Dynamic Optimization

An Introduction

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Outline

1. Background
   - What is Optimization?
   - EITM: The Importance of Optimization

2. Dynamic Optimization in Discrete Time
   - A Simple Two-period Consumption Model
   - The Bellman Equation
   - Cake Eating Problem
   - Profit Maximization

3. Dynamic Optimization in Continuous Time
   - The Method of Hamiltonian Multiplier
   - Cake Eating Problem Revisited

4. An EITM Example
   - Dynamics in a Money-in-the-Utility Model
   - TM: Theoretical Model
   - EI: Empirical Implications
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- Key assumptions: (1) Rationality, and (2) Efficiency.
  - If rationality is assumed, firms will maximize their profits (supply) and households will maximize their utility (demand).
  - When their results are achieved, the market is in equilibrium (efficiency).
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There are two general methods of optimization:

- Analytical optimization
  - Solving the optimal solution(s) mathematically.

- Numerical (or computational) optimization
  - Searching for the optimal solution(s) according to different algorithms (using computers).
  - For example, simulations, calibrations, and maximum likelihood estimations.
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- **Optimization without Constraints**
  - First-order conditions (FOCs)

- **Optimization with Constraints**
  - The method of Lagrangian multiplier

- **Dynamic Optimization (with/without Constraints)**
  - Discrete time: The Bellman Equation
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Social scientists are interested in causal effects:

How do x’s affect y?

Empirical studies (e.g., regression analysis) can show us, at most, correlations among variables (not causality!) If the coefficient on x is significant, it could imply that:

- x causes y; or
- y causes x; or
- there is another unobservable variable, called z, which contributes x and y to move simultaneously.

But, how do we know if x’s really cause y?
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But, how do we know if $x$’s really cause $y$?

- NOBODY TRULY KNOWS!!

We need to use our logical thinking and reasoning to describe why $x$’s can cause $y$.

- But, the world is just too complex!
- An easy way to do so is to build a theoretical model which describes some aspect of the market (or the society) that includes only those features that are needed for the propose at hand.
- It is necessary to impose assumptions to make a model simpler.

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A Famous Quote from Robert Solow (1956, page 65):

- "All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive."

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- If this is the case, probably the assumptions we make are too sensitive (too strong) to the final results.
- Removing those assumptions / imposing some more realistic assumptions would be necessary.
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This assumption helps us formulate human behavior in order to predict outcomes in aggregate markets. This is called the microfoundation of macroeconomics.

Microfoundations refers to the microeconomic analysis of the behavior of individual agents such as households or firms that underpins a macroeconomic theory. (Barro, 1993)

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In this lecture, we study two methods of dynamic optimization: (1) Discrete-time Optimization - the Bellman equations; and (2) Continuous-time Optimization - the method of Hamiltonian multiplier.

Examples:
- **Discrete-time case:**
  1. Cake-eating problem
  2. Profit maximization
- **Continuous-time case:**
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Two-period Consumption Model

Two periods in the model
Period 1: The present; and Period 2: The future

The two-period utility function can be written as:

\[ U = u(c_1) + \frac{1}{1+\rho} u(c_2). \]

- We call \(1/(1+\rho)\) as the discount factor, where \(\rho\) is called the discount rate (or the degree of impatience).
- If an agent is more impatient \((\rho \uparrow \Rightarrow 1/(1+\rho) \downarrow)\), then she would put less weight on the utility of future consumption.
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The two-period utility function can be written as:

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Assuming that the agent has a first-period budget constraint:

\[ Y_1 + (1 + r) A_0 = c_1 + A_1, \]

where \( Y_t = \) exogenous income at time \( t \), \( A_t = \) assets / debts that the individual accumulates at time \( t \), and \( r = \) an exogenous interest rate.
The complete model is:

The two-period utility function:

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The first-period and second-period budget constraints:

\[ Y_1 = c_1 + A_1 \quad (1\text{-st-period BC}), \quad \text{and} \]
\[ Y_2 + (1 + r) A_1 = c_2 \quad (2\text{-nd-period BC}). \]

We assume that the individual does not have any inheritance/debt in period 1 (i.e., \( A_0 = 0 \)) and does not leave any bequest/debt after period 2 (i.e., \( A_2 = 0 \)).
Two-period Consumption Model

Lagrangian Multiplier

The system:

\[ U = u(c_1) + \frac{1}{1+\rho} u(c_2). \]

and

\[ Y_2 + (1+r) A_1 Y_1 = c_1 + A_1, \text{ and } Y_2 + (1+r) A_1 = c_2. \]

To maximize the system of equations, we can apply the method of Lagrangian multiplier to solve the model:

\[ L = u(c_1) + \frac{1}{1+\rho} u(c_2) + \lambda_1 (Y_1 - c_1 - A_1) + \lambda_2 (Y_2 + (1+r) A_1 - c_2), \]

where \( \lambda_1 \) and \( \lambda_2 \) are the Lagrangian multipliers.
We have 5 choice variables: $c_1$, $c_2$, $A_1$, $\lambda_1$, and $\lambda_2$. We can solve for those variables based on the 5 first-order conditions:

\[
\frac{\partial L}{\partial c_1} = 0 \Rightarrow u'(c_1) - \lambda_1 = 0
\]

\[
\frac{\partial L}{\partial c_2} = 0 \Rightarrow \frac{1}{1 + \rho} u'(c_2) - \lambda_2 = 0
\]

\[
\frac{\partial L}{\partial A_1} = 0 \Rightarrow -\lambda_1 + (1 + r) \lambda_2 = 0
\]

\[
\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow Y_1 = c_1 + A_1
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\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow Y_2 + (1 + r) A_1 = c_2.
\]
From equations (1)–(3), we have:

\[ \lambda_1 = u'(c_1), \tag{6} \]

\[ \lambda_2 = \frac{1}{1 + \rho} u'(c_2), \text{ and} \tag{7} \]

\[ -\lambda_1 + (1 + r) \lambda_2 = 0. \tag{8} \]
Now we can plug (6) and (7) into (8), we have the following equation:

\[ u'(c_1) = \frac{1+r}{1+\rho} u'(c_2). \]  

Equation (9) is called the Euler equation. By combining equations (4) and (5), we have the lifetime budget constraint:

\[ Y_1 + \frac{Y_2}{1+r} = c_1 + \frac{c_2}{1+r}. \]  

Finally, given a certain functional form of \( u(\cdot) \), we can use equations (9) and (10) to obtain the optimal levels of \( c_1 \) and \( c_2 \), (i.e., \( c_1^* \) and \( c_2^* \)).
Two-period Consumption Model

What does the Euler equation: \( u'(c_1) = \frac{1+r}{1+\rho} u'(c_2) \) tell us?

- Suppose that an agent gives up $1 consumption today (the present), the utility cost to her will be \(-u'(c_1)\). In return, she will get \((1+r)\) additional consumption tomorrow (the future) so that her utility gain for the tomorrow will be \((1+r)u'(c_2)\).
- However, since we assume that agents are impatient, the totally gain from giving up today’s consumption for tomorrow would be \(\frac{1}{1+\rho} \times (1+r) u'(c_2)\).
- In equilibrium, the agent will not give up more or less today’s consumption for tomorrow at the optimal level only if \( u'(c_1) = \left[\frac{(1+r)}{(1+\rho)}\right] u'(c_2)\).
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Two-period Consumption Model: An Example

Let the utility function be the power function, \( u(c) = \frac{1}{\alpha} c^\alpha \), where \( \alpha \in (0, 1) \), and \( \alpha = 1/2 \), we have:

\[
    u'(c) = c^{-1/2}.
\]  

(11)

We can plug equation (11) into equation (9), we have:

\[
    c_1^{-1/2} = \frac{1 + r}{1 + \rho} c_2^{-1/2} \Rightarrow c_2 = \left( \frac{1 + r}{1 + \rho} \right)^2 c_1.
\]  

(12)
Plug eq.(12) into the lifetime budget constraint (eq.(10)), we have:

\[
Y_1 + \frac{Y_2}{1+r} = c_1 + \frac{1+r}{(1+\rho)^2} c_1
\]

\[
Y_1 + \frac{Y_2}{1+r} = c_1 \left( 1 + \frac{1+r}{(1+\rho)^2} \right)
\]

\[
\Rightarrow c_1^* = \left( 1 + \frac{1+r}{(1+\rho)^2} \right)^{-1} \left( Y_1 + \frac{Y_2}{1+r} \right). \tag{13}
\]

Now we plug equation (13) into equation (12), we have:

\[
c_2^* = \left( \frac{1+r}{1+\rho} \right) \left( 1 + \frac{1+r}{(1+\rho)^2} \right)^{-1} \left( Y_1 + \frac{Y_2}{1+r} \right). \tag{14}
\]
The optimized consumption levels in period 1 and period 2 are:

\[ c_1^* = \left( 1 + \frac{1+r}{(1+\rho)^2} \right)^{-1} \left( Y_1 + \frac{Y_2}{1+r} \right) \]

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   - Dynamics in a Money-in-the-Utility Model
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   - EI: Empirical Implications
In the previous section, we can use the method of Lagrangian multiplier for solving a simple dynamic optimization problem. However, such the method can be sometime tedious and inefficient.

This alternative technique is based on a recursive representation of a maximization problem, which is called the Bellman equation.

The Bellman equation represents a maximization decision based on the forward (or backward) solution procedure with the property of time consistency.

This time consistency property of the optimal solution is also known as Bellman’s optimality principle.
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The Bellman Equation

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This time consistency property of the optimal solution is also known as **Bellman’s optimality principle**.
Let’s consider the following maximization problem:

$$\max_{c_t} \left[ U = \sum_{t=1}^{\infty} \left( \frac{1}{1+\rho} \right)^{t-1} u(c_t) \right],$$

subject to the following budget constraint:

$$A_t = (1+r)A_{t-1} + Y_t - c_t.$$  

- $A_t$ is the state variable in each period $t$, which represents the total amount of resources available to the consumer;
- $c_t$ is the control variable, where the consumer is choosing to maximize her utility. Note that $c_t$ affects the amount of resources available for the next period, that is, $A_t$. 
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Consumption Dynamics

- For this technique of dynamic optimization, the maximum value of utility not only depends on the level of consumption at time $t$, but also the resource left for future consumption (i.e., $A_t$).

- In other word, given the existing level of asset $A_{t-1}$, the level of consumption chosen at time $t$ (that is, $c_t$) will affect the level of assets available at time $t+1$, (that is, $A_t$).
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In other word, given the existing level of asset $A_{t-1}$, the level of consumption chosen at time $t$ (that is, $c_t$) will affect the level of assets available at time $t+1$, (that is, $A_t$).
Consumption Dynamics

Since the consumer would like to maximize her utility from time $t$ onwards, we define the following value function $V_1(A_0)$ which represents the maximized value of the objective function from time $t = 1$ to the last period of $t = T$:

$$V_1(A_0) = \max_{c_1} \sum_{t=1}^{T} \left( \frac{1}{1+\rho} \right)^{t-1} u(c_t)$$  \hspace{1cm} (17)$$

$$V_1(A_0) = \max_{c_1} \left( u(c_1) + \left( \frac{1}{1+\rho} \right) u(c_2) + \cdots + \left( \frac{1}{1+\rho} \right)^{T-1} u(c_T) \right)$$

$$V_1(A_0) = \max_{c_1} \left\{ u(c_1) + \left( \frac{1}{1+\rho} \right) \left[ \sum_{t=2}^{T} \left( \frac{1}{1+\rho} \right)^{t-2} u(c_t) \right] \right\}$$  \hspace{1cm} (18)$$

subject to

$$A_t = (1+r)A_{t-1} + Y_t - c_t.$$  \hspace{1cm} (19)$$
Consumption Dynamics

From equation (17), we see that $V_1(A_0)$ is the maximized value of the objective function at time $t = 1$ given an initial stock of assets $A_0$.

After the maximization in the first period ($t = 1$), the consumer repeats the same procedure of utility maximization according to the objective function in period $t = 2$, given an initial stock of assets in period 1:

$$V_2(A_1) = \max_{c_2} \sum_{t=2}^{T} \left( \frac{1}{1+\rho} \right)^{t-2} u(c_t),$$  \hspace{1cm} (20)

subject to equation (19).

By plugging equation (20) into equation (18), we have:

$$V_1(A_0) = \max_{c_1} \left[ u(c_1) + \left( \frac{1}{1+\rho} \right) V_2(A_1) \right].$$
Consumption Dynamics

The Bellman Equation

Assuming the consumer is maximizing her utility every period, we rewrite the maximization problem recursively. Therefore, we can present the well-known *Bellman equation* as follows:

\[ V_t(A_{t-1}) = \max_{c_t} \left[ u(c_t) + \left( \frac{1}{1 + \rho} \right) V_{t+1}(A_t) \right], \]

where \( A_t = (1 + r) A_{t-1} + Y_t - c_t \).
Consumption Dynamics

How to solve the Bellman equation in the problem?

Please see the whiteboard!
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Cake Eating Problem

Question

Suppose the size of a cake at time $t$ is $\Pi_t$; and the utility function is presented as $u(c_t) = 2c_t^{1/2}$. Given that $\Pi_0 = 1$, $\Pi_T = 0$, and the discount rate is $\rho$, what is the optimal path of consumption?
Answer

This optimization problem can be written as:

$$\max_{c_1, c_2, \ldots, c_T} \sum_{t=1}^{T} \left( \frac{1}{1 + \rho} \right)^{t-1} u(c_t),$$

subject to $\Pi_t = \Pi_{t-1} - c_t$, for $t = 1, 2, \ldots, T$, and $\Pi_0 = 1$ and $\Pi_T = 1$.

In this case, we see that the choice variable is $c_t$ and the state variable $\Pi_t$. 
Now we can formulate the Bellman equation:

\[
V(\Pi_{t-1}) = \max_{c_t} \left[ u(c_t) + \frac{1}{1+\rho} V(\Pi_t) \right],
\]

subject to

\[
\Pi_t = \Pi_{t-1} - c_t.
\]

Since \( u(c_t) = 2c_t^{1/2} \), we can rewrite the Bellman equation as:

\[
V(\Pi_{t-1}) = \max_{c_t} \left[ 2c_t^{1/2} + \frac{1}{1+\rho} V(\Pi_t) \right].
\]
How to solve the Bellman equation in the problem?

Please see the whiteboard!
Cake Eating Problem

The Optimal Path of Cake Consumption

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<th>A</th>
<th>B</th>
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<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
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<td>Cake Size (Pᵢₜ)</td>
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Assuming that a representative firm maximizes the present value of all future profits by choosing the levels of investment $I_t$ and labor $L_t$ for $t = 1, 2, \ldots, T$. Therefore, we have the following maximization problem:

$$
\max_{I_1, I_2, \ldots, I_T, L_1, L_2, \ldots, L_T} \sum_{t=1}^{T} \left( \frac{1}{1+r} \right)^t \pi_t \\
\max_{I_1, I_2, \ldots, I_T, L_1, L_2, \ldots, L_T} \sum_{t=1}^{T} \left( \frac{1}{1+r} \right)^t (F(K_t, L_t) - w_t L_t - I_t),
$$

subject to

$$K_{t+1} = K_t - \delta K_t + I_t$$

for $t = 1, 2, \ldots, T$, and $K_1$ and $K_T$ are given, $\pi_t$ is the level of profit at time $t$, $K_t$ is the stock of capital at time $t$, $F(K_t, L_t)$ is a production function, $w_t$ is the wage rate, $r$ and $\delta$ is the interest rate and depreciate rate in the market, respectively.
In this case,

- the choice variable is: \( I_t \) and \( L_t \), and
- the state variable is \( K_t \).

According to the above system, we can formulate the following Bellman equation:

\[
V_t(K_t) = \max_{I_t, L_t} \left[ F(K_t, L_t) - w_t L_t - I_t + \frac{1}{1+r} V_{t+1}(K_{t+1}) \right], \tag{25}
\]

where

\[
K_{t+1} = (1 - \delta) K_t + I_t. \tag{26}
\]
In this optimization problem, we have two important conditions:

1. \[ \frac{\partial F(K_t,L_t)}{\partial L_t} = w_t. \]
   - This result suggests that the optimal amount of labor satisfies the condition where marginal product of labor (MPL) equals the real wage rate in each period, that is \( MPL_t = w_t \).

2. \[ \frac{\partial F}{\partial K_t} = r + \delta. \]
   - This result suggests that the firm must choose a level of investment such that the marginal product of capital (MPK) equals the sum of interest rate and depreciation rate in each period, that is \( MPK_t = r + \delta \).
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In the previous sections, we study the dynamic optimization based the discrete-time dynamic model, where the change in time $\Delta t$ is positive and finite (for example, we assume that $\Delta t = 1$ for all $t \geq 0$, such that $t$ follows the sequence of $\{0, 1, 2, 3, \ldots\}$).

In the section, we consider the method of dynamic optimization in continuous time, where $\Delta t \rightarrow 0$. Therefore, we assume that agents make optimizing choices at every instant in continuous time.

The method is called the technique of Hamiltonian multiplier.
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In the section, we consider the method of dynamic optimization in continuous time, where $\Delta t \to 0$. Therefore, we assume that agents make optimizing choices at every instant in continuous time.

The method is called the **technique of Hamiltonian multiplier**.
A general continuous-time maximization problem can be written as follows:

$$\max_{x_t} \int_0^T e^{-\rho t} f (x_t, A_t) \, dt, \quad (27)$$

subject to the constraint:

$$\dot{A}_t = g (x_t, A_t), \quad (28)$$

where $\dot{A}_t$ is a time derivative of $A_t$ defined as $dA_t/dt$, and $\rho$ is the discount rate in the model, which is equivalent to the $\rho$ we use in the discrete time models.

- If $\rho = 0$, then $e^{-\rho t} = 1$. It implies that the agent does not discount the activities in the future. On the other hand, if $\rho \uparrow$, then $e^{-\rho t} \downarrow$. It implies that the agent becomes more impatient and discounts the value (or utility) of future activities more.
We set up the following Hamiltonian function:

\[ H_t = e^{-\rho t} \left[ f(x_t, A_t) + \lambda_t \dot{A}_t \right], \quad (29) \]

where \( \lambda_t = \mu_t e^{-\rho t} \) is called the Hamiltonian multiplier.

The three conditions for a solution:

1. The FOC with respect to the control variable \( (x_t) \):
   \[ \frac{\partial H_t}{\partial x_t} = 0; \quad (30) \]

2. The negative derivative of the Hamiltonian function w.r.t. \( A_t \):
   \[ -\frac{\partial H_t}{\partial A_t} = \frac{d(\lambda_t e^{-\rho t})}{dt}; \quad (31) \]

3. The transversality condition:
   \[ \lim_{t \to \infty} \lambda_t e^{-\rho t} A_t = 0. \quad (32) \]
The Method of Hamiltonian Multiplier

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Let us consider the same cake eating problem in continuous time:

$$\max_{c_t} \int_0^T e^{-\rho t} u(c_t) \, dt,$$

subject to

$$\dot{\Pi}_t = -c_t,$$

and $\Pi_0$ and $\Pi_1$ are given. Again, the choice variable is $c_t$, and the state variable is $\Pi_t$. 
Cake Eating Problem

We set up the Hamiltonian as follows:

$$ H_t = e^{-\rho t} \left( u(c_t) + \lambda_t \Pi_t \right), \quad (35) $$

where $\lambda_t$ is the co-state variable. Now, we can plug equation (34) into equation (35), we have the final Hamiltonian equation:

$$ H_t = e^{-\rho t} (u(c_t) - \lambda_t c_t) \quad (36) $$
Cake Eating Problem

We obtain the following conditions:

1. The FOC w.r.t. (with respect to) $c_t$:

\[
\frac{\partial H_t}{\partial c_t} = e^{-\rho t} (u'(c_t) - \lambda_t) = 0
\]

\[
\Rightarrow u'(c_t) = \lambda_t;
\]

(37)

2. The negative derivative w.r.t. $\Pi_t$ equals the time derivative of $\lambda_t e^{-\rho t}$:

\[
-\frac{\partial H_t}{\partial \Pi_t} = \frac{d (\lambda_t e^{-\rho t})}{dt};
\]

(38)

and

3. The transversality condition:

\[
\lim_{{t \to \infty}} \lambda_t e^{-\rho t} \Pi_t = 0.
\]

(39)
Cake Eating Problem

We obtain the following conditions:

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\[
\frac{\partial H_t}{\partial c_t} = e^{-\rho t} \left( u'(c_t) - \lambda_t \right) = 0
\]

\[\Rightarrow u'(c_t) = \lambda_t; \quad (37)\]

2. The negative derivative w.r.t. \( \Pi_t \) equals the time derivative of \( \lambda_t e^{-\rho t} \):

\[
-\frac{\partial H_t}{\partial \Pi_t} = \frac{d(\lambda_t e^{-\rho t})}{dt}; \quad (38)
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Cake Eating Problem

How to solve the Hamiltonian equation in the problem?

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Why do people want to hold money?

- Holding cash does not generate any returns!

Two possible arguments:

- People feel good and safe if they hold some cash in their pocket (Sidrauski, 1967)
- Money can provide transaction services (transaction purpose!) (Baumol, 1952, Tobin, 1956)

Here we study a basic neoclassical model where agents’ utility depends directly on their consumption of goods and their holdings of money (money demand).

- Assumption: Money yields direct utility.

This is called the Money-in-the-Utility model (Chari, Kehoe, and McGrattan, *Econometrica* 2000; Christiano, Eichenbaum, and Evans, *JPE* 2005)
Why do people want to hold money?
- Holding cash does not generate any returns!

Two possible arguments:
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1. Background
   - What is Optimization?
   - EITM: The Importance of Optimization

2. Dynamic Optimization in Discrete Time
   - A Simple Two-period Consumption Model
   - The Bellman Equation
   - Cake Eating Problem
   - Profit Maximization

3. Dynamic Optimization in Continuous Time
   - The Method of Hamiltonian Multiplier
   - Cake Eating Problem Revisited

4. An EITM Example
   - Dynamics in a Money-in-the-Utility Model

   TM: Theoretical Model
   EI: Empirical Implications
Consider the following Money-in-the-Utility Model:

\[ W = \sum_{t=0}^{\infty} \left( \frac{1}{1 + \rho} \right)^t u(c_t, m_t), \]

where \( m_t = \frac{M_t}{P_t N_t} = \) real money holding per capita in an economy. The budget constraint in the whole economy is:

\[ GDP_t = Consumption_t + Investment_t + GovtSpending_t + \]
\[ NewNationalDebt_t + NewMoney_t. \]

We can translate as:

\[ Y_t = C_t + [K_t - (1 - \delta) K_{t-1}] - \tau_t N_t + \left[ \frac{B}{P_t} - (1 + i_{t-1}) \frac{B_{t-1}}{P_t} \right] + \]
\[ \left[ \frac{M_t}{P_t} - \frac{M_{t-1}}{P_t} \right]. \]
An MIU Model

\[ Y_t = C_t + [K_t - (1 - \delta) K_{t-1}] - \tau_t N_t + \left[ \frac{B}{P_t} - (1 + i_{t-1}) \frac{B_{t-1}}{P_t} \right] + \left[ \frac{M_t}{P_t} - \frac{M_{t-1}}{P_t} \right]. \]

where \( Y_t \) = aggregate output (GDP), \( \tau_t N_t \) = total lump-sum transfers (positive) or taxes (negative), \( i \) = interest rate, and \( N_t \) = population.
Finally, the complete system is:

\[ W = \sum_{t=0}^{\infty} \left( \frac{1}{1+\rho} \right)^t u(c_t, m_t), \]

and the budget constraint can be presented as follows:

\[ Y_t + \tau_t N_t + (1 - \delta) K_{t-1} + (1 + i_{t-1}) \frac{B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} = C_t + K_t + \frac{M_t}{P_t} + \frac{B_t}{P_t}. \]

We also define the production function as \( Y_t = F(K_{t-1}, N_t) \). The per capita income is:

\[ y_t = \frac{1}{N_t} F(K_{t-1}, N_t) = F\left( \frac{K_{t-1}}{N_t}, \frac{N_t}{N_t} \right) = F\left( \frac{K_{t-1}}{(1+n)N_{t-1}}, 1 \right) \]

\[ = f\left( \frac{k_{t-1}}{1+n} \right), \text{ where } n = \text{population growth rate}. \]
Finally, the complete system is:

$$ W = \sum_{t=0}^{\infty} \left( \frac{1}{1+\rho} \right)^t u(c_t, m_t), $$

and the budget constraint (per capita) is:

$$ f \left( \frac{k_{t-1}}{1+n} \right) + \tau_t + \frac{1-\delta}{1+n} k_{t-1} + \frac{(1+i_{t-1}) b_{t-1} + m_t}{(1+\pi_t)(1+n)} = c_t + k_t + m_t + b_t, $$

where $\pi_t$ is the inflation rate, such that, $P_t = (1+\pi_t) P_{t-1}$, $b_t = B_t/(P_t N_t)$, and $m_t = M_t/(P_t N_t)$. 
An MIU Model

We can now formulate the model as the Bellman equation:

\[ V(W_t) = \max \left[ u(c_t, m_t) + \frac{1}{1+\rho} V(W_{t+1}) \right], \]

subject to

\[ W_{t+1} = f \left( \frac{k_t}{1+n} \right) + \tau_{t+1} + \left( \frac{1-\delta}{1+n} \right) k_t + \frac{(1+i_t) b_t + m_t}{(1+\pi_{t+1})(1+n)} = k_{t+1} + c_{t+1} + m_{t+1} + b_{t+1}. \]
How to solve the Bellman equation in the problem?

Please see the whiteboard!
Finally we have the following equilibrium:

\[ \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{i}{1 + i}. \]

Now assume that the utility function is of the constant elasticity of substitution (CES) form:

\[ u(c_t, m_t) = \left[ ac_t^{1-b} + (1-a) m_t^{1-b} \right]^{1/(1-b)}. \]

The final solution based on the CES utility function is:

\[ m_t = \left( \frac{a}{1-a} \right)^{-1/b} \left( \frac{i}{1+i} \right)^{-1/b} c_t, \text{ or} \]

\[ \ln m_t = \frac{1}{b} \ln \left( \frac{1-a}{a} \right) + \ln c_t - \frac{1}{b} \ln \gamma, \]

where \( \gamma = i/(1+i) = \) the opportunity cost of holding money.
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Empirical Estimation of Money Demand

According to the theoretical solution:

\[
\ln m_t = \frac{1}{b} \ln \left( \frac{1-a}{a} \right) + \ln c_t - \frac{1}{b} \ln \gamma, \tag{40}
\]

where \( \gamma = i/(1+i) \) which can be called the opportunity cost of holding money.

Therefore, we empirical model can be written as:

\[
\ln m_t = \alpha_0 + \alpha_1 \ln c_t + \alpha_2 \ln \gamma + \epsilon_t. \tag{41}
\]

- According to equation (41), we expect that the coefficient on \( \ln c_t \) is \( \alpha_1 = 1 \). It implies that consumption (income) elasticity of money demand is equal to 1.
- The coefficient on \( \ln \frac{i}{1+i} \) is \( \alpha_2 = -1/b \). For simplicity, we call it as the interest elasticity of money demand.
Empirical Estimation of Money Demand

The empirical result of the money demand function for the United States based on quarterly data from the period of 1984:1 – 2007:2.

Estimated Money Demand (MZM), U.S., 1984:1–2007:2

<table>
<thead>
<tr>
<th></th>
<th>Const</th>
<th>ln C</th>
<th>ln Y</th>
<th>ln(i/(1+i))</th>
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<tr>
<td>1.</td>
<td>-8.482</td>
<td>1.357</td>
<td></td>
<td>-0.090</td>
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<tr>
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<td>(0.192)</td>
<td>(0.024)</td>
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<td>(0.010)</td>
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<td>2.</td>
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<td>1.500</td>
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<tr>
<td></td>
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<td>(0.028)</td>
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<td>3.</td>
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<td>-0.016</td>
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<tr>
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<td>(0.040)</td>
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<td>0.149</td>
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<tr>
<td></td>
<td>(0.275)</td>
<td>(0.030)</td>
<td></td>
<td>(0.004)</td>
</tr>
</tbody>
</table>

*Note: Standard errors in parentheses.*

*Source: Walsh (2010: 51)*
Thank you!

Questions?