LAST TIME: Nonlinear oscillators, Hamilton’s principle, calculus of variations, Euler-Lagrange equation

An alternate view of classical mechanics: Hamilton’s principle – The actual path that a particle follows between two points in a given time, $t_1$ to $t_2$, is such that the action integral

$$ S = \int_{t_1}^{t_2} L \, dt $$

is stationary when taken along the actual path.

$L$ is known as the Lagrangian function with $L = T - U = \text{difference between the kinetic and potential energies}$. Therefore we want $\delta S = 0$, but we do not want to have to repeat the process every time we calculate the variation. This led to the Euler-Lagrange equation

$$ \frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0. $$

We defined generalized coordinates with the transformations given by $r = r(q_i)$ and $q_i = q_i(r)$. If more than one variable is involved, then

$$ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0. $$

There will be a Lagrange equation for each of the generalized coordinates. What is the best procedure for solving problems where the Lagrangian approach is the best approach, and how do we know when the Lagrangian approach is the best method? Comments.

Procedure: Write $T = T(\dot{x}, \dot{y}, \dot{z}) = (1/2)m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Reason

Write $r = r(q_1, q_2, q_3)$ It is usually obvious what the coordinates should be.

Get $\dot{x}, \dot{y}, \dot{z}$ in terms of $\dot{q}_1, \dot{q}_2$, and $\dot{q}_3$.

Write the potential energy using the coordinates you set up. Write $L = T - U$ in terms of $q, \dot{q}$, and $t$.

Here is an example of the motion of a particle in two dimensions using polar coordinates. In this figure, the velocity is already shown in polar coordinates, but to illustrate my
approach, let’s start by writing the transformation equations and getting the kinetic energy.

\[ x = r \cos \phi, \quad y = r \sin \phi, \quad \dot{x} = \dot{r} \cos \phi - r \dot{\phi} \sin \phi; \quad \dot{y} = \dot{r} \sin \phi + r \dot{\phi} \cos \phi \]

Therefore, \[ T = (1/2)m(\dot{r}^2 \cos^2 \phi - 2r \dot{r} \sin \phi \cos \phi + r^2 \dot{\phi}^2 \sin^2 \phi) \]

\[ + \dot{r}^2 \sin^2 \phi + 2r \dot{r} \sin \phi \cos \phi + r^2 \dot{\phi}^2 \cos^2 \phi \] = \( (1/2)m(\dot{r}^2 + r^2 \dot{\phi}^2) \).  Comments about derivatives.

\[ U = U(r, \phi) \text{ in general so } L = (1/2)m(\dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi). \]

Now we get the equations of motion directly from Lagrange’s equations, one equation for \( r \) and one for \( \phi \).

\( r \) equation: \( \frac{d}{dr} \frac{\partial L}{\partial \dot{r}} = m \dot{r}^2 - \frac{\partial U}{\partial r}; \quad \frac{\partial L}{\partial r} = m \ddot{r}; \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m \ddot{r} \)

Therefore, \( m \dot{r}^2 - \frac{\partial U}{\partial r} - m \ddot{r} = 0 \) \( \Rightarrow \) \( m(\ddot{r} - r \ddot{\phi}^2) = F_r. \)

\( \phi \) equation: \( \frac{\partial L}{\partial \phi} = -\frac{\partial U}{\partial \phi}; \quad \frac{\partial L}{\partial \phi} = m r^2 \ddot{\phi}; \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = mr^2 \dddot{\phi} \)

Therefore, \( -\frac{\partial U}{\partial \phi} - \frac{d}{dt} mr^2 \ddot{\phi} = 0. \) Notice that \( \frac{\partial U}{\partial \phi} \) is not the \( \phi \) component of \( \nabla U = \frac{\partial U}{\partial r} \dot{r} + \frac{1}{r} \frac{\partial U}{\partial \phi} \dot{\phi}. \)

\( F_\phi = -\frac{1}{r} \frac{\partial U}{\partial \phi} \) This means that \( r F_\phi = \frac{d}{dt} mr^2 \ddot{\phi} \) \( \Rightarrow \) \( \Gamma = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}. \) This points out a more general feature of Lagrange’s equations.

\[ \frac{\partial L}{\partial \dot{q}_i} = i^{th} \text{ component of the generalized force} \]

\[ \frac{\partial L}{\partial \dot{q}_i} = i^{th} \text{ component of the generalized momentum} \]

Therefore, \( \frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}. \)

Generalized force = the time derivative of the generalized momentum. The example above is for a free particle without constraints. So far, we have encountered normal forces and strings as constraints.
Holonomic constraints and degrees of freedom: A holonomic constraint is one that can be expressed as \( f(q_1, q_2, q_3, \ldots, q_n, t) = 0 \). We will deal with problems where that is the case. If an object is constrained to move on the surface of a sphere of radius \( a \), then \( r = a \) is a holonomic constraint. Each constraint reduces the number of degrees of freedom by one. Sometimes, constraints may be “hidden.”

Summary: For any holonomic system with \( n \) degrees of freedom and \( n \) generalized coordinates, and with nonconstraint forces derivable from a potential,

\[
\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0.
\]

Now let’s tackle a classic problem that will have numerous interesting features and allow us to understand the value of the Lagrangian approach.

Bead on a spinning wire loop without friction: Here is the figure. First, let’s analyze what we think might happen with this bead. What would happen if \( \omega = 0 \)? Now what changes occur if it spins? Normal force and \( mg \) considerations. If we wish to consider the problem of how the bead behaves with respect to \( \theta \), why do we run into problems with Newton’s second law?

\[
T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)
\]

What are our transformation equations? \( \rho = R \sin \theta \) and

\[
x = \rho \cos \phi \text{ with } y = \rho \sin \phi \text{ and } \phi = \omega t; \quad z = R \cos \theta
\]

Finally, \( x = R \sin \theta \cos \phi; \quad y = R \sin \theta \sin \phi \)

Calculate \( \dot{x}, \dot{y}, \text{ and } \dot{z} \) and then their squares. Here they are.

\[
\dot{x} = R\dot{\theta} \cos \theta \cos \phi - R\dot{\phi} \sin \theta \sin \phi
\]

\[
\dot{y} = R\dot{\theta} \cos \theta \sin \phi + R\dot{\phi} \sin \theta \cos \phi
\]

\[
\dot{z} = -R\dot{\theta} \sin \theta
\]

\[
\dot{x}^2 = R^2\dot{\theta}^2 \cos^2 \theta \cos^2 \phi - 2R^2\dot{\theta}\dot{\phi} \sin \theta \cos \theta \cos \phi \sin \phi + R^2\dot{\phi}^2 \sin^2 \theta \sin^2 \phi
\]
\[ \ddot{y}^2 = R^2 \dot{\theta}^2 \cos^2 \theta \sin^2 \phi + 2R^2 \dot{\phi} \sin \theta \cos \theta \cos \phi \sin \phi + R^2 \dot{\phi}^2 \sin^2 \theta \cos^2 \phi \]

\[ \dot{z}^2 = R^2 \dot{\theta}^2 \sin^2 \theta \]

The middle terms cancel and some of the other terms combine to give

\[ \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = R^2 \dot{\theta}^2 + R^2 \dot{\phi}^2 \sin^2 \theta = R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2 \theta. \]

Setting \( U = 0 \) at the bottom, \( U = (R - R \cos \theta)mg = mgR(1 - \cos \theta). \)

Finally, \( \mathcal{L} = (1/2)m(R^2 \dot{\theta}^2 + R^2 \omega^2 \sin^2 \theta) - mgR(1 - \cos \theta). \)

Notice that the only variable that appears is \( \theta \), indicating that we have only one degree of freedom. What are the two constraints? \( R = \text{constant} \) is one and \( \phi = \omega t \) is the other. So the one variable is reasonable.

\[ \frac{\partial \mathcal{L}}{\partial \theta} = (1/2)mR^2 \omega^2 (2 \sin \theta \cos \theta) - mgR \sin \theta \]

\[ \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = (1/2)mR^2 (2\dot{\theta}) = mR^2 \dot{\theta} \text{ and } \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mR^2 \ddot{\theta} \]

Last

\[ \ddot{\theta} = \sin \theta \left( \omega^2 \cos \theta - g/R \right) \]

What to do now!!

Equilibrium occurs when the acceleration is zero, so whenever \( \ddot{\theta} = 0 \), there are four values of \( \theta \) for equilibrium. 0, \( \pi \), and \( \theta = \pm \cos^{-1}(g/R) \). From \( U = mgR(1 - \cos \theta) \), we get

\[ dU/d\theta = mgR \sin \theta = 0 \text{ at } 0 \text{ and } \pi. \]

The second derivative \( d^2U/d\theta^2 = mgR \cos \theta \) tells us the stability at the two values. To use Taylor’s method for determining the stability of the remaining points, we let \( \theta = \theta_o + \epsilon \) and expand the sines and cosines.

This gives \( \ddot{\theta}_o + \ddot{\epsilon} = [\omega^2 \cos (\theta_o + \epsilon) - g/R] \sin(\theta_o + \epsilon) \) for \( \epsilon \ll 1 \) and \( \theta \approx \theta_o. \)

\[ = [\omega^2 (\cos \theta_o \cos \epsilon - \sin \theta_o \sin \epsilon) - g/R] (\sin \theta_o \cos \epsilon + \cos \theta_o \sin \epsilon) \]

\[ = [\omega^2 (\cos \theta_o - \epsilon \sin \theta_o) - g/R] (\sin \theta_o + \epsilon \cos \theta_o). \]

Now we use the value of \( \theta_o \) in the equation to get

\[ \ddot{\epsilon} \approx -\epsilon \omega^2 \sin^2 \theta_o \text{ since the term in } \epsilon^2 \text{ is neglected.} \]
But \( \sin^2 \theta_o = 1 - \left( g/R \omega^2 \right) \) so define \( \Omega^2 = \omega^2 - (g/\omega R)^2 \).

Suppose we make the small angle approximation in the original equation. Then

\[
\ddot{\theta} = \theta (\omega^2 - g/R).
\]

If \( \omega^2 - g/R > 0 \), then we have an equation of the form \( \ddot{\theta} = k\theta \), so the equilibrium is unstable and no oscillation occurs. If, however, \( \omega^2 - g/R < 0 \), then the equation is of the form \( \ddot{\theta} = -k\theta \) and we get SHM. Now let’s deal with this issue I mentioned near the end of the last period concerning why this potential energy does not yield the equilibrium points at \( \theta = \pm \cos^{-1}(g/\omega^2 R) \).

We rewrite \( \frac{d^2 \theta}{dt^2} = \frac{d}{dt} \omega_p \frac{d \omega_p}{d \theta} = \sin \theta \left( \omega^2 \cos \theta - g/R \right) \). This expression may be directly integrated to find the kinetic energy of the bead as it moves along the hoop.

\[
\frac{\omega_b^2}{2} = \int \left[ \sin \theta \left( \omega^2 \cos \theta - g/R \right) \right] d\theta = \frac{\omega^2 \sin^2 \theta}{2} + \frac{g}{R} \cos \theta + \text{constant}
\]

Therefore, the new constant of the motion is

\[
\frac{\omega_b^2}{2} - \frac{\omega^2 \sin^2 \theta}{2} - \frac{g}{R} \cos \theta = \text{constant}
\]

Now we call the second two parts of this \( U_{\text{eff}} \). Therefore,

\[
U_{\text{eff}} = -\frac{\omega^2 \sin^2 \theta}{2} - \frac{g}{R} \cos \theta.
\]

Assign homework to get all equilibrium positions from \( U_{\text{eff}} \).

**NEXT TIME**: More examples of calculations using the Lagrangian.