$y_t = .8y_{t-1} + .20I_t, \tag{20}$$

where we know the initial effect of the intervention is .20. Applying the rule in (19) we get the accumulated effect in Table 1:

Table 1

| $t$ | $\rho^t\beta$ | $\mathrm{I}_{t+n}$ |
|---|---|---|
| 0 | $.8^0\,(.20)$ | .20 |
| 1 | $.8^1\,(.20)$ | .16 |
| 2 | $.8^2\,(.20)$ | .13 |
| 3 | $.8^3\,(.20)$ | .11 |
| 4 | $.8^4\,(.20)$ | .08 |
| 5 | $.8^5\,(.20)$ | .07 |

## Permanent Interventions

An alternative intervention specification for $I_t$ is one that is *permanent*, where the policy shift continues from point $t$ to the end of the data series. Consider the sequence:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & . & . & . & 1 \end{bmatrix}, \tag{21}$$

which can also be written as a column vector and take on the identity of a dummy variable:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}. \tag{22}$$

Consider, as before, an AR(1) process $I_t$ is the intervention and $|\rho| < 1$:

$$y_t = \rho y_{t-1} + \beta I_t + \varepsilon_t \tag{23}$$

$\beta$ gives the point estimate or initial effect of the intervention, but the accumulation of the intervention's effect is derived as:

$$y_t = \rho y_{t-1} + \beta\,(1) = \beta. \tag{24}$$

The identity for $y_{t+1}$ is:

$$y_{t+1} = \rho y_t + \beta\,(1) = \rho\,(\beta) + \beta, \tag{25}$$

and repeating the substitution for future periods:

$$\begin{aligned} y_{t+2} &= \rho y_{t+1} + \beta(1) = \rho\,(\rho\beta + \beta) + \beta & (26) \\ &= \rho^2\beta + \rho\beta + \beta, & (27) \end{aligned}$$

$$\begin{aligned} y_{t+3} &= \rho y_{t+2} + \beta(1) = \rho\left(\rho^2\beta + \rho\beta + \beta\right) + \beta & (28) \\ &= \rho^3\beta + \rho^2\beta + \rho\beta + \beta. & (29) \end{aligned}$$

From these substitutions we can come up with the rule for permanent interventions:

$$y_{t+n} = (1 + \rho + ... + \rho^n)\,\beta,$$

which shows the cumulative effect of the intervention after "n" periods. The total effect of the intervention is:

$$\begin{aligned} \sum_{t=1}^{N} \rho^t &= \beta \times \left[\frac{1}{1-\rho}\right] & (30) \\ &= \frac{\beta}{1-\rho}. & (31) \end{aligned}$$

**A Permanent Intervention: An Example**

Consider the same example, but this time with a permanent intervention in $I_t$:

$$y_t = .8y_{t-1} + .20I_t \tag{32}$$

Since $\rho = .8$, we get the accumulated effect in Table 2:

Table 2

| $t$ | $\rho^t\beta$ | $\sum_{t=1}^{N}\rho^t\beta$ |
|---|---|---|
| 0 | .20 | .20 |
| 1 | $.8^1\,(.20)$ | .36 |
| 2 | $.8^2\,(.20)$ | .49 |
| 3 | $.8^3\,(.20)$ | .59 |
| 4 | $.8^4\,(.20)$ | .67 |
| 5 | $.8^5\,(.20)$ | .73 |

It is also useful when investigating permanent interventions to get the total effect of the intervention.

$$\frac{\beta}{1-\rho} = \frac{.2}{1-.8} = 1.0. \tag{33}$$

Note the difference between the point estimates and the total effect.

# Assignment

Consider equation (1a). Show the long-run, steady-state identities for $\alpha$, $z_t$, $x_t$ :

$$y_t = \alpha + \rho_1 y_{t-1} + \rho_2 y_{t-2} + \delta_1 z_t + \delta_2 z_{t-1} + \gamma_1 x_{t-1} + \gamma_3 x_{t-3} + \upsilon_t. \tag{1a}$$

Now, as an example, say the result for (1a) is:

$$y_t = 1.2 + 1.1 y_{t-1} - .6 y_{t-2} + 1.5 z_t - .2 z_{t-1} + 2.1 x_{t-1} + .6 x_{t-3} + \upsilon_t \tag{1b}$$

Calculate the long-run, steady-state results for (1b).

For (1a), the long-run, steady-state result identities are as follows:

$$y_t = \alpha + \rho_1 y_{t-1} + \rho_2 y_{t-2} + \delta_1 z_t + \delta_2 z_{t-1} + \gamma_1 x_{t-1} + \gamma_3 x_{t-3} + \upsilon_t,$$

with the long-run results for the constant:

$$\frac{\alpha}{1 - \rho_1 L - \rho_2 L^2},$$

for $z_t$:

$$\frac{\delta_1 + \delta_2}{1 - \rho_1 L - \rho_2 L^2} z^*,$$

and for $x_t$:

$$\frac{\gamma_1 + \gamma_3}{1 - \rho_1 L - \rho_2 L^2} x^*,$$

for $x_{t-1} = x_{t-2} = x_{t-3} = ... = x^*$ and $z_t = z_{t-1} = z_{t-2} = ... = z^*$ in steady state.

For (1b) the values are:

$$y_t = 1.2 + 1.1 y_{t-1} - .6 y_{t-2} + 1.5 z_t - .2 z_{t-1} + 2.1 x_{t-1} + .6 x_{t-3} + \upsilon_t$$

The numerical results are: for $\alpha$:

$$\frac{1.2}{(1 - 1.1 - (-.6))} = 2.4$$

for $z_t$:

$$\frac{1.5 - .2}{(1 - 1.1 - (-.6))} z^* = 2.6 z^*$$

and for $x_t$:

$$\frac{2.1 + .6}{(1 - 1.1 - (-.6))} x^* = 5.4 x^*$$

for $x_{t-1} = x_{t-2} = x_{t-3} = ... = x^*$ and $z_t = z_{t-1} = z_{t-2} = ... = z^*$ in steady state.

Note we can combine the coefficients of the same variable, regardless of the lag. In addition, the denominator can give us a clue as to how much different the steady state result is from the combined point estimates. Here we are dividing by .5, which is the same as multiplying by 2 $\rightarrow \left(\frac{1}{.5} = 2\right)$.

1. Assume that the behavior of $y_t$ follows a simple cobweb process where the exogenous shock follows an AR(1) process:

$$y_t = \alpha_0 + \alpha_1 E_{t-1} y_t + u_t$$

and

$$u_t = \rho u_{t-1} + \varepsilon_t,$$

for $|\rho| < 1$. Based on a conjecture that $y_t = \phi_0 + \phi_1 u_{t-1} + \phi_2 \varepsilon_t$, use the method of undetermined coefficients to solve the REE in this model.

2. Assume that the behavior of $y_t$ is governed by:

$$y_t = \alpha_0 + \alpha_1 E_t y_{t+1} + u_t,$$

where $u_t$ is white noise. Use the method of undetermined coefficients to solve the rational expectations equilibrium in this model.

3. Now assume that $u_t$ is not white noise but instead obeys an AR(1) process:

$$u_t = \rho u_{t-1} + \varepsilon_t,$$

where $|\rho| < 1$ and $\varepsilon_t$ is white noise. Based on a conjecture that $y_t = \phi_0 + \phi_1 u_{t-1} + \phi_2 \varepsilon_t$, use the method of undetermined coefficients to solve the REE in this model.

1. Assume that the behavior of $y_t$ follows a simple cobweb process where the exogenous shock follows an AR(1) process:

$$y_t = \alpha_0 + \alpha_1 E_{t-1} y_t + u_t \tag{34}$$

and

$$u_t = \rho u_{t-1} + \varepsilon_t, \tag{35}$$

for $|\rho| < 1$. Based on a conjecture that $y_t = \phi_0 + \phi_1 u_{t-1} + \phi_2 \varepsilon_t$, use the method of undetermined coefficients to solve the REE in this model.

- $y_t = \alpha_0 + \alpha_1 E_{t-1} y_t + u_t$ and $u_t = \rho u_{t-1} + \varepsilon_t$, we have:

$$y_t = \alpha_0 + \alpha_1 E_{t-1} y_t + \rho u_{t-1} + \varepsilon_t. \tag{36}$$

- conjecture: $y_t = \phi_0 + \phi_1 u_{t-1} + \phi_2 \varepsilon_t$
- Take expectations $E_{t-1}$ on $y_t$, we have:

$$\begin{aligned} E_{t-1} y_t &= E_{t-1} \left(\phi_0 + \phi_1 u_{t-1} + \phi_2 \varepsilon_t\right) && (37) \\ E_{t-1} y_t &= \phi_0 + \phi_1 u_{t-1} && (38) \end{aligned}$$

- Plug (38) into (36) :

$$\begin{aligned} y_t &= \alpha_0 + \alpha_1 \left(\phi_0 + \phi_1 u_{t-1}\right) + \rho u_{t-1} + \varepsilon_t && (39) \\ y_t &= \left(\alpha_0 + \alpha_1 \phi_0\right) + \left(\alpha_1 \phi_1 + \rho\right) u_{t-1} + \varepsilon_t && (40) \end{aligned}$$

- Therefore:

$$\begin{aligned} \phi_0 &= \left(\alpha_0 + \alpha_1 \phi_0\right) \\ \phi_0 &= \alpha_0 / \left(1 - a_1\right) \end{aligned}$$

  and

$$\begin{aligned} \phi_1 &= \alpha_1 \phi_1 + \rho \\ \phi_1 &= \rho / \left(1 - a_1\right) \end{aligned}$$

  and

$$\phi_2 = 1.$$

- The REE is:

$$y_t = \left(\alpha_0 / \left(1 - a_1\right)\right) + \left(\rho / \left(1 - a_1\right)\right) u_{t-1} + \varepsilon_t.$$

2. Assume that the behavior of $y_t$ is governed by:

$$y_t = \alpha_0 + \alpha_1 E_t y_{t+1} + u_t, \tag{41}$$

where $u_t$ is white noise. Use the method of undetermined coefficients to solve the rational expectations equilibrium in this model.

- $y_t = \alpha_0 + \alpha_1 E_t y_{t+1} + u_t$
- Conjecture: $y_t = \phi_0 + \phi_1 u_t$
- Take expectations $E_t$ on $y_{t+2}$, we have:

$$
\begin{aligned}
E_t y_{t+1} &= E_t \left(\phi_0 + \phi_1 u_{t+1}\right) && (42) \\
&= \phi_0 && (43)
\end{aligned}
$$

- Plug (43) into (41) :

$$y_t = \alpha_0 + \alpha_1 \phi_0 + u_t.$$

- Therefore,

$$
\begin{aligned}
\phi_0 &= \alpha_0 + \alpha_1 \phi_0 \\
\phi_0 &= \alpha_0 / \left(1 - \alpha_1\right).
\end{aligned}
$$

  and

$$\phi_1 = 1.$$

- The REE is:

$$y_t = \frac{\alpha_0}{1 - \alpha_1} + u_t.$$