Problem Set 2

Due date: Thursday, February 11, in class.

Consider the simple linear regression model with one regressor:

\[ y_i = \beta x_i + u_i , \]

where \( y_i \) and \( x_i \) are expressed in deviations from their means. Assume that the standard classical assumptions hold. Specifically, the \( x_i \)'s are fixed in repeated sampling, and \( u_i \sim iid(0, \sigma^2) \).

1. Consider the following linear estimator of \( \beta \):

\[ \tilde{\beta} = \frac{1}{N} \sum_{i=1}^{N} \frac{y_i}{x_i} \]

Derive the mean and variance of \( \tilde{\beta} \). Compare the variance of \( \tilde{\beta} \) to \( \frac{\sigma^2}{\sum x_i^2} \). Which is larger? Are the residuals, \( y_i - \tilde{\beta} x_i \), uncorrelated with the explanatory variables?

2. Answer the same question with \( \tilde{\beta} = \frac{y_2 - y_1}{x_2 - x_1} \).

3. Consider the following regression model \( y_i = \beta x_i + u_i \), with \( u_i \sim iidN(0, \sigma^2) \), where \( i = 1, 2, \ldots, 10 \);

Suppose that \( \sum x_i y_i = 17900 \), \( \sum x_i^2 = 39400 \), and \( \sum u_i^2 = 283.27 \)

Consider the following hypothesis and 2-sided alternative:

\[ H_0 : \beta = 0.50 \]
\[ H_1 : \beta \neq 0.50 \]

Test the above hypothesis using a confidence interval and the test of significance approach. Set the size of your test to 0.05 and 0.01. Report the outcome of both tests. Also report the p-value of your t-statistic.
4. Prove the Frisch-Waugh Theorem. That is, in the linear regression model

\[ y = X_1\beta_1 + X_2\beta_2 + u , \]

demonstrate that

\[ \hat{\beta}_1 = (X_1^* X_1^*)^{-1} X_1^* y^* , \]

where \( X_1^* \equiv M_2 X_1 \), \( y^* \equiv M_2 y \), and \( M_2 = I - X_2 (X_2' X_2)^{-1} X_2' \).

5. Consider a nonsingular linear transformation of the regressors, \( XA \), where the matrix \( A \) is a \( k \times k \) and invertible.

Show that the fitted values and the residuals from a regression of \( y \) on \( XA \) are the same as from a regression of \( y \) on \( X \).

The following results from matrix algebra will be helpful:

\( (BC)' = C' B' \)

\( (BC)^{-1} = C^{-1} B^{-1} \), whenever all three inverses exist.