German Cubas, Hyunju Lee
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Comprehensive Exam in Macroeconomic Theory–Procedural Instructions

1. There are total of 6 questions. The last question ends on page 8, 6.(c).
2. Write your answers only on the paper we provide.
3. We are distributing a numbered sign–in sheet in a moment. The number next to your signature will be your student number.
4. Every sheet of paper you turn in to us must have your student number written at the top–center of the sheet and circled.
5. Every sheet of paper you turn in to us must have a page number written at the top–right corner of the sheet.
6. You have 4 hours to complete the exam. When you have finished, or when it is 4:00 pm (whichever comes first), prepare a cover sheet for your exam. This cover sheet should not have a page number, but must have the following things on it:
   (a) Your student number at the top–center, circled.
   (b) The phrase "Macroeconomics Comprehensive Exam".
   (c) The sentence "My last page is page number X", where X is your total number of pages of answers.

1. **Pure exchange economy (10%):** The economy has two agents and infinite time horizon. There is one perishable consumption good in the economy. The utility function of agent $i \in \{1, 2\}$ is given as the following:

$$U(c^i) = \sum_{t=0}^{\infty} \beta^t \log(c_t^i).$$

Each consumer is endowed with a sequence of consumption goods $\{e_t^i\}_{t=0,1,2,\ldots}$ such that:

$$\begin{aligned} Time &: 0, 1, 2, 3, \ldots \\ Consumer\ 1\ (e^1) &: 2, 4, 2, 4, \ldots \\ Consumer\ 2\ (e^2) &: 4, 2, 4, 2, \ldots \end{aligned}$$

There is NO free disposal of consumption goods.

(a) Define a (Arrow-Debreu) Competitive Equilibrium, characterize the solution, and solve it. **(5%)**

(b) Define Pareto Efficiency. **(2%)**

(c) Now, there is a one-period risk-free asset a_{t+1}^i that promises to deliver 1 period of consumption good in $t + 1$. The risk-free interest rate is denoted as r_{t+1}. Let's assume that the initial asset is given as 0 for all agents. These assets are IOUs - there is zero net supply of assets. It is assumed that no agents can borrow more than $\bar{A}$. In this setting, define a Sequential Markets Equilibrium. **(3%)**

2. **Neoclassical Growth Model: balanced growth path (15%):**

Consider an economy with a constant rate of population growth and a constant rate of productivity growth. Specifically, population (N) grows at rate n, and productivity (A) grows at rate γ. There is a representative firm and all households are identical. The households live forever, and time is discrete. Each household's flow utility is given as $u(c) = \log(c)$. Consider a utilitarian social planner that maximizes the sum of individual utility with equal weight:

$$\sum_{t=0}^{\infty} \beta^t N_t u(c_t),$$

where $\beta \in (0, 1)$ is a discount factor.

The firm's production function uses two input factors, capital service and labor. The firm's production function is given as $F(K, AN) = K^{\alpha}(AN)^{1-\alpha}$, where K is the capital service input, (AN) is efficiency units of labor, and $\alpha \in (0, 1)$ is a capital share. In each period, the capital stock depreciates with a rate $\delta \in [0, 1]$. The household's consumption and capital stock cannot be negative. There is a single final good that can be used for either consumption or investment. NO free disposal is allowed.

(a) Define a Social Planner's problem. **(5%)**

(b) Write down the resource constraint in "per efficiency units of labor". **(5%)**

(c) Derive the Euler equation per efficiency units of labor. **(5%)**

3. **Neoclassical Growth Model: advanced (25%):** There is a representative household and a representative firm. The household lives forever, and time is discrete. Assume that households own all the production factors (labor (n), capital (x)) and firms rent them from the households. Households are the ultimate owners of the firms.

   The household chooses her consumption (c), labor hours (n), capital services (k), and capital investment (i). In each period, the capital stock depreciates with a rate $\delta \in [0, 1]$. The household is endowed with an initial stock of capital $\bar{K}_0$ in period 0 and maximum labor hours $\bar{N} = 1$ in each period. The household maximizes the expected present value of utility,

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c(s^t), n(s^t)),$$

   where $\beta \in (0, 1)$ is a discount factor, $s^t \in S^t$ is the history of states up to time t, and $\pi(s^t)$ is the time-0 probability of history s^t occurring. Her flow utility is given as a separable power utility function:

$$u(c, n) = \frac{c^{1-\gamma}}{1-\gamma} - \frac{n^{1+\theta}}{1+\theta}.$$

   The firm's production function is given as $F(K, N; A) = A \cdot K^{\alpha} N^{1-\alpha}$, where A is the productivity level, K is the capital service input, N is the labor input, and $\alpha \in (0, 1)$ is a capital share. The productivity A is stochastic and follows a log-linear AR1 process, where $\log A_t = \rho \log A_{t-1} + \varepsilon_t$, and $\varepsilon_t \sim N(0, \sigma)$, i.i.d. Firms rent capital service (k) from households and hire labor (n) every period and maximize their profit.

   (a) Define a Social Planner's problem. **(5%)**

   (b) Characterize the Social Planner's problem. **(5%)**

   (c) Define a Recursive Competitive Equilibrium. **(5%)**

   (d) Characterize the Recursive Competitive Equilibrium. **(5%)**

   (e) Solve for the steady state. **(5%)**

4. **Problem 4 (20%)**

An individual lives for an infinite number of discrete time periods $t = 0, 1, 2, \ldots$. In each period, the individual chooses a share of time $e_t \in [0, 1]$ to allocate to education (human capital investment), and the remainder $1 - e_t$ is used to work and earn income.

The objective is to maximize discounted lifetime utility:

$$\max_{\{e_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t), \quad \text{with } \beta \in (0, 1)$$

Wages and Human Capital:

- Wage per unit of human capital is w (constant).
- Human capital evolves via: $h_{t+1} = (1 - \delta)h_t + \gamma e_t^{\alpha}$, where $\delta \in (0, 1)$, $\gamma > 0$, and $\alpha \in (0, 1)$.
- Consumption each period: $c_t = w(1 - e_t)h_t$

(a) **(5%)** Set up the dynamic programming problem. Derive the first-order condition (FOC) for the optimal effort using the Bellman equation.

(b) **(2.5%)** Discuss the role of α, β, and δ in shaping the optimal policy $e^{\star}(h)$.

(c) **(2.5%)** Solve for the steady state level of human capital $h^{\star}$ and education effort $e^{\star}$, assuming the value function is differentiable and concave.

(d) **(5%)** How do proportional taxes on labor income, τ, affect the optimal choice of e_t? Rewrite the Bellman equation under taxation and analyze the distortion.

(e) **(2.5%)** Suppose instead of a proportional tax, a lump-sum tax is imposed. Does this change the individual's decision regarding e_t? Why or why not?

(f) **(2.5%)** Discuss policy implications: What types of taxation are neutral with respect to human capital accumulation? What types are distortionary?

5. **Problem 5 (15%)**

A representative household maximizes:

$$\max_{\{c_t, x_t, k_{t+1}, d_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, d_t)$$

subject to:

$$c_t + x_t + k_{t+1} = f(k_t) + (1 - \delta_k)k_t \quad (1)$$

$$d_{t+1} = (1 - \delta_d)d_t + x_t \quad (2)$$

where: - c_t: nondurable consumption - x_t: investment in the durable good - d_t: stock of the durable good - k_t: physical capital - $f(k)$: production function (assume Cobb-Douglas $f(k) = Ak^{\alpha}$) - δ_k, δ_d: depreciation rates - $u(c, d)$: period utility (e.g., $u(c, d) = \frac{(c^{\gamma} d^{1-\gamma})^{1-\sigma}}{1-\sigma}$)

(a) **(5%)** Set up the dynamic programming problem.

(b) **(5%)** Derive the first-order condition(s) using the Bellman equation. Obtain the Euler equation(s). Discuss the trade-offs the household is facing. What aspects of the real world this model captures compared the standard one-good neoclassical growth model?

(c) **(5%)** Solve for the steady state.

6. **Problem 6 (15%)**

Consider a worker who lives for two periods and lives in a labor market that features different occupations ($j \in J$). In the first period, given the initial human capital h_0, workers work in their current occupations, $j \in J$. Also, workers accumulate human capital via an "effort" (e) decision. In period 1:

$$h_1 = h_0(1+e)^\alpha$$

where $0 < \alpha < 1$. Workers can switch their occupations in the second period. If they switch, they lose a fraction $\delta(h)$ of occupation-related human capital:

$$\delta(h) = 1 - \frac{1}{h^\xi}.$$

where $0 < \xi < 1$.

The pre-tax earnings of a worker in occupation j in period t are given by:

$$y_{jt} = w_{jt}\tilde{h}_t z_t$$

where $\tilde{h}_1 = h_0$, and

$$\tilde{h}_2 \equiv (1-\delta(h)) \times \mathbb{1}(switch)h_1$$

is the human capital where $\mathbb{1}(switch)$ is an indicator variable that takes the value one if the worker switch occupations. The variable w_{jt} is the wage rate in occupation j and z_t is the shock from a uniform distribution $U[\underline{a}, \overline{a}]$. Assume that $z_0 = 1$. Also, for simplicity, if workers switch their occupation, they won't face a shock, i.e. $z_1 = 1$. Yet, if workers stay in their occupation and $z_1 = a$.

We assume taxes are collected by the function: $T(y) = y - \lambda y^{1-\tau}$, with $0 < \lambda < 1$ and $0 < \tau < 1$. Then,

$$c_{jt} = \lambda y_{jt}^{1-\tau}.$$

For simplicity, we shut down heterogeneity in salaries across occupations, i.e. $w_{jt} = w_t = 1$. Also, assume households' preferences for per period consumption is represented by $u(c)$ (increasing and concave) and disutility of effort for human capital improvement is: $h(e) = \mu\frac{e^2}{2}$.

(a) **(5%)** What α and ξ represent? Please provide economic intuition.

(b) **(5%)** Use backward induction, i.e. given the effort decision, e, and obtain the decision rule for the worker to move to the other occupation. This is the same as solving for the threshold level of a such that the worker moves or stays in its current occupation. Is the mobility decision affected by the parameters of the tax function? Please provide intuition.

(c) **(5%)** Given your previous response, set up the problem to derive the worker's optimal effort choice in the first period. Obtain the first order conditions.