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August 14, 2025

Comprehensive Exam in Macroeconomic Theory–Procedural Instructions

1. There are total of 6 questions. The last question ends on page 6 (6.c).
2. Write your answers only on the paper we provide.
3. We are distributing a numbered sign–in sheet in a moment. The number next to your signature will be your **student number**.
4. Every sheet of paper you turn in to us **must** have your **student number** written at the **top–center** of the sheet and **circled**.
5. Every sheet of paper you turn in to us **must** have a **page number** written at the **top–right corner** of the sheet.
6. You have 4 hours to complete the exam. When you have finished, or when it is 4:00 pm (whichever comes first), prepare a cover sheet for your exam. This cover sheet should not have a page number, but must have the following things on it:
   (a) Your **student number** at the top–center, circled.
   (b) The phrase "Macroeconomics Comprehensive Exam".
   (c) The sentence "My last page is page number X", where X is your total number of pages of answers.

1. **(20%)**

   A transportation company operates buses whose engines wear out with use. Each period t the company chooses whether to continue using the current engine or replace it with a new one. Let x_t be the mileage (or age) of the engine at time t, being a continuous variable. The engine of a bus has maintenance or running costs given by the function $c_{\text{run}}(x)$ which is increasing in x. If the company decides to replace the engine, it will incur in a high and fixed replacement cost denoted by c_{rep}. The mileage process is random and it evolves according to a distribution. When replacing the engine, miles are reset to 0. Time goes to infinity and the objective of the company is to minimize the expected discounted sum of costs given a discount factor β.

   (a) **(5%)** Set up the dynamic programming problem for the company.

   (b) Now assume that mileage is a discrete variable, i.e. $x_t \in \{0, 1, 2, 3\}$ (coarsened bins from "new" 0 to "very high" 3). Also, that operating cost when continuing: $c_{\text{run}}(x) = 2 + x$ and; replacement cost when replacing: $K = 6$, after which mileage resets to 0. Also assume $\beta = 0.9$.

   As for the mileage transitions.If $d = 1$ (Replace): next mileage is $x' = 0$ with probability 1. If $d = 0$ (Continue): x' follows:

current x	$P(x' = x \mid x)$	$P(x' = x + 1 \mid x)$
0	0.5	0.5
1	0.4	0.6
2	0.3	0.7
3	0.6	0.4

   i. **(5%)** Set up the dynamic programming problem, stating the state-contingent choice-specific value functions.

   ii. **(5%)** Consider the initial guess $V^{(0)}(x) = 0$ for all x. Compute *one* Bellman update to obtain $V^{(1)}(x)$ for each $x \in \{0, 1, 2, 3\}$. Show your intermediate calculations (expected continuation values).

   iii. **(5%)** Suppose the company is (incorrectly) myopic and sets $\beta = 0$. Derive the implied decision rule. For which x would it replace? Please provide economic intuition.

2. **(15%)**

   Consider an infinitely lived consumer who has time-separable preferences, with single-period utility given by $u(c) = \log(c)$, and a discount factor of $\beta = 0.95$. Consumption is *exogenously* determined–this is not a dynamic programming problem in which it is optimally chosen – by a 3 states i.i.d distribution. Consumption can take the value of 1 with probability 0.25, 2 with probability 0.50 and 3 with probability 0.25. In other words, at the beginning of period t the consumer knows the realization of consumption and form its expectations according to the probabilities described above.

(a) **(5%)** Write down the functional equation that characterizes the consumer's discounted expected utility from the exogenous consumption stream. Please specify all the elements, including the expectation operator. What are the dimensions of the value function?

(b) **(5%)** Solve the functional equation in the previous step. You don't need to calculate the exact numbers for your functional equation, but to get the solution in terms of primitives.

(c) **(5%)** Explain the steps of an iteration algorithm to solve the functional equation you specified.

3. **(15%)**

Consider an infinite-horizon economy populated by two types of agents indexed by $i \in \{1,2\}$. Time is discrete: $t = 0,1,2,\ldots$. The process $\{s_t\}$, where $s_t \in S_t = \{1,2\}$ for all t, is i.i.d. with $\pi(s_t) > 0$ for all s_t. The initial state s_0 is known. Let S^t denote the set of partial histories up to t: $s^t = (s_0, \ldots, s_t)$. All agents observe these histories. The probability of s^t is denoted by $\pi(s^t)$.

Agents have preferences:

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\ln c_{i,t} - \frac{l_{i,t}^{\phi}}{\phi} \right) \right\}, \quad \text{where } \beta \in (0,1),\ \phi > 1$$

$c_{i,t}$ is consumption, $l_{i,t}$ is labor supply.

Each period, agents are endowed with labor productivity $e_i(s_t)$, given by:

$$e_i(s_t) = \varepsilon_i + \tau_i(s_t), \quad \varepsilon_i > 0, \quad \varepsilon_1 + \varepsilon_2 = 1, \quad \tau_1(s_t) + \tau_2(s_t) = 0$$

This means that if agent i works l_i hours with a productivity level e_i, his total supply of effective units of labor is $e_i l_i$. There is an aggregate linear technology to produce the consumption good where aggregate production at time t is:

$$Y(s_t) = B[e_1(s_t) l_1(s_t) + e_2(s_t) l_2(s_t)], \quad B > 0$$

Agents are endowed with no financial wealth at date 0.

Let α be the welfare weight on agent 1. Define the planner's recursive problem and characterize the solution justifying your answer. *Hint: optimal policy functions will depend on welfare weights.*

4. **Solow growth model with population and technology growth (10%):** The economy has discrete and infinite time horizon, $t = 0, 1, 2, \ldots$. There is one final output in the economy, which can be either consumed or invested. Output is produced with labor and capital input. Output function is given as a Cobb-Douglas function with a capital share α.

$$F(K_t, A_t L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha},\ \alpha \in (0,1)$$

Assume that the economy has a constant savings rate s, investing a constant fraction of output every period. Capital stocks depreciate by rate δ every period. The economy is endowed with an initial level of capital $\bar{K}_0$ and the maximum labor hours is given as 1. Population (N_t) grows at rate n, given the initial level of population $N_0 = 1$. Labor productivity A_t grows at rate g, and initial technology level is given as $A_0 = 1$.

(a) Express the capital law of motion and output function in terms of "per effective worker" unit. **(3%)**

(b) Derive the steady state levels of capital, output, consumption, and investment of the normalized (per effective worker) economy. **(4%)**

(c) What is the golden rule of savings rate, which maximizes the consumpion at the steady state? **(3%)**

5. **Neoclassical Growth Model: Recursive Competitive Equilibrium (20%):**

Let's consider the infinite horizon economy. There is a representative household and a representative firm. The household lives forever, and time is discrete. Household's flow utility is given as $u(c,l)$, where $u(\cdot,\cdot)$ is increasing, twice differentiable, and concave with respect to the first argument (consumption) and decreasing, twice differentiable, and convex with respect to the second argument (labor hours). The household maximizes the present value of utility, $\sum_{t=0}^{\infty}\beta^t u(c_t,l_t)$, where $\beta \in (0,1)$ is a discount factor. The firm's production function is given as $F(K,L) = K^{\alpha}L^{1-\alpha}$, where $\alpha \in (0,1)$ is a capital share. Household is endowed with initial level of capital $\bar{K}_0$ in period 0 and maximum labor hours $\bar{L} = 1$ in each period. Firms rent capital from households and hire labor every period and maximize their profit. In each period, the capital stock depreciates with a rate $\delta \in [0,1]$.

(a) Define a Recursive Competitive Equilibrium. **(6%)**

(b) Now consider a special case where there is no labor disutility and the household's utility function is given as $u(c)$. Under this special case, characterize the Recursive Competitive Equilibrium, including the Envelope condition and functional Euler equation. **(7%)**

(c) Now consider a special case where there is no labor disutility and the household's utility function is given as $u(c) = \log(c)$ and the capital stock depreciates fully in each period ($\delta = 1$). Under this special case, derive a closed-form solution for the capital policy function. **(7%)**

6. **Pure exchange economy with uncertainty and IOUs (20%):** Consider the following economy. There are two agents, household 1 and 2, and time is discrete. There are stochastic endowments $\{e_t^i(s^t)\}$ given to household i in each event history node $s^t \in S^t$. In each event history node, households can trade Arrow securities $a_{t+1}^i(s^t, s_{t+1})$, $\forall i = 1, 2,\ s^t \in S^t,\ s_{t+1} \in S$. Prices of arrow securities are denoted as $q(s^t, s_{t+1})$. There is a zero net supply of IOUs ($\sum_{i=1}^2 a_{t+1}^i(s^t, s_{t+1}) = 0$). There is a borrowing limit $-\bar{A}^i$, which never binds in equilibrium. Initial position of IOU is zero.

   The flow utility of each household $u(c)$ is given as an increasing, twice differentiable, and concave function. Each household maximizes her expected lifetime utility with a common discount factor $\beta \in (0, 1)$.

   (a) Define a Sequential Markets Equilibrium. **(6%)**

   (b) Characterize the Sequential Markets Equilibrium. **(6%)**

   (c) Now, consider the following environment. The utility function is given as a log utility ($\log(c)$) for both agents. There are two possible realizations for endowments, High (e_H) and Low (e_L). Endowments are i.i.d. with the following probability:

$$Prob(e_t^i = e_H) = 1/4 \tag{1}$$

$$Prob(e_t^i = e_L) = 3/4,\ \forall i = 1, 2,\ \forall t = 0, 1, 2, \ldots \tag{2}$$

   Under this specification, write down the equilibrium allocations and prices in the Sequential Markets Equilibrium. **(8%)**