Capital Gains and Dividends in the Black-Scholes Model

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Abstract: Under a no arbitrage condition, the value of a stream of payments equals a conditional expectation of those payments. In this note, we observe that if there is a fixed risk-free rate, then the no arbitrage condition implies a linear relationship between the stock price process and the payment stream. If the stock price is Markov, then so too is the dividend. If the stock price is a random walk, so is the dividend. If the stock price is a diffusion such as the popular geometric Brownian motion, then the payout must also be a diffusion. This simple result exposes a flaw in the classical Black-Scholes (1973) model in which Black-Scholes assume that dividends are not risky prior to some future date, *T*. The dividends must be a geometric Brownian motion as well.

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1. Introduction

As a consequence of the martingale representation established in Harrison and Kreps (1979) and Harrison, Pliska (1981), the value of a claim on a stream of payments $\{D_t\}$ is equal to

$$(1) \qquad P_t = E_t^* \int_0^\infty D_{t+s} ds$$

where $E_t^*[\cdot]$ is a conditional expectation not necessarily equal to the true, objective density. Equation (1) is valid in both continuous and discrete time. If dividends are paid in discrete intervals, the integral collapses to a summation.

In this note, we observe that if there is a risk-free rate, r, and the stock price (1) is a Geometric Brownian Motion, then the dividend payment D_t is a diffusion (Ito process). Dividends are *continuously risky*.

This simple result exposes a flaw in the Black-Scholes (1973) model. The classic Black-Scholes (1973) option pricing model assumes no arbitrage opportunities are available in the economy and the price of the underlying asset is a geometric Brownian motion. To price options, Black-Scholes also "assume" that dividends are not risky prior to some future date, T. That is, they also assume the dividend process are *not continuously risky*. Their model is thus based on a set of assumptions which are mutually contradictory. They simultaneously assume dividends are not a diffusion and that they are a diffusion.

Intuitively, Black-Scholes' methodology relies on there being just one source of uncertainty in the economy. In order to form a hedge portfolio (which leads to the Black-Scholes PDE), asset prices must depend on a single source of risk. In the case of a stock price, the single shock must be the dividend payment. A stock is a claim on dividends. In fact, the stock price is linear in the dividend as a consequence of risk-neutral valuation. Assuming the stock is a Geometric Brownian motion pins down the dividend process as a diffusion too.

Section 2 presents a proof of the main theorem, a subset of the assumptions made by Black-Scholes imply the dividend is a diffusion. Section 3 discusses the consequences of dividends being continuously risky over time. An alternative to the standard Black-

Scholes model is derived by fixing dividends and letting the discount rate vary. Section 4 concludes.

2. If the stock price is a geometric Brownian motion, the dividend is a diffusion

In this section we derive the relationship between the stock price process and the dividend payment. We begin by listing the standard assumptions of interest.

Assumption 1. No arbitrage opportunities exist.

The value of a claim a to random payoff at T, which can be an asset and has no liability,

$$E_t\ P_T > 0, \quad P_T \geq 0 \tag{2}$$

must have a non-zero price, $P_t(P_T) > 0$. The prices $\{P_t\}$, $t \in [0,\infty]$, are real valued random variables adapted with respect to a filtration Ω_t which has the usual interpretation as the time t information set.

Harrison and Kreps (1979) and Harrison, Pliska (1981) show that under no arbitrage, the current price P_t can be written

$$P_t(P_T) = E_t^*\ P_T \tag{3}$$

where $E_t^*[\cdot]$ is a conditional expectation not necessarily equal to the true, objective density. Equation (3) is valid in both continuous and discrete time.

Assumption 2. Capital markets are complete and frictionless

Assumption 3. There is a riskless interest rate.

Black-Scholes assumed that there is a riskless interest rate, r_t, which is interpreted as the payoff to a riskless bond. For a zero-coupon bond that pays 1 dollar upon expiration,

$$\begin{aligned} 1 &= B_t e^{r(T-t)} \\ \Rightarrow B_t &= e^{-r(T-t)}. \end{aligned} \tag{4}$$

A share of stock is a claim on a stream of future dividends $\{D_t\}$ which can be bought or sold for price P_t. The dividend stream $\{D_t\}$ is non-negative and at least one of them is random. From the martingale-equivalent representation of Harrison, Pliska (81), the stock price is equal to

(5) $$P_t = E_t^* \int_0^\infty D_{t+s} ds$$

(6) $$= \int \int_0^\infty D_{t+s}^\omega ds \ f_s(\omega | \Omega_t(z_t)) d\omega$$

where $\Omega_t(z_t)$ is the history of the conditioning information. If $\{D_t\}$ are paid discretely, the integral collapses to a summation. We do not assume the process $\{D_t\}$ is continuous but rather show it as a result.

Assumption 4. The stock price is the geometric Brownian motion

(7) $$dP_t = P_t(\mu dt + \sigma dW_t)$$

where (μ , σ) are fixed, known parameters.

This process is appealing for modeling a stock price. It is the continuous time analog of a random walk in percentages. Moreover, Black and Scholes (1973) note a derivative asset V_t (P_t) could then be valued by appealing to Ito's lemma. In particular, we can form a hedge portfolio $(1, \beta_t)$ in $V_t(P_t)$ and P_t such that the return:

(8) $$d(V_t - \beta_t P_t)$$

is *riskless*. This leads immediately to the partial differential equation (PDE)

(9) $$d(V_t - \beta_t P_t) = rdt$$

which can be solved for dV_t in terms of P_t.

Our main result is that under assumptions A1-A4, the dividend payment is a diffusion process.

Theorem 1. Assumptions A1-A4 imply that the dividend payment on which a stock is a claim,

(10) $$dD_t = \mu_t^0(P_t)dt + \sigma_t^0(P_t)dP_t$$

is a diffusion process which is instantaneously linear in the stock price dP_t.

Our strategy is to first show that P_t=$G(D_t)$, the price is a function of the dividend payment which is invertible,

(11) $$D_t = G^{-1}(P_t).$$

It then follows, by applying Ito's lemma, that the dividend is also a diffusion. Economically, we can treat the dividend process as any other derivative written on the stock. This means the stock price and dividend must commove.

Proposition 1. Arbitrage pricing of $V_t(P_t)$ requires that stock price P_t be a function $P_t=G(z_t)$ of a scalar random variable z_t.

Arbitrage pricing means both assets must depend on a single source of risk.

Proof. Suppose not, i.e. the stock depends on two (or more) variables,

$$P_t = G(x_t, y_t). \tag{12}$$

The return to any portfolio is:

$$\begin{aligned} & dV(P_t) - \beta_t dP_t \\ &= dV(x_t, y_t) - \beta_t dP_t(x_t, y_t) \\ &= a_1(x_t, y_t)dt + b_{1t}dy_t + c_{1t}dx_t \\ &\quad - \beta_t\left[a_2(x_t, y_t)dt + b_{2t}dy_t + c_{2t}dx_t\right]. \end{aligned}$$

There is no β_t for which

$$\begin{aligned} b_{1t} - \beta_t b_{2t} &= 0 \\ c_{1t} - \beta_t c_{2t} &= 0 \end{aligned} \tag{13}$$

both sources of risk can be hedged. There is no hedge portfolio. It is not possible to solve for V_t in terms of P_t. *QED.*

The arbitrage portfolio requires $P_t=G(z_t)$ where z_t is the unique state variable. We now show that z_t must equal D_t, the dividend payment.

Proposition 2. The stock price is a function of the dividend payment $P_t = G(z_t) = G(D_t)$.

Proof. Writing the stock price using the martingale-equivalent measure,

$$P_t = E_t^*\left(\int_0^\infty D_{t+s}\, ds\right)$$

$$= \underbrace{D_t}_{measurable\ at\ time-t} + E^*\Big[\underbrace{\int_0^\infty (D_{t+s} - D_t \delta_{s=0})\, ds \mid x_t}_{not\ measurable\ at\ time-t}\Big]$$

(14) $$= D_t + h(x_t)$$

where $h(x_t)$ is a time-invariant function $h(\cdot)$ of the variable x_t used in forming conditional expectations.

We have that $G(z_t)$ is linear in D_t. The stock can only depend on one state so that the stock price (14) must be of the form

(15) $$G(z_t) = D_t + \hat{h}(D_t) .$$ QED.

The only state variable permitted under arbitrage pricing is the current dividend payment. The reason is the stock is a claim on dividends. Arbitrage pricing implies both that

1) the stock is a function of a single state variable
2) the stock is linear in the dividend (through risk-neutral valuation).

Taken together, the stock is a function only of the dividend. We can now prove the main result, Theorem 1. Under the assumptions A1-A4,

A1. No arbitrage

A2. Frictionless capital markets

A3. There is a riskfree rate.

A4. The stock price is a diffusion

the dividend payment is the diffusion process,

$$dD_t = \tilde{\mu}_t\ (P_t)dt + \tilde{\sigma}_t\ (P_t)dP_t .$$

Proof. Under no arbitrage, the stock price

$$P_t = E_t^*(\int_0^\infty D_{t+s}\ ds)$$

(16) $$= D_t + \int_{dt}^\infty E_t^*(D_{t+s})\ ds .$$

Since the conditional expectation is a linear operator, $E_t\ (X_{t+s} + Y_{t+s}) = E_t\ (X_{t+s}) + E_t(Y_{t+s})$ equation (16) can be written as

(17) $$= D_t\ + A(D_t\ )$$

where A is a linear operator.[1] Since $P_t = (1+A)D_t$ and A is linear, there is an inverse

(18) $$D_t = (1+A)^{-1}(P_t)\,.$$

The existence of the inverse in equation (19) can be shown formally using the Fredholm Alternative (e.g., Riesz, Sz.-Nagy (1990), p.164-165). If A is a linear operator and P_t has finite second moments (ensured by assumption A3) then either

A. there is some $D_t \neq 0 \; s.t. \quad (1+A)D_t = 0$

or

B. the equation $(1+A)D_t = P_t$ can be inverted : $D_t = (1+A)^{-1}(P_t)$ for any P_t.

No arbitrage implies that if $D_t \neq 0$ (at least one dividend is risky and dividends are non-negative), then $P_t > 0$. This eliminates the first Fredholm alternative and implies that

(19) $$D_t = (A+1)^{-1}(P_t) = V(P_t)$$

where V is a linear functional.

Applying Ito's lemma, since the stock dP_t is a diffusion, we have that

(20) $$dD_t = dV(P_t) = a_t dt + V'(P_t)dP_t$$

the dividend is a diffusion which is instantaneously linear in the stock.QED.

3. Consequences for option pricing

Assumptions A1-A4 which together imply the dividend process is continuously risky. Black-Scholes (1973) simultaneously "assume" that dividends are not risky prior to option expiration at some future date, T.

It is clear that their model, at least as originally constructed, is based on assumptions which contradict each other. The pricing formula and any other conclusion is therefore invalid.

More generally, the riskiness of dividends presents a substantial obstacle to the valuation of Euoprean options. A European option, F_t, is a claim on a random payoff at a future date, T. On the other hand, our main result shows the stock is a claim on random payoffs prior to T.

[1] To be precise, A is an linear operator on the Hilbert space of square integrable random variables.

Arbitrage pricing does not apply – we are no longer comparing two distributions at *T*. In general, preferences must be known to compare random claims paid at different times. The very reason Black-Scholes wanted dividends to be riskless during the life of the option is to avoid the risky claim prior to *T*.

Dividends are in fact continuously risky if the stock is continuously risky. Therefore *even in the absence of arbitrage*, preferences (intertemporal substitution) prevent the pricing of European derivatives.

3b. A Solution

Suppose that rather than dividends revealing information over time, the discount rate does. In particular, suppose that we drop assumption A3 and allow for a risky discount rate. We replace A3 with

Assumption A3'. The dividends are purely deterministic, D_t are never risky for any t. Under assumptions A1-A2, the stock price can again be written

$$P_t = E_t^* \int_0^\infty D_{t+s} ds \, . \tag{21}$$

Assumption A3' implies the dividends are deterministic, so that $E_t^* \int_0^\infty D_{t+s} ds$

$$\begin{aligned} &= E_t^* K \\ &= K \; E_t^* 1 \end{aligned} \tag{22}$$

where K is a constant.

Since the stock price (the LHS of equation (21)) is a diffusion, so must the conditional expectation in (22). The stochastic discount factor is a diffusion.

In this model, dividends are never risky but the stochastic discount factor is continuously risky. All variation in the stock price is driven by changes in the SDF.

It is not clear whether such an approach is sufficiently in touch with empirical realities. The idea that dividends are purely deterministic may not be a good approximation. But the approach has two attractive features to recommend it. First, it avoids the inconsistent treatment of dividends. Second, it seems more empirically realistic than the alternative with risky dividends being continuously paid to shareholders.

4. Conclusions

If we insist on a condition as strong as no arbitrage, the value of a risky claim takes its value from the underlying payments alone. Variation in the price of a claim can only come from uncertainty about the stochastic discount factor or about the payments. If we fix the SDF, the stock price is linear in the dividend process. If the stock price is a random walk with or without drift, then so too is the dividend. This holds in both discrete and continuous time and is a consequence of the extremely restrictive approach of arbitrage pricing.

This may be important for valuation of European options. In order to value a derivative, it is necessary to fix either the SDF or the dividend so that only 1 shock drives asset prices. Asset prices are typically modeled as some sort of Markovian stochastic process in order to satisfy market efficiency. Doing so pins down the dividend payout as a Markov process as well.

The riskiness of dividends presents an obstacle to the valuation of options. However, one possible solution is given by pretending dividends are purely deterministic. This implies that all variation in stock prices are explained by changes in the stochastic discount factor.

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