Formal Concepts and Analogues

Expectations: Basics

- Forecasts or views agents hold about future variables of interest.
- Substantial effect on current choices and overall activity.
- Evolution of Modeling Expectations: Cobweb, Adaptive, and Rational Expectations.
- One form of expectations, <u>**rational expectations**</u> (RE), is a particular equilibrium concept.
- RE represents the optimal choice of the decision rule used by agents depending on the choices of others.
- An RE equilibrium (REE) condition imposes a consistency condition that each agents' choice <u>**is a best response**</u> to the choices of others.

- In contrast to a mathematical expectation, which is a summary measure (expected value), a **conditional expectation** is mathematical expectation with a modified probability distribution ("information set").

Solution Procedure(s): Method of Undetermined Coefficients and MSV

- "Closing the model" involves taking unknown variables (i.e., expectations) and expressing them in terms of other "known" variables.

- The **method of undetermined coefficients** is a particular solution process that closes a model.

- A **minimum state variable (MSV)** solution is the simplest solution when using the method of undetermined coefficients.

Operational Issues to Model and Test Expectations

1. How Do We Express Conditional Expectations?

We use expectation operators:

$E_{t-1}Y_t$ means agents use all available information in the model (up to period t-1) to forecast Y_t,

Or

E_tY_{t+1} means agents use all available information in the model (up to period t) to forecast Y_{t+1}.

1 Expectation Operators

Consider the equation:

$$y_t = \alpha + \beta E_{t-1} y_{t+1} + \delta x_t + u_t \qquad (1)$$

The expectations operator, E_{t-i} , means (under rational expectations) use all available information up to period $t - i$ (assumed to be in the model – in (1) that is very simple) – in making predictions.

Variations on (1):

1. $E_{t-1} y_t = \alpha + \beta E_{t-1} y_{t+1} + \delta E_{t-1} x_t$

And more generally:

2. $y_{t+1} = \alpha + \beta E_t y_{t+2} + \delta x_{t+1} + u_{t+1}$
3. $E_{t-1} y_{t+1} = \alpha + \beta E_{t-1} y_{t+2} + \delta E_{t-1} x_{t+1}$

Continuing:

4. $y_{t+2} = \alpha + \beta E_t y_{t+3} + \delta x_{t+2} + u_{t+2}$
5. $E_{t-1} y_{t+2} = \alpha + \beta E_{t-1} y_{t+3} + \delta E_{t-1} x_{t+2}$

Furthermore, what happens if we use E_t instead of E_{t-1} in (1)?

6. $E_t y_t = \alpha + \beta E_t y_{t+1} + \delta E_t x_t$

7. $E_t y_{t+1} = \alpha + \beta E_t y_{t+2} + \delta E_t x_{t+1}$

2 Unconditional Expectations

Model

$$y_t = \alpha + \beta x_t + \rho(y_{t-1} - E(y_{t-1})) + \varepsilon_t \qquad (2)$$

where ε_t is iid with $E(\varepsilon_t) = 0$. Taking unconditional expectations (that is, $y_{t-1} - E(y_{t-1}) = 0$) of (2), we get:

$$E(y_t) = \alpha + \beta x_t$$

$$E(y_{t-1}) = \alpha + \beta x_{t-1}$$

Now, close (2):

$$y_t = \alpha + \beta x_t + \rho y_{t-1} - \rho(\alpha + \beta x_{t-1}) + \varepsilon_t$$

Collect terms:

$$y_t - \rho y_{t-1} = \alpha + \beta x_t - \rho\alpha - \rho\beta x_{t-1} + \varepsilon_t$$

$$y_t - \rho y_{t-1} = \alpha - \rho\alpha + \beta x_t - \rho\beta x_{t-1} + \varepsilon_t$$

Using “L” as the lag operator, we get:

$$(1 - \rho L)y_t = (1 - \rho L)\alpha + (1 - \rho L)\beta x_t + \varepsilon_t \qquad (3)$$

Divide both sides by $(1 - \rho L)$:

$$y_t = \alpha + \beta x_t + (1 - \rho L)^{-1}\varepsilon_t$$

$$= E(y_t) + (1 - \rho L)^{-1}\varepsilon_t$$

This means therefore that:

$$y_t - E(y_t) = (1 - \rho L)^{-1}\varepsilon_t$$

$$= \varepsilon_t + \rho\varepsilon_{t-1} + \rho^2\varepsilon_{t-2} + \cdots$$

which is an MA(q) process and by invertibility is an AR(1) process.

3 Rational Expectations

If we now use rational expectations and use the information in the model at up to time t-1, we will find the following:

$$y_t = \alpha + \beta x_t + \rho(y_{t-1} - E_{t-1}y_{t-1}) + \varepsilon_t \tag{4}$$

where ε_t is iid with $E_{t-1}(\varepsilon_t) = 0$. Under rational expectations, it is assumed that $y_{t-1} - E_{t-1}y_{t-1} = 0$. Therefore, under rational expectations, (4) is equivalent to:

$$y_t = \alpha + \beta x_t + \varepsilon_t$$

4 Rational Expectations

Under Rational Expectations

We can see that unconditional expectations provide an alternative dynamic process to a conditional process that assumes rational expectations. Note the difference between conditional and unconditional expressions resides in what period information is available to agents. And does this difference under alternative timing lead to different results for rational expectations and "persistence"? In other words, what occurs if the expression $(y_{t-1} - E_{t-1}y_{t-1})$ is instead $(y_{t-1} - E_{t-2}y_{t-1})$?[1] In the latter case, agents only have information available up to one period BEHIND, when they make forecasts.

Rewrite (4) as:

$$y_t = z_t + \rho(y_{t-1} - E_{t-2}y_{t-1}) + \varepsilon_t \tag{5}$$

where $z_t = \alpha + \beta x_{t-1} + \eta_t$ for purpose of simplicity.

[1] The equivalent is $(y_t - E_{t-1}y_t)$.

First, solve for y_{t-1}:

$$y_{t-1} = z_{t-1} + \rho(y_{t-2} - E_{t-3}y_{t-2}) + \varepsilon_{t-1} \tag{6}$$

and $E_{t-2}y_{t-1}$ equals:

$$E_{t-2}y_{t-1} = E_{t-2}z_{t-1} + \rho(E_{t-2}y_{t-2} - E_{t-2}(E_{t-3}y_{t-2})) \tag{7}$$

$$= E_{t-2}z_{t-1} + \rho(y_{t-2} - E_{t-3}y_{t-2}) \tag{8}$$

Subtract (8) from (6):

$$y_{t-1} - E_{t-2}y_{t-1} = z_{t-1} - E_{t-2}z_{t-1} + \varepsilon_{t-1} \tag{9}$$

Plug (9) into (5):

$$y_t = z_t + \rho(z_{t-1} - E_{t-2}z_{t-1}) + \rho\varepsilon_{t-1} + \varepsilon_t \tag{10}$$

Now close: $E_{t-2}z_{t-1}$ and note that $z_{t-1} = \alpha + \beta x_{t-2} + \eta_{t-1}$ (add a stochastic term for purpose of allowing for updating), we find that $z_{t-1} - E_{t-2}z_{t-1}$ is:

$$z_{t-1} - E_{t-2}z_{t-1} = \eta_{t-1} \tag{11}$$

With this result, we can rewrite (10), expand z_t and note the lagged appearance of persistent shocks:

$$y_t = \alpha + \beta x_{t-1} + (1 - \rho L)\eta_t + (1 - \rho L)\varepsilon_t$$

How Do We Solve for Models with Expectations?

- This is called “closing” a model. “Closing the model” involves taking unknown variables (i.e., expectations) and expressing them in terms of other “known” variables. There are many ways to do this.

- The simplest way is to simply substitute for the expectations operator.

- A second, more general way is to use the **method of undetermined coefficients**.

Consider the Cobweb Model:

$$y_t = \alpha + \beta E_{t-1} y_t + \gamma \omega_{t-1} + \varepsilon_t \tag{1}$$

Method of Substitution

From (1), we have:

$$E_{t-1}y_t = \alpha + \beta E_{t-1}y_t + E_{t-1}\gamma\omega_{t-1}$$
$$= \alpha + \beta E_{t-1}y_t + \gamma\omega_{t-1} \quad (2)$$

From (2), we have:

$$E_{t-1}y_t - \beta E_{t-1}y_t = \alpha + \gamma\omega_{t-1}$$
$$(1-\beta)E_{t-1}y_t = \alpha + \gamma\omega_{t-1}$$
$$E_{t-1}y_t = (\frac{\alpha}{1-\beta}) + (\frac{\gamma}{1-\beta})\omega_{t-1} \quad (3)$$

Then substitute (3) into (1), we have:

$$y_t = \alpha + \beta\left[\left(\frac{\alpha}{1-\beta}\right) + \left(\frac{\gamma}{1-\beta}\right)\omega_{t-1}\right] + \gamma\omega_{t-1} + \varepsilon_t$$
$$= \left[\alpha + \left(\frac{\alpha\beta}{1-\beta}\right)\right] + \left[\left(\frac{\beta\gamma}{1-\beta}\right) + \gamma\right]\omega_{t-1} + \varepsilon_t$$
$$= \left(\frac{\alpha-\alpha\beta+\alpha\beta}{1-\beta}\right) + \left(\frac{\beta\gamma+\gamma-\beta\gamma}{1-\beta}\right)\omega_{t-1} + \varepsilon_t$$
$$= \frac{\alpha}{1-\beta} + \frac{\gamma}{1-\beta}\omega_{t-1} + \varepsilon_t \text{ (Rational Expectations Equilibrium (REE))}$$

The intuition of the REE is as follows:

A rational expectations equilibrium imposes the consistency condition that each agent’s choice is a best response to the choices by others (Evans and Honkapohja 2001: 11).

Method of Undetermined Coefficients (See Enders)

We can first assume the REE of equation (1) is:

$$y_t = \Pi_0 + \Pi_1 \omega_{t-1} + \varepsilon_t \tag{4}$$

- We do not know what the Π's are, but we first assume that there exists an equilibrium (REE) in the Cobweb Model. That is what we call "the method of undermined coefficients."[2]

Now take E_{t-1} in equation (4), we have:

$$E_{t-1} y_t = \Pi_0 + \Pi_1 \omega_{t-1}$$

Then plug it in equation (1), we have:

$$y_t = \alpha + \beta(\Pi_0 + \Pi_1 \omega_{t-1}) + \gamma \omega_{t-1} + \varepsilon_t$$

[2] Note that the Minimum State Variable (MSV) Solution is just the simplest parameterization one can choose using the method of undetermined coefficients.

We collect terms and have:

$$y_t = (\alpha + \beta \Pi_0) + (\gamma + \beta \Pi_1)\omega_{t-1} + \varepsilon_t \tag{5}$$

- The expression of equation (5) is identical to that of equation (4), so the coefficients of equation (4) should be consistent with equation (5), such that

[From equation (4)] $\Pi_0 = \alpha + \beta \Pi_0$ [From equation (5)]

$$\Pi_1 = \gamma + \beta \Pi_1$$

Now solve for Π_0 and Π_1, we have:

$$\Pi_0 = \frac{\alpha}{1-\beta}$$

$$\Pi_1 = \frac{\gamma}{1-\beta} \tag{6}$$

Therefore, we get the rational expectations equation by putting Π's (result 6) back into equation (4):

$$y_t = \frac{\alpha}{1-\beta} + \frac{\gamma}{1-\beta}\omega_{t-1} + \varepsilon_t$$

Conclusion

- Again, both methods are consistent. But the method of undetermined coefficients is more general and easier to solve for the REE (if we have a different expression of the DGP, other than the Cobweb expression (e.g. $E_t y_{t+1}$instead of $E_{t-1} y_t$).