Final Exam, December 3rd, 2003—7 questions. All sub-questions carry equal weight.)

1. (18%) Consider two random variables X and Y. Assume they both are discrete and that both X and Y can take the values 1, 2, and 3. The probabilities for (X,Y) are shown in the following table:

<table>
<thead>
<tr>
<th>X=1</th>
<th>X=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y=1</td>
<td>1/12</td>
</tr>
<tr>
<td>Y=2</td>
<td>1/12</td>
</tr>
<tr>
<td>Y=3</td>
<td>2/12</td>
</tr>
</tbody>
</table>

i) Find the marginal probabilities of X.
ii) Find the mean and the variance of X.
iii) Are the events X = 1 and Y = 1 independent events?
iv) Are the random variables X and Y independent?
v) Find the probability \( P(\{X > 1\} \cap \{Y \leq 2\}) \)
vi) Find the conditional distribution of X given Y = 2.

2. (12%) Assume \( X_1, X_2, \) and \( X_3 \) are identically and independently exponentially distributed with mean 1. Let Y be the largest of these 3 random variables \( Y = \max\{X_1, X_2, X_3\} \). Derive the density (PDF) for Y.

3. (12%) Assume \( X \sim N(0,9), Y \sim N(2,9), \) and \( Z \sim N(2,9) \). Further assume that the co-variance between X and Y is 2, while both X and Y are independent of Z.
   i) What is \( E(X|Y = 2, Z = 3) \)? (State the formula you use and then the number.)
   ii) What is the conditional variance \( \text{Var}(X|Z = 3) \)?

4. (20%) Assume \( X_1, X_2, \ldots, X_n \) are all iid normally distributed with mean 0 and variance \( \sigma^2 \).
   i) State and derive the distribution of the average \( \overline{X} \)?
   ii) State and derive the distribution of \( s^2 \).
   iii) Normalize \( \overline{X} \) with something [you need to state what, I will call it \( W \) for now] such that you get a t-distribution. What are the degrees of freedom?
   iv) Demonstrate that \( \overline{X}/W \) [where you explained in part iii] what \( W \) is] is t-distributed.

5. (12%) Prove the law of iterated expectations (you can do the discrete or the continuous case).
6. (16%) In some random experiment, $\hat{\theta}_n$ is a consistent estimator of $\theta$.
   i) Is $\log \hat{\theta}_n$ a consistent estimator of $\log \theta$?

   Assume $X_n$ is a sequence of random variables which converges in distribution to $X$.
   ii) Is $\theta_n X_n$ a consistent estimator of $\theta X$ (why or why not)?

7. (10%) Formulate and derive Bayes’ Law.