Graduate Macroeconomics I

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Contents

Preface

These notes are based mainly on the first-year graduate macro class of David Weil, with additional material from Sebnem Kalemli-Ozcan, Chris Carroll, and Brian Krauth that I've assimilated. There are sure to be errors and omissions, but those are mine alone.

Graduate Macro in Fifteen Minutes or Less

What Do We Care About?

Stuff. We care about how much stuff we have. Cars, boats, sandwiches, computers, clothes, and books among other things. The underlying assumption in this class is that we derive all our utility from consuming things. So we care about how much stuff we can consume at any given point in time. For the most part this is determined by the ability of the economy to produce stuff, and that depends on things like the capital stock (things like factories and tools), the technology level (internal combustion engines versus horse-drawn carriages), and the efficiency of our financial system (how easy it is for people to get loans and invest in their businesses). In the long run these are all that matters for our standard of living.

In intermediate macro, you would have learned how money gets used to buy stuff. That is, money makes it easier for the right stuff to find the right person in the economy. However, for the purposes of this course, we're going to ignore money (and all associated nominal concepts) completely. We will focus totally on the real side of the economy - on how much stuff we can produce and how our consumption decisions today affect how much stuff we can have tomorrow. This doesn't mean that we won't be able to talk about business cycles, but those cycles will have to be introduced through real shocks to the economy, as opposed being caused by the sluggish adjustment of the price level (a nominal thing).

One could argue that only the long-run real aspect of the economy matters, because it's trend dominates the path of income. Figure 1 plots US real GDP since 1970, and you can see that the trend growth swamps the cyclical movements. Understanding why the US trend growth was so big is arguably more important that understanding why there were slight dips and surges in output over the last 35 years. Across countries, the story seems much the same. Nigeria isn't poor because it has lots of recessions. Nigeria is poor because it's long term growth trend is essentially zero. This class will spend a lot of time talking about the long-run growth trends of economies, and not very much about business cycles. This is part of the reason why we ignore money, because if you

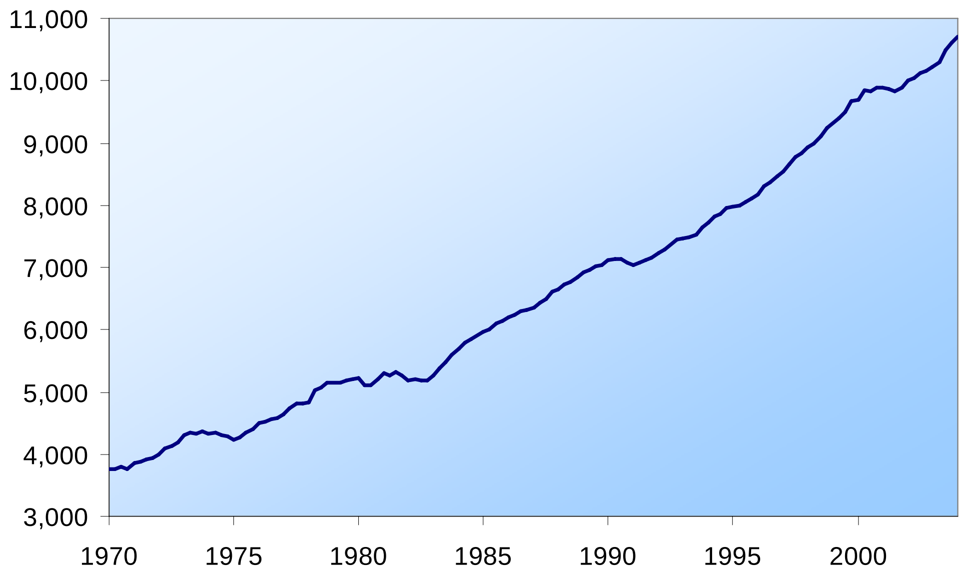

Figure 1: Growth in log income per capita in the U.S., 1970-2005

recall, in the long-run money is neutral. Also, we only have one semester, so we can't talk about everything.

Finally, the whole structure of modern macro is built around optimization over time. The models we build are centered around the idea that people take into account the future repercussions of their current actions. Consuming more stuff today means saving less stuff, and that may mean lower consumption in the future. We will be focusing on people's decisions about how to allocate their consumption (and savings, and work effort) across different periods of their lives.

The Structure of Study

The course will break down as follows:

1. **The Consumption/Savings Decision**. We start out by examining why we need to consider intertemporal decision-making at all. This involves looking into the properties of utility functions and the consideration of how to add up utility over time and how to handle uncertainty about utility.

2. **Mechanics of Economic Growth**. We step away from optimization for a while to consider the basics of economic growth, as this will allow us to think about how the interest rate and wages are actually determined in the economy. We'll cover production functions and look at the simplest

model of growth, the Solow model. These basics will then be useful for us in determining how individuals choose their consumption paths when they realize that their actions affect their future wages and interest rates.

3. **Essential Models of Dynamic Optimization**. This constitutes the heart of the class. We'll look at the two workhorse model structures of modern macro: infinitely lived households and the overlapping generations model. You'll see how to mathematically set up these problems and what the results actually mean. We'll cover solutions to these problems both with and without uncertainty about the future. Each model includes the possibility of output growth over time.

4. **The Open Economy Interpretation**. The models we consider can be used to examine the external positions of economies in the world. We'll spend a short time showing how these positions come out of models that are essentially identical to the previous models of dynamic optimization. As you'll have plenty of chances for open economy macro later in your training, this section won't be terribly long.

5. **Endogenous Growth Models**. In these growth models, there are again dynamically optimizing agents who take the mechanics of the growth process into account. In some cases, this won't result in much of a difference. The decisions that people make in these models are not simply between consumption versus savings, but also consumption versus research effort or consumption versus fertility. It also will introduce several common ways of modelling firms in markets without perfect competition.

Goals of this Class

From my perspective, the goal is to teach you the important intuitions and mechanics of modern macroeconomic research. This will allow you to start reading journal articles - the medium through which you'll actually teach yourself about macroeconomic issues and facts. This subtle difference exists in all of your first year courses. A useful analogy for graduate school is this: first year courses are similar to taking courses in a foreign language. You'll learn syntax and grammar, but you won't be reading any great works of literature. Your upper level classes will introduce you to the 'classics' in this new language. Finally, your own reading and research will be like finding a favorite author or genre and becoming fluent in this new language.

This means that you might feel frustrated by this first year, because it focuses a lot on mathematical techniques. This is not an indicator of how the profession feels about research. Research economists are not impressed with technique. But you cannot speak about the important empirical issues of the day without having this technique in place. So if you become frustrated, remind yourself that you are learning a new language, and you have to be patient.

From your perspective, the goals of this class are to learn enough macroeconomics to pass the comprehensive exams, and to understand the material well enough that you can begin reading journal articles. I have several words of advice for you.

1. This is your job. You are a poorly paid or unpaid intern in the economics profession, but you are a member of this profession now. Act professionally and take this seriously.

2. This is not at all like your undergraduate classes. In those classes we were trying to get across a small number of very general concepts. In graduate school we are trying to get across a large number of very specific concepts. This requires you to study more evenly throughout the semester, as opposed to cramming everything in just prior to tests.

3. Work with your classmates. You'll all see different aspects of the problems you'll be working on, and you'll learn from their insights while they learn from yours. Also, it helps to have other people who like to make fun of the professor.

4. Ask questions and interrupt class. If you aren't getting what I'm saying, stop me. Sometimes all it takes is for me to explain things in a slightly different manner for things to click.

5. Don't compare yourself to your classmates. You all have vastly different backgrounds and preparations for this. I am perfectly happy to give all of you superior ratings on your comprehensive exams. There is no competition going on here.

6. Do not ask "will this be on the test?" The answer is always yes. If your attitude is that you want to pass with as little effort as possible, then I'd suggest you find another line of work. If you really want to get a PhD, you should want to know everything.

7. Do as many problems as possible. Do the homework problems I assign, and then do the extra problems you have access to. Do old midterm and final questions. Do old comprehensive questions. After you've done all these problems, do them again. They are the best way to understand this material and the best way to study for your comprehensive tests.

Chapter 1

The Consumption/Savings Decision

In the beginning of macro, there was Keynes. And Keynes said that people consumed according to the following function (roughly)

$$C_t = c_0 + c_1 (Y_t - T_t) \tag{1.1}$$

and that c_1 represents the marginal propensity to consume. Notice that I've written the Keynsian function with a time subscript, t. This is so that you can see explicitly that consumption in period t is a function of disposable income in period t only. Future income, and expectations of future income, do not enter the Keynsian consumption function. This generates several problems.

- Theoretical: It seems weird that future income doesn't enter at all into someones decision to consume. Imagine that you know (with certainty) that you'll receive \$1,000,000 exactly one year from today. Wouldn't you consume more today? I would, but the Keynesian function in (1.1) says you won't. It says that when you receive the million dollars, your consumption will rise, but not before. That seems weird, so we're going to want to create a model of consumption that explicitly takes into account future income.

- Empirical: The Keynsian consumption function implies that the savings rate is affected by the income level

  $$s = \frac{Y - T - C}{Y - T} = 1 - \frac{C}{Y - T} = 1 - \frac{c_0}{Y - T} - c_1 \tag{1.2}$$

  which shows that savings rates should rise with income. In the cross-section, this seems to work. Bill Gates saves a higher proportion of his income than you do, for example. But over time, this doesn't seem to hold for countries or even necessarily for people. In Keynes defense, the data didn't exist at the time he was writing that could have showed him his function didn't work.

So the basic Keynsian model of consumption is flawed, and this flaw comes primarily from the fact that it does not account for the intertemporal nature of the savings decision. We want to construct a better model of savings, and to do so we'll want to explicitly consider optimizing agents. That is, we want an economic model of savings in which someone is maximizing their utility subject to a budget constraint. The budget constraint will involve the trade-off of consumption and savings, as opposed to the trade-off of good X and good Y. To start with, we'll need to understand the nature of the utility function a little better.

1.1 Properties of the Utility Function

Since we'll be looking at an intertemporal utility function, we'll need to distinguish between overall utility and period-specific utility. The period specific utility function, call it $U(c_t)$, tells us how much the consumption you do in period t makes you happy. Sometimes the period specific utility is referred to as "felicity". We assume that $U(c_t)$ has the following properties

$$\frac{\partial U(c_t)}{\partial c_t} = U'(c_t) > 0 \tag{1.3}$$

$$\frac{\partial^2 U(c_t)}{\partial c_t^2} = U''(c_t) < 0 \tag{1.4}$$

which are the basic properties that felicity is monotonically increasing in c_t, and that felicity exhibits decreasing marginal utility. Without these properties, and especially without (1.4), there really isn't anything interesting to say about dynamic optimization. The decreasing marginal utility of consumption has two important intepretations. First, a negative second derivative implies risk aversion - that is, people dislike uncertainty about consumption. Second, people like to smooth their consumption over time - that is, they dislike jumps or spikes in consumption. These two properties are really just the same thing, and they arise solely because of this negative second derivative.

This might be easier to see in diagrams. If one maps utility against consumption - $U(c)$ versus c - then what we are assuming is that this is a concave function. Utility goes up with more consumption, but at a decreasing rate. Graphing the first derivative against c shows us a convex function, or marginal utility is large when c is low (and approaching infinity as c approaches zero) and marginal utility is very low as c is high (and approaching zero as c approaches infinity). Finally, graph the second derivative and you get a line that is always negative, and is close to negative infinity when c is close to zero and close to zero when c is very large.

These graphs can give us the insight we need into why people are risk averse and prefer smooth consumption. The essence of the intution is this: the average of utilities is lower than the utility of the average. Take consumption of 10 and consumption of 20, with an average of 15. The nature of the utility function tells us that the average of the utility of 10 and utility of 20 is less than the utility of 15. You can see this easily on the graph.

Definition 1 *A way to measure the degree to which these effects bite is the* ***coefficient of relative risk aversion*** *and it measures the curvature of the utility function. Relative risk aversion measures the elasticity of marginal utility with respect to consumption, or*

$$-\frac{\partial U'/\partial C}{U'/C} = -c\frac{U''}{U'} \tag{1.5}$$

The larger is this value, the more rapidly marginal utility declines as consumption increases, or the utility curve "curves" more. The larger this curve, the more a person wants to smooth consumption and the more they'd pay to get rid of uncertainty.
An alternate measure is the ***coefficient of absolute risk aversion****, which is not an elasticity. It is the percent change in utility given a unit change in c, or* $-U''/U'$.

1.1.1 Total Utility and Consumption Smoothing

If $U(c_t)$ is felicity, or the utility of consumption in period t, what is lifetime, or total utlility? Let's call this V. We will generally assume that V exhibits a property defined now as

Definition 2 ***Additive separability*** *is a property of a utility function, and means that total utility* V *can be written as follows*

$$V = \sum_{t=0}^{T} U(c_t) \tag{1.6}$$

or in a continuous time situation that

$$V = \int_0^T U(c_t)\,dt \tag{1.7}$$

The definition just given assumes that each period is just as important as any other. Later, we'll introduce the idea of discounting to take into account the idea that you might not care about the future as much as today.

Another way of looking at additive separability is that the marginal rate of substitution between two periods is assumed to be independent of any other period, or

$$MRS_{t,t+1} = \frac{U'(c_t)}{U'(c_{t+1})} \tag{1.8}$$

is independent of the marginal utility from any other period besides t and $t+1$. This makes the life of the dynamic optimizing much easier. However, it may not be a great assumption, and we'll cover some situations later on where this is violated[1].

So let's think about what this additive separability means for how people will allocate their consumption across periods. Let's say you have \$300 and two periods to spend it in. Your utility function is therefore $V = U(c_1) + U(c_2)$. What is the optimal way to split up the \$300? Given the additive nature, the answer is to **smooth** it, or consume \$150 in each period. To see this, consider what the marginal utility of consumption would look like in each period if you didn't smooth it. Let's say you consume $c_1 = 200$ and $c_2 = 100$. Then the marginal utilities, given the properties we described above, relate like this $U'(200) < U'(100)$, or the marginal utility of first period consumption is lower than second period consumption. So you could move a dollar from period 1 to period 2 and have a net gain in utility. You could keep doing this until marginal utility was equalized across periods, which only happens when consumption is 150 in each period.

This shouldn't be too surprising. The price of consumption in each period is identical, so our intermediate microeconomist training tells us that the ratio of marginal utilities must equal the price ratio. In other words, the marginal utilities must be identical, and that can only happen when consumption is 150 in each period.

So one primary consequence of additive separability and the property of the utility function is that people want to smooth consumption over time.

1.1.2 Uncertainty and Risk Aversion

Now let's start over and consider utility only within one period, but there is uncertainly about what consumption might be. Suppose that I have a 50% chance of having \$100 to consume and a 50% chance of having \$200 to consume. How do I calculate my expected utility? There are two ways you could consider doing this. One, you could take the utility of the expected level of consumption which would look like this

$$V = U(0.5 * 100 + 0.5 * 200) \tag{1.9}$$

and in this case you would be completely wrong. This is something that people commonly make mistakes on, so stop and drill it into your head that this is incorrect.

The second method of computing expected utility is the right way and deserves its own definition.

Definition 3 *The Von Neumann Mortgenstern (VNM) method of computing utility takes the expected utility of consumption, written as*

$$V = \sum_{i=1}^{N} p_i U(c_i) \tag{1.10}$$

[1] A simple example involves habit formation. If your utility depends not only on what you consume today, but the size of this consumption relative to all your previous consumption, then additive separability doesn't hold.

where p_i is the probability of situation i occuring (out of a possible N outcomes) and $U(c_i)$ is the utility of consumption in situation i. To make sense, we have to have as well that $\sum_{i=1}^{N} p_i = 1$. Note that VNM utility is very similar to the simple additively separable utility function from the previous definition. They are both linear combinations of felicities.

In our example, the correct VNM specification for utility is

$$V = 0.5 * U(100) + 0.5 * U(200). \tag{1.11}$$

What does the VNM utility specification imply about our utility? Consider comparing the utility in (1.11) to the utility you get in (1.9). Which is higher? By the fact that the U function is concave (has a negative second derivative), Jensen's inequality tells us that

$$0.5 * U(100) + 0.5 * U(200) < U(0.5 * 100 + 0.5 * 200) \tag{1.12}$$

or that the expected utility of consumption - $E(U(C))$ - is lower than the utility of expected consumption - $U(E(C))$. That is, if you could have \$150 with certainty, you'd prefer that to the 50/50 situation where you could get \$100 or \$200. This is, people dislike uncertainty, and we call this dislike **risk aversion**.

The consequences of risk aversion are many. Insurance and most financial products wouldn't exist if people weren't risk averse. The property of risk aversion means that you will actually pay to remove uncertainly about your consumption (which is what you do when you buy insurance). You can figure out how risk averse you are by asking how much certain money you'd give up in order to avoid a lottery. In other words, since \$150 with certainty is preferred to the \$100/\$200 coin flip, then is \$140 preferred? Is \$130 preferred? You can figure this out by solving the following

$$0.5 * U(100) + 0.5 * U(200) = U(X) \tag{1.13}$$

and the lower the value of X that you find, the more risk averse you are. Note that X won't be any lower than \$100, because it would be silly to take less than the minimum pay out. The difference between \$150 and X is the amount you'd pay to avoid the lottery.

Risk aversion is a result of the properties of the utility function and the use of the VNM utility. As I said in the definition, VNM utility is functionally identical to additive separability. That is, in VNM utility, the marginal rate of substitution between consumption in two states of the world (i and $i+1$, for example) is independent of any other state. This is true even though utility is weighted by the probabilities p_i. So ultimately, risk aversion is identical to consumption smoothing. They are the same property! At times we'll talk about how people's risk aversion affects their path of consumption even when there is no uncertainty. That is because by measuring the degree of risk aversion, we are also measuring people's desire to smooth consumption.

To see how this all fits together, let's look at a specific functional form for utility that will be very useful throughout the course.

Example 4 *The CRRA Utility Function. This utility form is called constant relative risk aversion and has the mathematical form of*

$$U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma} \tag{1.14}$$

and the value of $\sigma > 0$. You may notice that if $\sigma > 1$, then utility is actually always negative, but it becomes less negative as consumption rises. People often get confused by this. If $\sigma = 0$ then the function is just linear. Finally, if $\sigma = 1$ then the function reduces to $U(C_t) = \ln(C_t)$[2]. If you apply the definition of relative risk aversion, you'll see that it is exactly σ for this utility function.
You can see how this works by going back to our little lottery example. Given that utility is CRRA, what amount are you willing to pay to avoid the lottery? We need to solve equation (1.13) *or*

$$X^{1-\sigma}/(1-\sigma) = 0.5 * 100^{1-\sigma}/(1-\sigma) + 0.5 * 200^{1-\sigma}/(1-\sigma) \tag{1.15}$$

for X. This gives us

$$X = \left(0.5 * 100^{1-\sigma} + 0.5 * 200^{1-\sigma}\right)^{1/(1-\sigma)} \tag{1.16}$$

and the actual value of X depends on the size of σ. If you calculate this out for several values, this is what you get

σ	X
1	141.40
2	133.30
3	126.50
4	121.20
5	117.10
6	114.20

and you can see that the amount you'd take with certainty (X) decreases with σ, or the more risk averse you are. Empirical attempts to measure σ seem to find the value of around 3*, but there is a lot of disagreement on this. Some of the financial puzzles (like the high equity premium) seem to be explained only by unreasonably high values of σ. In theoretical work, we will use the $\sigma = 1$, or log consumption, a lot, even though this is probably not reasonable.*

1.2 The Fisher Model

Now we can start thinking about the optimal mix of savings and consumption. We'll do this in a simple two period framework now, and then in the next chapter

[2] To see this, rewrite utility by adding a constant as $U(C_t) = \left(C_t^{1-\sigma} - 1\right)/(1-\sigma)$ and you can rewrite the function as $(C_t \exp(-\ln(C_t)\sigma) - 1)/(1-\sigma)$. As σ goes to 1, this function goes to $0/0$. So use L'Hopital's rule and find that $\lim_{\sigma\to 1}(-C_t \ln(C_t)\exp(-\ln(C_t)\sigma))/-1 = \ln(C_t)$.

look at how this generalizes to give us the core models of macroeconomics. The Fisher model involves the following facts. There are two periods of life. People earn income in both periods of W_1 and W_2. They have no assets when they enter the world, and they leave nothing behind when they die (that is, they consume all that they earn). They'll consume some amount C_1 and some amount C_2 such that

$$C_1 + C_2 = W_1 + W_2 \tag{1.17}$$

and we can define their savings in period 1 as

$$S_1 = W_1 - C_1 \tag{1.18}$$

which can be negative (meaning that the person is borrowing). You could talk about second period savings too, but these are always zero.

Notice from the budget constraint in (1.17) that the cost of consumption in the two periods is identical. What are preferences? They are

$$V = U(C_1) + U(C_2) \tag{1.19}$$

and we can look at the solution to this problem on a graph just like in intermediate micro.

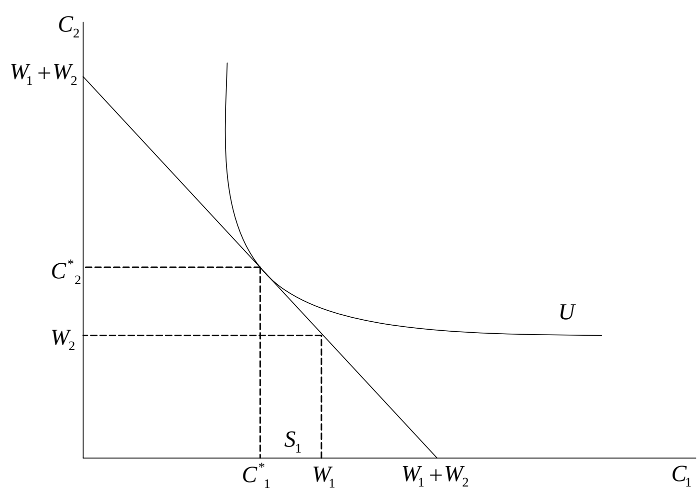

Figure 1.1:

What the diagram shows us is the budget constraint tangent to the indifference curve. Notice that the budget constraint goes through the point W_1, W_2

because it is always feasible to just consume your endowment of income. It has a slope of negative one, and it shows that you could consume all your income in period 1 if you wanted to, or all income in period 2 if you wanted. There is no cost to moving income between periods here.

The indifference curve (and you can confirm this shape is correct by looking at how the marginal rate of substitution changes at different levels of C_1 and C_2), has the typical form. The tangency shows $C_1 < W_1$, which implies that savings is positive. So this person has large W_1 relative to W_2. and because they generally like to smooth consumption, they save something to consume in period 2. In fact, because the price of consumption is the same in each period, this person will consume exactly $C_1 = C_2$. The big kicker is that this doesn't depend at all on endowment, either its size or its distribution over time. The absolute size of C_1 and C_2 depends on the size of the total endowment, but not on the distribution of it.

This is actually easier to see in math. So let's set up the constrained optimization problem for this person as a Lagrangian.

$$L = U(C_1) + U(C_2) + \lambda (W_1 + W_2 - C_1 - C_2) \tag{1.20}$$

which has FOC of

$$\begin{aligned} \partial L/\partial C_1 &= U'(C_1) - \lambda = 0 & (1.21) \\ \partial L/\partial C_2 &= U'(C_2) - \lambda = 0 & (1.22) \\ \partial L/\partial \lambda &= W_1 + W_2 - C_1 - C_2 = 0 & (1.23) \end{aligned}$$

and solving the first two FOC conditions together gives

$$\begin{aligned} U'(C_1) &= U'(C_2) & (1.24) \\ C_1 &= C_2 & (1.25) \end{aligned}$$

or that consumption must be equal in both periods. Note that this part of the solution didn't depend at all on using the budget constraint. Now combine this with the budget constraint and you can get that

$$C_1 = C_2 = \frac{W_1 + W_2}{2} \tag{1.26}$$

which is about what you'd expect.

What happens to your consumption if W_2 goes up? Then C_1 goes up and C_2 goes up. Which is contrary to the Keynsian consumption function we started this chapter with. By taking the intertemporal nature of consumption into account, we've found that the Keynsian version of the world doesn't quite hold up. Consumption today depends crucially on income today *and* in the future.

1.2.1 Interest Rates

Now we start by adding some more realism. Let's call r the real interest rate that is earned on money saved in period 1 - or alternately the interest rate paid by people who borrow. Savings is still $S_1 = W_1 - C_1$ but consumption in the second period is now $C_2 = S_1(1+r) + W_2$ and we can combine this information to get the new budget contraint which is

$$W_1 + W_2/(1+r) = C_1 + C_2/(1+r) \tag{1.27}$$

and notice that now the price of consumption in the second period is actually lower than that of consumption in the first period (because $1/(1+r) < 1$).

What does this do to consumption? Well, if we did another Lagrangian, except with the new budget constraint, we'd get the following FOC

$$\begin{aligned}
\partial L/\partial C_1 &= U'(C_1) - \lambda = 0 & (1.28)\\
\partial L/\partial C_2 &= U'(C_2) - \lambda/(1+r) = 0 & (1.29)\\
\partial L/\partial \lambda &= W_1 + W_2/(1+r) - C_1 - C_2/(1+r) = 0 & (1.30)
\end{aligned}$$

and solving the first two conditions together gives us

$$\begin{aligned}
U'(C_1) &= (1+r)U'(C_2) & (1.31)\\
C_1 &< C_2 & (1.32)
\end{aligned}$$

where the second step follows because $1 + r$ is greater than one, so that the marginal utility of C_2 must be smaller than the marginal utility of C_1, which given the nature of the utility functions means that $C_1 < C_2$. So with interest involved, we see that now consumption tilts towards the second period. Notice again that this doesn't depend on the size or distribution of the endowment.

What about savings? Did savings rise or fall? This depends on what the person was doing without interest rates.

- Person was saving. Recall that there are two effects of a price change: the income and substitution effects. The decrease in price of C_2 is like having a higher income, so that this pushes up consumption in period one and period two. However, the substitution effect says that the person should start consuming more C_2 and less C_1 (which we saw). So the effect on C_1 is ambiguous, and therefore the effect on S_1 is ambiguous. This analysis holds in any case where the interest rate is increasing and the person is a first period saver.

- Person was borrowing. In this case the income effect is negative for both C_1 and C_2. The substitution effect is negative as well for C_1 when r goes up, so that it is unambiguous that C_1 falls and savings goes up (or borrowing goes down).

Example 5 *We can complicate things slightly more in macro because we actually have that the price of one good* (r) *affects your actual income (think of this as an endowment effect). So we can decompose the effects of a change in* r

into more parts. Let's take a typical CRRA utility function and solve it with a budget constraint. The Euler equation we get is

$$C_2 = (1+r)^{1/\sigma} C_1 \tag{1.33}$$

and a budget constraint of

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r} \tag{1.34}$$

yields the following answer for first period consumption.

$$C_1 = \frac{1}{1+(1+r)^{1/\sigma-1}}\left[Y_1 + \frac{Y_2}{1+r}\right]. \tag{1.35}$$

First, notice that consumption in the first period is a constant fraction of total lifetime income. If we hold the total $Y_1 + \frac{Y_2}{1+r}$ constant but move around the pattern of income, first period consumption doesn't change. So back to the decomposition. The tension in between the income and substitution effects is found in the $(1+r)^{1/\sigma-1}$ term. If $1/\sigma - 1 > 0$, or $\sigma < 1$, then the substitution effect dominates, and an increase in r will shift consumption towards period 2 because people are really willing to substitute. If $1/\sigma - 1 < 0$, then the income effect dominates and an increase in r will raise current consumption. There is a third effect, though, that we can see from this equation. That is the wealth effect - or the fact that $1+r$ going up means that the PDV of lifetime earnings is now lower. So while the income effect might win out, and it normally will given that our assumption is that $\sigma > 1$, we still might have current consumption falling if income is mainly gained in the future.

1.2.2 Discount Rates

So far we've assumed that the utility of consuming in period 2 is just as good as the utility of consuming in period 1. Except that you have to wait until period 2 to do the consumption, and this delay might make it less satisfying *today.* That is, you are optimizing today, and delaying consumption means that you have to wait, which may not be very fun. So let's say that now we discount future utility by the factor of θ in the following manner.

$$V = U(C_1) + \frac{U(C_2)}{1+\theta} \tag{1.36}$$

which affects the slope of the utility function (but does nothing to the budget constraint). As θ goes up, this makes utility in the second period less and less desirable, and the indifference curves start to slope more steeply, pushing the optimal choice towards C_1. If you set up the Lagrangian again, including the interest rate and the discount rate, you get the following FOC

$$\partial L/\partial C_1 = U'(C_1) - \lambda = 0 \tag{1.37}$$

$$\partial L/\partial C_2 = U'(C_2)/(1+\theta) - \lambda/(1+r) = 0 \tag{1.38}$$

$$\partial L/\partial \lambda = W_1 + W_2/(1+r) - C_1 - C_2/(1+r) = 0 \tag{1.39}$$

which can be solved for

$$U'(C_1) = \frac{(1+r)}{(1+\theta)} U'(C_2) \tag{1.40}$$

$$C_1 < C_2 \Longleftrightarrow \frac{(1+r)}{(1+\theta)} > 1 \tag{1.41}$$

$$C_1 > C_2 \Longleftrightarrow \frac{(1+r)}{(1+\theta)} < 1 \tag{1.42}$$

and we see that the pattern of consumption depends on the relative sizes of r and θ. Also, if $r = \theta$, then we're right back at having $C_1 = C_2$, although for a much different reason than we started with.

1.2.3 Modified Budget Constraints

So far we've assumed that the consumer can borrow and save at the identical interest rate. What if there are differential interest rates so that $r_b > r_s$ or the real interest rate to borrow exceeds that of savings. Then the budget set is kinked and there are three possible optima. The indifference curve could be tangent to one of the arms, so that the differential doesn't matter. Or the solution could be to cosume at the kink point. The interesting thing about this is that if either interest rate changes, consumption and savings may not change at all, meaning that savings are invariant to interest rates. You'll have some problems to do that involve this modification.

Another possibility is that the consumer is liquidity constrained, and cannot borrow but can only save (something like being a college student). Now the budget set has a vertical section below the endowment point, and the individual cannot set $C_1 > W_1$. Does this constraint bind? Only if the person would have borrowed in the first place. If so, then they'll consume at the kink point again, and any increase in their W_1 will translate one for one into increases in C_1, giving something like a Keynsian relationship.

This section seems really short, and you might be inclined to think that this means that modified budget constraints are relatively unimportant. This couldn't be further from the truth. Much of what is added to consumption theory to make it match the facts involves limitations to peoples ability to borrow or save. This kind of problem requires you to think a lot about peoples optimization - and can't always be solved in a straightforward Lagrangian or with calculus. Which is why it makes for great problem sets and test questions. Be warned.

1.2.4 Ricardian Equivalence

This is a concept that David Ricardo originally mentioned, and then dismissed. Robert Barro revived the discussion in modern times by asking whether government debt actually constituted wealth. To see what they were both talking about, consider a two period Fisher world, but now you have to pay taxes in each period (the government collects this money and throws it in the ocean, so there is no affect of government spending here). Your budget constraint is now

$$C_1 + \frac{C_2}{1+r} = W_1 - T_1 + \frac{W_2 - T_2}{1+r} \tag{1.43}$$

which is no different than saying that the actual size of your wages changed. From before, we know that this will have no effect on the optimal consumption path, and that we should still have $U'(C_1)/U'(C_2) = (1+r)/(1+\theta)$, which shows that the size of your wages and their distribution doesn't affect your optimal path.

What happens if we introduce a change in the tax collection scheme? We will say that taxes change as follows

$$\begin{aligned} \Delta T_1 &= -Z \\ \Delta T_2 &= (1+r)Z \end{aligned}$$

or that the present value of taxes collected is unchanged. If Z were positive, then taxes are being cut today, and raised in the future. Plug this into the budget constraint and you get

$$C_1 + \frac{C_2}{1+r} = W_1 - [T_1 - Z] + \frac{W_2 - [T_2 + (1+r)Z]}{1+r}$$

and if you do the algebra you see that the $Z's$ cancel out completely and you're left with the same exact budget constraint as before. What happens to first period savings? Well, the Euler equation is identical, and the budget constraint is identical to before the change in taxes, so your consumption must be completely unchanged by this change in taxes. Then

$$S_1 = W_1 - T_1 - C_1 \tag{1.44}$$

and C_1 is fixed. So any decrease in taxes must raise savings by the same amount. People do not choose to consume any of their tax break. Does this increase in savings have any effect on the capital stock? No. Because to finance the tax cut, the government has to issue bonds, on which it will pay an interest rate of r. So from an individuals perspective, these bonds are wealth, in that they provide a way of earning r on their savings. But from the aggregate perspective they are not wealth, because they are just government liabilities which have to be paid back by the economy later.

So the upshot is that if taxes go down today, that should have no effect on consumption because people are far-sighted enough to understand that they'll need to pay higher taxes in the future. Ricardian equivalence is about the timing of taxes - it does not say that government spending will have no effect on consumption. So if the government raises spending, this will lower your absolute consumption levels as they take more money out of the system. But it won't matter to you whether this new spending is financed by direct taxes or by bonds. The optimal path of consumption will remain the same, only the level will change.

There a host of objections to Ricardian equivalence:

1. Different interest rates for government borrowing and individual saving /borrowing

2. If people are liquidity constrained in the first period, then government borrowing (taxes going down) will raise their consumption

3. If people are myopic it doesn't work. Probably true, but how exactly do you model myopia?

4. If people will be dead before they have to pay back the taxes, then the tax decrease will allow them to increase their consumption. Later generations will pay the extra taxes and consume less. This has raised a lot of debate and runs into a whole long-winded debate about intergenerational relations. Before touching on that, note that most of the present value of a tax cut will be paid back by people alive today, so RE should hold pretty close to absolutely.
   The intergenerational argument in defense of RE is that we see people giving bequests when they die, so they must care about their children's consumption. The tax decrease is like taking money away from their children and giving it to them, so they'll just leave that as a bequest for their children and consumption won't change in the current period.
   A lot of people attack this reasoning. Parents may not get utility from their childrens consumption but from the actual giving of a bequest, and then the bequest is kind of like consumption for the parent and when taxes fall they'll increase both their bequest and their income. Alternately, you could argue that most bequests are accidental, not intentional, because people die before they expect to.

A last thought about RE. If people were completely myopic and never expected to pay back their tax cut, what would happen. If they followed our usual model of consumption smoothing, they would still only raise their consumption by a little, spreading the tax decrease out over their whole lives (remember that the timing of your income doesn't matter). So RE would be very close to true. Compare that to Keynsian consumption, where people would consume almost the whole tax cut immediately.

1.3 Uncertainty and Consumption

We now have the basic framework in place to analyze a lot of different problems relating to the consumption/savings decision and how uncertainty changes things.

1.3.1 The Permanent Income Hypothesis

The permanent income hypothesis of Milton Friedman developed two modern lines of thinking about consumption. First, he suggested that uncertain income should be treated different from certain income. Second, he suggested that people should take into account their whole lifetime path of income and consumption into their decision process (extending the Fisher model to T periods, and similar to Modigliani's life-cycle hypothesis). In this section, we'll consider the first point - the role of uncertainty in income. Friedman proposed that you could divide income into two components: the permanent part and the transitory part

$$Y = Y^P + Y^T.$$

The permanent component of income is what you could think of as average lifetime income, or your expected income in any period. The transitory component is al those additional random factors that occur. So permanent income may be your expected salary (which can be rising, but is generally known and you expect to receive it) while transitory income would be like winning a lottery or having your car break down (the shock can be either positive or negative, but is not expected).

What Friedman proposed was that only the permanent component of income should matter for consumption. The transitory component has a mean of zero (if it didn't, then it would have a permanent component to it), so that you expect over your life that the shocks to income will balance out. Which means that if you recieve a positive shock, you'll save it to cover yourself when you have a negative shock. So no transitory shock will ever affect your consumption. Any change in permanent income, though, will materially impact your consumption because you are raising your expectation of all your future incomes. So consumption looks like this

$$C = \alpha Y^P$$

Over the long run, as income increases, so does consumption. So this seems to match with Keynes. What about in the short run, or looking across households? If all the variation in income in the short run was from transitory income shocks, then there would be NO relationship between C and Y in the short run. As more and more of the variation across households is explained by permanent income differences, then C would start to show a positive relationship with Y.

This matches the data much better than Keynes did, who predicted that the short run and long run relationship between C and Y was the same. The data shows that C is related to Y much more strongly in the long run than in the short run. And this we attribute to the larger transitory component of income in the short run. This type of thinking leads us into the uncertainty portion of consumption theory, and we'll see how more variability in income leads to different savings behavior.

Let's think about a really simple two period model. In the first period you earn $\bar{W}$ with certainty, and in the second your earn $\bar{W}+\delta$, where δ is some random variable (transitory income) with mean zero. Your expected utility is

$$EV = U(C_1) + E(U(C_2)) \tag{1.45}$$

and the budget constraint can be written as

$$C_2 = 2\bar{W} - C_1 + \delta \tag{1.46}$$

so that utility is now

$$EV = U(C_1) + E\left(U\left(2\bar{W} - C_1 + \delta\right)\right).$$

You can maximize this over C_1 and you'll find that you get

$$\begin{aligned} U'(C_1) &= EU'\left(2\bar{W} - C_1 + \delta\right) \\ &= EU'(C_2) \end{aligned}$$

which tells us that you'll equate marginal utility of consumption in period one with the expected marginal utility of consumption in period 2. Not surprising, really. Your income will vary in period 2, but you still want to smooth consumption in the same manner as before. This is highly important to note. From this Euler equation - uncertainty has not changed anything fundamental about how you dynamically optimize. It doesn't indicate that your behavior actually changes (we'll get to if and why it might). All uncertainty should do is reduce your overall utility, but it won't *necessarily* change how much you consume.

Recall a handy rule from the world of statistics. Namely, that the actual realization of a random variable can be written as follows

$$X = E(X) + \varepsilon \tag{1.47}$$

where ε is a random variable with mean zero (and is different than the δ random variable). So we can write

$$U'(C_2) = EU'(C_2) + \varepsilon \tag{1.48}$$

and then use (1.48) in (**??**) to show that

$$U'(C_2) = U'(C_1) + \varepsilon \tag{1.49}$$

or that the marginal utility of consumption follows a random walk. If you actually have positive r and θ, then you'd get an expression that shows that marginal utility should follow a random walk with drift.

So what does all this do for us? Well, equation (1.49) tells us that the change in marginal utility between periods - $U'(C_2) - U'(C_1)$ - should be *random*. If the change in actual marginal utility between periods is random, then the change in consumption itself should be random. More specifically, there is no information out there in the world that should be able to predict how your consumption will change from period to period.

This is the test of the PIH proposed by Robert Hall. Suppose you have a set of variables Z that predict changes in income (things like past income, the stock market, consumer confidence, etc..). These Z items may also predict the *level* of consumption in any period. But these Z factors should have *no* relationship to the *change* in consumption between periods. If you had any Z variable that predicted changes in consumption, then the PIH would be violated.

The great thing about Hall's idea is that is doesn't require the econometrician to know much about how the consumption decision is being made. For example, we don't have to know anything about how expectations of future income are made.

1.3.2 Lifespan Uncertainty

So far, we've assumed that there was certain end date of someones life. What if we allow for the proabability of dying prior to period 2? Let's start with a really simple example. Let's say that I have a two period Fisher model, but that I have a $1-p$ percent change of dying before I get to consume anything in period 2. Then what is my expected utility?

$$EV = U(C_1) + p * \frac{U(C_2)}{1+\theta} \tag{1.50}$$

which still has a pure time discount of θ, but has this 0.5 term to account for the fact that I only have half a chance of living to consume. Well, this is essentially just a modification of the discount rate, so my answer should be that

$$\frac{U'(C_1)}{U'(C_2)} = \left(\frac{1+r}{1+\theta}\right) p \tag{1.51}$$

and what this tells me is that, holding r and θ constant, I should scale down $U'(C_1)/U'(C_2)$ by the factor p. This means I should raise first period consumption (so that marginal utility falls) and lower second period consumption (so that marginal utility rises). Which isn't surprising. If I might be dead next year, I don't want to save too much money because I might not get to enjoy it at all.

1.3.3 Precautionary Saving

We know that uncertainty makes you worse off. That's a result of having $U'' < 0$ which implies that $E(U(C)) < U(E(C))$. But the fact that you are worse off with uncertainty doesn't necessarily mean you'll make any different choice about how to allocate your consumption. In the PIH, having transitory income wouldn't necessarily affect the amount that you consume each period at all. So the question we want to ask now is: does uncertainty change your consumption behavior?

We'll look at a simple two period model. We'll assume that the interest rate and discount rate are both zero, for simplicity. In period 1 you get $\bar{W}$ and consume C_1. In period 2 you get $\bar{W}$ again and you also have a 50% chance of getting L dollars in income and a 50% chance of having to pay out L dollars of your income. $\bar{W}$ is permanent income and L represents transitory income. So consumption in period 2 is

$$\begin{aligned} C_2 &= \bar{W} - C_1 + \bar{W} + L \text{ with a 50\% chance} \\ C_2 &= \bar{W} - C_1 + \bar{W} - L \text{ with a 50 \% chance} \end{aligned}$$

so now your actual budget constraint is uncertain. The only choice you have is C_1. Expected utility is

$$EV = U(C_1) + E(U(C_2)) \tag{1.52}$$

and notice that there is no E operator in front of $U(C_1)$ because we'll know period one consumption with certainty. Expanding on the uncertainty in period 2 consumption gives us

$$EV = U(C_1) + 0.5 * U\left(2\bar{W} - C_1 + L\right) + 0.5 * U\left(2\bar{W} - C_1 - L\right) \tag{1.53}$$

which we can maximize with respect to C_1. This gives the FOC of

$$0 = U'(C_1) - 0.5 * U'\left(2\bar{W} - C_1 + L\right) - 0.5 * U'\left(2\bar{W} - C_1 - L\right) \tag{1.54}$$

which can be solved for an optimal value of C_1. So the equation (1.54) defines an implicit function of C_1 as a function of L. We can use the implicit function theorem to find the derivative of C_1 with respect to L as

$$\frac{\partial C_1}{\partial L} = -\frac{F_L}{F_{C_1}} = \frac{0.5\left[U''(Y - C_1 + L) - U''(Y - C_1 - L)\right]}{U''(C_1) + 0.5 * U''(Y - C_1 + L) + 0.5 * U''(Y - C_1 - L)}. \tag{1.55}$$

The term $U'' < 0$, so the whole denominator is negative. The key term is therefore

$$\left[U''(Y - C_1 + L) - U''(Y - C_1 - L)\right]$$

. When this term is positive, then the whole expression is negative and therefore $\frac{\partial C_1}{\partial L} < 0$ and as uncertainty increases (the size of the lottery L goes up), then consumption falls. The term in the numerator is positive when $U''' > 0$. Which is like saying that marginal utility is convex. Because of the convexity of marginal utility, it means that as L increases, the expected value of marginal utility in the second period goes up. So with higher expected marginal utility in the second period, you would move consumption to the second period.

Think about it this way. If I wanted to limit the variation of my income in period 2, then I would consume as much as possible in period 2. If the size of the L were \$1000, then if I had saved only \$1000, I would be looking at a 50/50 chance of either zero dollars or \$2000, which seems like a big difference. However, if I had saved \$1,000,000, then I would have a 50/50 chance of either \$1,001,000 or \$999,000, and I'd be pretty well off with either of those results. So to limit the relative variability of my utility in period 2, I transfer money to that period. In essence, I buy certainty in period 2 with money from period 1.

This result holds for models beyond two periods as well. As long as $U''' > 0$, you get precautionary saving, and this holds for things like log utility or any CRRA utility function. On the other hand, if you have quadratic utility, then $U''' = 0$ and there is no precautionary saving. The person in this case acts as if income in the second period were certain to arrive at its expected value. The uncertainty doesn't alter his optimal consumption choice.

1.4 Violations of Utility Assumptions

1.4.1 Habit Formation

We said that $U'' < 0$ was crucial, but we have also been operating with additive separability of utility. While we generally need $U'' < 0$ to say anything interesting, there is nothing magic about additive separability. Durable goods violate additive separability because they have lingering utility even after their consumption (think of food or a vacation). The theory of habit formation suggests that not only does current consumption matter for current utility, but so does past consumption, although not because of durability, but because you become accustomed to a certain lifestyle. Utility is now

$$U(c, z) = \frac{(c/z^{\gamma})^{1-\sigma}}{1-\sigma} \tag{1.56}$$

where z measures your habitual level of consumption, or in other words your reference level of consumption. This could also be thought of as the consumption level of people around you.

Notice that a higher habit makes you worse off, and γ indexes how important habits are. The closer γ is to zero, the more people care only about their absolute consumption level. For example, if $\gamma = 1/2$, then a person with a consumption of 4 and a habit stock of 4 will have the same utility as a person with the consumption of 2 but a habit stock of only 1.

We then set up an evolution of the habit stock as follows

$$z_{t+1} = pc_t + (1-p) z_t \tag{1.57}$$

and the bigger is p, the faster habits adjust to current consumption. This leads to a really messing solution, so we aren't going to worry too much about it, but it sure seems like this is something worth considering when we think carefully about consumption.

1.4.2 Intertemporal Consistency

Now, we keep our preferences additively separable. But we consider the following question: In a world of certainty, would you ever want to stop and re-optimize your consumption path? That is, in our problems so far you solve for your entire lifetime consumption path at time zero, and then never deviate. Is this true? See Stroz (1956) for the classic treatment of this.

Stroz shows that the only form of the discount factor that does NOT lead to inconsistency (i.e. the desire to reoptimize) is our usual exponential discounting (e.g. the θ term in exactly the forms we've been using it). Any other form of discounting leads to the paradox that the relative importance of two period's consumption depends on the point in time we view the problem. For example, right now I would prefer that my consumption ten years from now be very similar to my consumption eleven years from now (smoothing). But that is because that is a long time away. Ten years from now, when I'm actually doing my consumption, I will probably prefer to have more consumption immediately (in year ten) than next year (in year eleven). So I would change my consumption path from what I set it to be ten years ago. I am dynamically inconsistent.

Some people argue that this means that only the exponential discounting we use is ration. But that seems a bit strong. People like David Laibson have looked at what happens if preferences are in fact hyperbolic, so that you discount the future a lot, but not much between the immediate future and the far future. The actual utility function he uses is

$$U_t = E_t\left(u\left(c_t\right) + \beta \sum_{s=1}^{T-t} \delta^s u\left(c_{t+s}\right)\right) \tag{1.58}$$

where $\delta, \beta < 1$. This form of discounting means that today, I'd like to precommit myself to a consumption plan to maximize my present utility. That is, force myself to save by having automatic withdrawals, buying of durable goods, putting my money in CD's that have big penalties for early withdrawals, etc.. This is because if I leave my future selves to make their own decisions, they'll have a different view of how important consumption in their period is. They'll spend too much, from the perspective of myself today.

Again, mathematically this stuff is ugly. But Laibson makes some arguments about how the decline in savings in the U.S. is a result of the ability of people to get around their precommitment devices (e.g. easier withdrawals of cash from brokerage accounts or lower minimum amounts for savings accounts).

1.5 Labor and Consumption Choices

To introduce a labor response, we will modify our typical utility function to include leisure (n). You'll get higher utility the more leisure time you have. Of course, you can't buy any consumption unless you work, so you'll have an intra-temporal choice to make about how much to work in addition to your inter-temporal choice about the path of consumption and leisure over time.

This section will also introduce you to a little general equilibrium model of the macro economy. We'll still take incomes as given, but we'll allow for the interest rate to be determined within the model itself. We do this by assuming that we have many individuals in the economy, all with the same optimization problem, and that the interest rate has to clear the market for loans and savings.

1.5.1 The Basic Consumption/Leisure Problem

Now utility is

$$U = U(c, n) \tag{1.59}$$

and you have one unit of time. The time spent working is $1 - n$ and so your total earnings in period t are $w_t(1 - n_t)$. There is an interest rate, r, and a discount rate θ, just as before. So the individual's utility is now

$$V = U(c_1, n_1) + \frac{U(c_2, n_2)}{1+\theta} \tag{1.60}$$

subject to the constraint that

$$c_1 + \frac{c_2}{1+r} = (1 - n_1) w_1 + \frac{(1 - n_2) w_2}{1+r} \tag{1.61}$$

and we can set up a Lagrangian to solve this.

$$L = U(c_1, n_1) + \frac{U(c_2, n_2)}{1+\theta} + \lambda \left((1 - n_1) w_1 + \frac{(1 - n_2) w_2}{1+r} - c_1 - \frac{c_2}{1+r} \right) \tag{1.62}$$

As before, we take FOC and get

$$\begin{aligned} U_c(c_1, n_1) &= \lambda && (1.63) \\ U_c(c_2, n_2) &= \lambda \frac{1+\theta}{1+r} && (1.64) \\ U_n(c_1, n_1) &= \lambda w_1 && (1.65) \\ U_n(c_2, n_2) &= \lambda w_2 \frac{1+\theta}{1+r} && (1.66) \end{aligned}$$

and if we combine the first two FOC's we get the Euler equation

$$\frac{U_c(c_2, n_2)}{U_c(c_1, n_1)} = \frac{1+\theta}{1+r} \tag{1.67}$$

and a similar equation for leisure across periods

$$\frac{U_n(c_2, n_2)}{U_n(c_1, n_1)} = \frac{w_2}{w_1} \frac{1+\theta}{1+r}. \tag{1.68}$$

So similar to the Euler equation for consumption, we have an intertemporal trade-off going on with leisure. Why is there an intertemporal trade-off?

Because the amount of leisure I take affects the earnings I make, and therefore I want my leisure to fit into my optimal consumption path. Notice, though, that optimal leisure also depends on the relative wage levels. Wages are how we translate the leisure choice into consumption, so the marginal cost of leisure in a period depends on its wage.

I can use the leisure FOC to think about the labor response to shocks. If there is a positive shock to output in period 1, then w_2 goes up, and the marginal cost of leisure is high, so I take less. In other words, I work more when the return to work is higher. So labor is procyclical, and output goes up not only because of the positive productivity shock, but also because of increased labor effort.

But that's not all, we also have to consider the static FOC that relates consumption and leisure within a given period. Consider the first and third conditions above which solve to

$$U_n(c_1, n_1) = w_1 U(c_1, n_1) \tag{1.69}$$

and says that the marginal utility of consumption in any given period has to be equal to the marginal utility of leisure in that period times the wage. In other words, the price of a unit of leisure relative to a unit of consumption is just w_1. You have to pay w_1 in consumption to buy an extra unit of leisure.

Note that you don't have to solve all the FOC to get your solutions. There are three essential FOC, two dynamic and one static. Equations (1.67), (1.68), and (1.69). Once you solve two of these, the other one must follow.

That's the essentials of a consumption model with leisure. With more specifics on the form of utility you can solve this - potentially. As you add in more elements to the optimization things start to get a little hairy, and that is why more complex models of leisure and consumption often end up having to be solved on a computer.

1.5.2 Fluctuations and Consumption

Let's start by dropping our leisure choice entirely, and just presume that everyone works full time. But we'll add in some randomness to the economy by having shocks to the individual's wages. Consumption is determined by

$$c_t = v_t \bar{w}$$

where v_t is the productivity shock by period. People are assumed to be maximizing consumption in a typical manner. The Euler equation tells us that

$$\frac{u'(c_2)}{u'(c_1)} = \frac{1+\theta}{1+r}$$

Now we're going to change our point of analysis and ask ourselves a macro question. That is, how does the interest rate respond to the shocks to consumption? Instead of asking how consumption of an individual will respond to a given process for r, we'll ask what r will make the Euler equation hold. How can we do this? Well, we have to assume everyone in the economy is identical, and is solving the identical problem. In that case, there can't actually be an borrowing or saving in equilibrium, because if one person wants to borrow, everyone wants to borrow, and there is noone who will provide savings. So we have to solve for the interest rate that will hold such that everyone is happy consuming exactly what they earn in each period. (This way of thinking about this problem leads to the general issue of asset pricing - i.e. solving for r).

Rewrite the Euler equation in this manner

$$1 + r = (1 + \theta) \frac{u' (v_1 \bar{w})}{u' (v_2 \bar{w})} \tag{1.70}$$

and we see that the size of the shock in period 2 determines the interest rate (since the shock in period one is known already). So if there is a large positive shock to productivity in period 2, then what happens? Consumption goes up, and therefore the marginal utility of consumption in that period falls, and to keep the Euler equation in equilibrium the interest rate must rise. In other words, to make people content with having consumption rise from period 1 to period 2, there must be a large interest rate.

Now what if the productivity shocks are random and people don't know what they will be? Then we get that

$$1 + r = (1 + \theta) \frac{u' (v_1 \bar{w})}{E (u' (v_2 \bar{w}))}$$

or that the expected interest rate that will hold depends on the expectation of shocks in the next period. Now recall what we know about precautionary savings. With $U''' > 0$ we know that the expected value of marginal utility is higher than the marginal utility of the expected outcome. That is, $E (u' (v_2 \bar{w})) > u' (E (v_2 \bar{w}))$. Thus the RHS of the above equation is lower, with uncertainty, than with certainty. Therefore the interest rate that holds under uncertainty is lower. Why? Because people do not need incentives to keep consumption in period 2, they want to do that anyway.

1.5.3 Labor, Consumption and Fluctuations

Now let's consider what happens when we include a labor choice into the problem of the previous section. Again, people are identical, so there actually is no trade in savings and loans, but we can still find the interest rate. We'll again take the wage rates as given to us exogenously with some random element. In equilibrium, again, we have to have that consumption equals income because everyone is identical. So $c_t = (1 - n_t) \bar{w} v_t$.

Our FOC including the labor choice are as follows

$$\frac{U_c(c_2,n_2)}{U_c(c_1,n_1)} = \frac{1+\theta}{1+r} \tag{1.71}$$

$$\frac{U_n(c_2,n_2)}{U_n(c_1,n_1)} = \frac{v_2}{v_1}\frac{1+\theta}{1+r} \tag{1.72}$$

$$U_n(c_1,n_1) = v_1\bar{w}U_c(c_1,n_1) \tag{1.73}$$

$$U_n(c_2,n_2) = v_2\bar{w}U_c(c_2,n_2) \tag{1.74}$$

and let's start by asking what happens if we have a positive shock to second period income. That is, v_2 goes up. Start with the final equation, the static FOC in period 2. The shock in period 2 means that you can earn more from each unit of work, raising the marginal cost of leisure, and lowering the amount of leisure you take. BUT, at the same time, the increase in productivity means that consumption goes up, lowering the marginal utility of consumption, which lowers the marginal cost of leisure. So the static FOC has an ambiguous answer about how leisure, and thus consumption, responds.

Without a clear answer from the static FOC, we can't figure out what happens to the interest rate. Unless we have more structure on the model.

Example 6 *Let's put some structure on the utility function and see what that tells us. Utility is now*

$$U(c_t,n_t) = \ln c_t + b\ln n_t \tag{1.75}$$

and that means that our FOC are as follows

$$\frac{c_1}{c_2} = \frac{1+\theta}{1+r} \tag{1.76}$$

$$\frac{n_1}{n_2} = \frac{v_2}{v_1}\frac{1+\theta}{1+r} \tag{1.77}$$

$$c_1 = v_1\bar{w}\frac{n_1}{b} \tag{1.78}$$

$$c_2 = v_2\bar{w}\frac{n_2}{b}. \tag{1.79}$$

Now recall that consumption has to be equal to income in each period, or $c_1 = (1-n_1)v_1\bar{w}$ *and* $c_2 = (1-n_2)v_2\bar{w}$. *Using this in the last two FOC gives us*

$$(1-n_1) = \frac{n_1}{b} \tag{1.80}$$

$$(1-n_2) = \frac{n_2}{b} \tag{1.81}$$

which solves to

$$n_1 = \frac{b}{1+b} \tag{1.82}$$

$$n_2 = \frac{b}{1+b} \tag{1.83}$$

or the choice of leisure is constant. This is the result of the log utility, which means that the offsetting impacts of any productivity shocks are completely equal. What this means is that we can solve for the interest rate using the inter-temporal leisure condition as

$$1 + r = \frac{v_2}{v_1}(1 + \theta) \tag{1.84}$$

and this tells us that the interest rate rises with a productivity shock in period 2. Consumption in period two has risen, and there is no change in leisure to offset this, so we have to have an increase in the interest rate to clear to financial market.

Let's think now for a moment about what would happen if you could actually save and borrow in equilibrium. This could either be because you have access to world financial markets, or because there is some asset like capital that you can accumulate yourself. What would happen to consumption and leisure due to a shock? Well, with access to financial markets, you'd smooth your consumption completely. So if v_2 was larger, you'd adjust to this by spreading your consumption around, and consumption in period 2 wouldn't rise by as much as wages actually did. So this would skew the reaction of your leisure. Now the marginal cost of leisure would rise and so leisure would fall, and this gives you a big response of labor to productivity shocks. The point is that in order to generate a large labor response to productivity shocks, you need some ability of individuals to move consumption between periods so that this blunts the consumption response to shocks.

Chapter 2

The Mechanics of Economic Growth

We've seen in the previous chapter how people will choose to make decision regarding their consumption path, given some return on assets (r) and some exogenous path of wages (w). They provided a lot of information about how individuals will act, but from a macro perspective they have some problems because they take both r and w as simply given. And in many cases what we want to do is actually describe how w and r change over time, given that people are optimizing. So we need to provide some mechanism for the setting of w and r in the economy. This chapter will present some fundamental concepts used in the growth literature, such as production functions and the Solow model. These concepts will then be joined together with the optimal consumption models in Chapter 3 to present to you the central models of dynamic optimization.

2.1 Production Functions

A production function is a mathematical function that tells us how much total output, Y, we can get for a given amount of inputs. For now, we'll divide up the inputs into capital K and labor L. The production function is then written as

$$Y = F(K, L) \tag{2.1}$$

There are several properties of the function F that we're going to be concerned with. The first is returns to scale

Definition 7 *The* ***returns to scale*** *of a production function are defined by the amount that output increases following an proportional increase of all the* *inputs. More specifically, the returns to scale are measured by the following*

$$\begin{aligned} CRS &: zY = F(zK, zL) \\ DRS &: zY < F(zK, zL) \\ IRS &: zY > F(zK, zL) \end{aligned}$$

where CRS stands for constant returns to scale, DRS for decreasing returns, and IRS for increasing return.

Now we need to consider the properties of the production function in terms of marginal products. We will generally assume that F has the properties that

$$MPK = \frac{\partial Y}{\partial K} = F_K(K, L) > 0 \tag{2.2}$$

$$MPL = \frac{\partial Y}{\partial L} = F_L(K, L) > 0 \tag{2.3}$$

where MPK stands for marginal product of capital and MPL is the marginal product of labor. The production function is also assumed to have the following second derivatives

$$F_{KK}(K, L) < 0 \tag{2.4}$$

$$F_{LL}(K, L) < 0 \tag{2.5}$$

$$F_{LK}(K, L) = F_{KL}(K, L) > 0 \tag{2.6}$$

which tells us several things. First, production is increasing in each factor separately, but there are decreasing returns to each individual factor. That is, if you add more capital, output goes up, but at a decreasing rate (output is concave with respect to K or L). The cross-derivative is positive, so that an increase in one factor increases the productivity of the other.

We now need to consider how factors get paid. The wage w is just the payment to labor, and the interest rate r is just the payment to capital. What are these values? Well, we assume that there are a number of competitive firms in the economy, each with the same F production function. Each firm maximizes

$$\pi = F(K, L) - rK - wL \tag{2.7}$$

which has the FOC of

$$F_K = r \tag{2.8}$$

$$F_L = w \tag{2.9}$$

so by profit maximization firms will pay the factors their marginal products. Nothing too surprising here.[1]

[1] One additional twist is to consider what the effect of capital depreciation has on the rates of return. Anyone who owns a piece of capital has a net return to that capital of $R - \delta$, where R is the gross return to a unit of capital. An alternative activity for a household, though, is to loan their money to a firm at rate r, rather than owning capital themselves. Since these two options are perfect substitutes, it must be that $r = R - \delta$ or that $R = r + \delta$. The marginal product of capital must equal R, so we get that $F_K = r + \delta$ or $r = F_K - \delta$.

Now, if labor and capital are paid their marginal products, is there any output left over? With CRS, we can see that in fact the payments to factors of production use up exactly all the output. Take the definition of CRS and differentiate with respect to z

$$dzF(zK, zL) = K(dz)F_K(zK, zL) + L(dz)F_L(zK, zL)$$

and notice that you can cancel the dz from both sides. Evaluate the equation at $z = 1$ and you find

$$F(K, L) = K \times F_K(K, L) + L \times F_L(K, L) \tag{2.10}$$

or all of output is paid out to capital or to labor.

The final thing we will do is consider the production function in per person terms, because what we'll ultimately care about is output per person rather than total output. This depends on the property of having CRS

Definition 8 *The **intensive form** of the production function is essentially the per person (or worker) version of the production function. With CRS, set* $z = 1/L$ *and you get*

$$y \equiv \frac{Y}{L} = F\left(\frac{K}{L}, 1\right) \equiv f(k) \tag{2.11}$$

where $f(k)$ *is just a short-hand way of writing* $F(K/L, 1)$*. Note that the function* f *retains the properties of* F *in terms of derivatives with respect to* K/L*. That is,*

$$f'(k) > 0 \tag{2.12}$$

$$f''(k) < 0 \tag{2.13}$$

Now, let's consider a more specific production function that we will use almost exclusively in our modelling.

Example 9 *The Cobb-Douglas production function is written as follows*

$$Y = K^{\alpha}L^{1-\alpha}$$

where $0 < \alpha < 1$ *and which has the following intensive form*

$$y = k^{\alpha}$$

and the following marginal products

$$\begin{aligned} MPK &= \alpha K^{\alpha-1}L^{1-\alpha} = \frac{\alpha}{k^{1-\alpha}} \\ MPL &= (1-\alpha)K^{\alpha}L^{-\alpha} = (1-\alpha)k^{\alpha} \end{aligned}$$

Furthermore, if we think about the division of output, we know that

$$\begin{aligned} Y &= K \times MPK + L \times MPL \\ &= \alpha(Y) + (1-\alpha)Y \\ &= Y \end{aligned}$$

or that the share of output that goes to capital is exactly α while the share of output paid to labor is exactly $(1-\alpha)$. So you can pick these shares right off from the production function.
Note that the actual interest rate in this economy is $r = MPK - \delta$, or $\frac{\alpha}{k^{1-\alpha}} - \delta$.

Now we'll think about adding a new element to production, total factor productivity (TFP). We're going to add a simple scalar term to the production function to allow us to scale up output. This term often gets called "technology" and that surely plays a part in it, but it is important to remember that in fact total factor productivity is just a way for us to account for how much we don't know about where output comes from.

Definition 10 ***Hicks neutral TFP*** *is denoted A and modifies the production function like this*

$$Y = AF(K, L)$$

giving you an intensive form of

$$y = Af(k)$$

Definition 11 ***Harrod neutral TFP*** *(or labor augmenting TFP) is denoted E and modifies the production function like this*

$$Y = F(K, EL)$$

which has an intensive form of

$$y = F(k, E)$$

When we use Cobb-Douglas production functions these two forms are identical, with $A = E^{1-\alpha}$. Notice that the TFP terms scale up the marginal products of capital and labor as well. Without TFP, the only way to alter the size of the marginal products of labor and capital was to change the quantity of either of those two. Now we have some outside way of raising marginal products of both.

2.2 The Solow Model

Robert Solow published this is 1956, in an attempt to explain how we could have two facts co-exist. One, both the capital stock and the labor supply were growing over time and two, the return on capital (interest rates) were roughly constant. The insight he had seems almost fantastically simple now. If we have both K and L growing at the same rate, then k is actually not changing at all, and since $r = MPK - \delta = \alpha/k^{1-\alpha} - \delta$ it means that the interest rate is constant as well. The model shows how some simple mechanics of capital accumulation, combined with $f'' < 0$ property of the production function leads the economy to always tend to a steady state where in fact K and L grow at the same rate.

2.2.1 Capital Accumulation

This is the heart of the Solow model and it requires us to think about how the capital stock changes over time. You may recall from intermediate that we said that $S = I$, or savings equals investment. Investment then becomes new capital. So the savings that individuals do adds to the capital stock. The simplest way to see this is to think of our economy as being completely based on tomatoes. All we produce is tomatoes, and we can eat them or leave them in the garden to sprout new tomato plants. The share that we don't eat (our savings) becomes an input into future tomato production (seeds turning into plants). That's it. Deferred consumption can be used in the interim to increase the output of the economy.

So the capital stock will go up by the amount of investment done. Does anything decrease the capital stock? Yes, depreciation. Each period some portion of the existing capital stock breaks down or becomes unusable. You could imagine that in any year, some fraction of all our tomato seeds never sprout.

So the change in capital per person over time is determined by investment and depreciation, or

$$\dot{k} = i - \delta k \tag{2.14}$$

where $\dot{k}$ is dk/dt, i is the amount of investment per person, and δ is the fraction of the existing capital stock that depreciates at any given point in time.

Since investment equals savings, we need to know how much savings is done. Chapter 1 tells us that savings is an optimal decision based on the future path of income, etc.. etc.., but this is more complicated than we want to deal with right now. So for the Solow model we just assume that savings is a constant fraction of income per person or

$$i = sf(k)$$

which we might get if people expected constant income over time. Therefore the growth of capital per person is

$$\dot{k} = sf(k) - \delta k \tag{2.15}$$

which is just a differential equation in k. Now, we said that Solow was interested in the case where k was constant, so that r was constant. When is k constant? Well, that is just when $\dot{k} = 0$. So the steady state of the Solow model is defined as the level of capital k^* that solves the following

$$sf(k^*) = \delta k^*. \tag{2.16}$$

You can draw this out by plotting investment and depreciation against k and seeing where they cross.

What is going on when k is actually less than k^*? In that case, investment is greater than depreciation, so that k is increasing. If k is greater than k^*, the opposite is true and k falling. So the steady state is stable. This stability is due to the fact that $f'' < 0$.

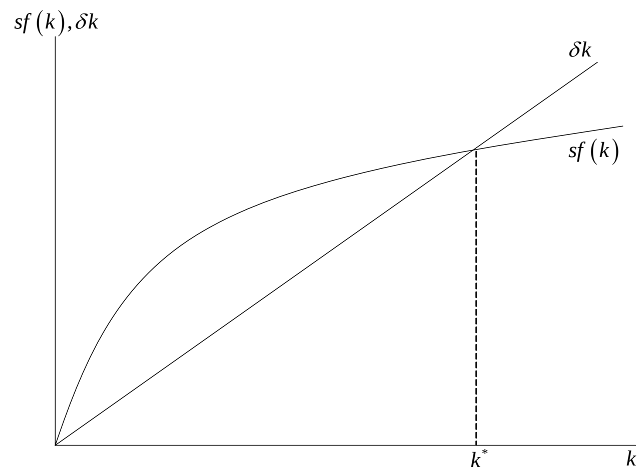

Figure 2.1:

How does an increase in s affect the system? It raises the $sf(k)$ curve, meaning that the steady state now lies farther to the right, or with a higher k^*. Given an increase in s, the capital per person will rise over time until reaching a new steady state, so r will fall over time until become constant again. A drop in s has the opposite effects.

Example 12 *With a Cobb-Douglas production function, the intensive form is*

$$y = f(k) = k^{\alpha}$$

so in the steady state we must have that

$$s(k^*)^{\alpha} = \delta k^*$$

which you can solve for

$$k^* = \left(\frac{s}{\delta}\right)^{1/(1-\alpha)}.$$

The steady state level of output is then just

$$y^* = (k^*)^{\alpha} = \left(\frac{s}{\delta}\right)^{\alpha/(1-\alpha)}.$$

Notice that the steady state levels of capital and output do not depend on your initial level of capital or output at all. Only the parameters of the Solow model make any difference to the steady state. What this tells us is that regardless *of where we start, we'll end up at the steady state. The actual growth rate of capital (and hence of output) depends on the size of the actual capital stock relative to the steady state. If $k > k^*$, then capital and output shrink (grow negatively). If $k < k^*$ then capital and output grow. At the steady state, neither capital nor output grow at all.*

2.2.2 Golden Rule Saving

So far we've had the savings rate be exogenous, but it's fair to ask what level of savings is actually optimal from the perspective of the individuals in this economy. In the next chapter we'll consider this much more fully by including a full dynamic optimization problem in the growth model, but for now we'll just examine the Solow model for the highest possible consumption level.

At a steady state, we know that the amount of savings done each period is δk^*. Therefore consumption is $c = f(k^*) - \delta k^*$. On a graph, c is then the gap between the $f(k)$ curve and the δk line. Then c is maximized where their slopes are equalized, or when $f'(k) = \delta$. You can see this more formally by taking the derivative of c with respect to k^*. You can solve this for some value k^{**} which is the golden rule capital stock. Notice that we can achieve the golden rule level of capital by setting the savings rate to whatever gets us to have our steady state equal the golden rule, or $k^* = k^{**}$.

Example 13 *In the CD world, the golden rule implies that*

$$\alpha k^{\alpha-1} = \delta$$

so that the golden rule level of capital is

$$k^{**} = \left(\frac{\alpha}{\delta}\right)^{1/(1-\alpha)}.$$

Comparing this to the steady state value of capital shows you that the optimal savings rate (or the consumption maximizing savings rate) is just $s = \alpha$.

Now for something that is pretty neat. Suppose that the rents from capital were all saved, and the wages to labor were all consumed, what would happen? Well, the saving rate is then

$$s = \frac{k \cdot f'(k)}{f(k)}$$

and in the steady state we know that $sf(k) = \delta k$ so we can plug this in to find

$$\begin{aligned} \delta k &= k \cdot f'(k) \\ \delta &= f'(k) \end{aligned}$$

or in other words we are at the golden rule level of capital. So saving the payments to capital will lead us to precisely the optimal consumption path.

2.2.3 Population Growth

So far we have left the size of L fixed. Now we consider how the model changes when we include a growth rate for population of $n = \dot{L}/L$. Now we want to find out what $\dot{k}$ is, but it isn't quite as easy as before. Here is what we get

$$\dot{k} = \frac{\partial (K/L)}{\partial t} = \frac{L\dot{K} - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L}.$$

We already know that the total capital stock evolves as follows

$$\dot{K} = sY - \delta K \tag{2.17}$$

and we can plug this and our definition of n into the equation for $\dot{k}$ to get

$$\dot{k} = sy - (\delta + n)\,k.$$

What is going on here is that as the population grows, the capital stock per person is falling, so population growth is just another form of depreciation. The analysis of the steady state and golden rule follow the same methods as before and we get

$$\begin{array}{rl} \text{Steady state:} & sf(k^*) = (n+\delta)\,k^* \\ \text{Golden rule:} & f'(k^{**}) = (n+\delta) \end{array}$$

A last question is what the growth rate of output is? Well, in the steady state we know that $\dot{k} = 0$ so that $\dot{y} = 0$, or output per person is not growing. But we also know that population is growing at rate n. So it must be the case that both $\dot{K}$ and $\dot{Y}$ are equal to n as well. We now have total output growing, but no per capita growth.

2.2.4 Technical Change

This last modification of the Solow model allows for technical change, which will turn out to be the only source of sustained growth in output per capita. This will be a feature of growth models in general. Sustained growth in output per capita is not possible just by accumulating more factors of production. The diminishing marginal returns to both labor and capital mean that you'll eventually exhaust the gains. So most of growth theory is now actually about how to model the process of technical change that leads to sustained growth.

For the Solow model, we're going to take the growth of technology as exogenous and just see how it alters the results. We'll say that the production function now looks like this

$$Y = F(K, EL)$$

or that we have Harrod neutral technology. E represents the efficiency of a unit of labor. This is simply the amount of work you can get done in a specified amount of time. As E increases, labor is more productive. More importantly for the model, the growth of E is like you are adding new workers to the economy.

The growth of E is given by $g = \dot{E}/E$. To analyze the Solow model now, we make a trick of notation and rephrase the intensive forms of the production function by saying that now

$$\begin{aligned} \tilde{k} &= K/EL \\ \tilde{y} &= Y/EL \end{aligned}$$

which allows us to write the intensive production function as

$$\tilde{y} = f\left(\tilde{k}\right).$$

Now let's analyze the change in $\dot{k}$ over time, just like before

$$\begin{aligned} \partial\tilde{k}/\partial t &= \frac{\partial K/EL}{\partial t} = \frac{EL\dot{K} - KE\dot{L} - KL\dot{E}}{(EL)^2} \\ &= \frac{\dot{K}}{EL} - \frac{K}{EL}\frac{\dot{L}}{L} - \frac{K}{EL}\frac{\dot{E}}{E}. \end{aligned}$$

We know again from (2.17) how to write $\dot{K}$ and we have definitions for $\dot{L}/L$ and $\dot{E}/E$ so we get the following

$$\partial\tilde{k}/\partial t = sf\left(\tilde{k}\right) - (n + \delta + g)\,\tilde{k} \tag{2.18}$$

which says that the capital stock per efficiency unit is subject to two forces. First, savings increases the capital stock, raising the level of capital per efficiency unit. Second, there is depreciation of the capital stock per efficiency unit, as δ decreases K, n increases L, and g increases E. This makes it sound like technology growth is bad, but recall that we are talking about the growth of the capital stock per efficiency unit, not the capital stock per person. Technology is making it seem as if there are more people working, but there in reality are not more people.

In the steady state we know that $\partial\tilde{k}/\partial t = 0$, and that $\partial\tilde{y}/\partial t = 0$. What does this mean for the growth rate of capital and output per worker? Let's look at the growth of output per worker

$$\begin{aligned} \frac{\partial Y/L}{\partial t} &= \frac{\partial Y/EL}{\partial t} + \frac{\partial E}{\partial t} \\ &= \partial\tilde{y}/\partial t + \dot{E}/E \\ &= 0 + g \\ &= g. \end{aligned}$$

So we have that output per person is growing at rate g in the steady state when we have technology growth. This result follows through for capital per worker as well.

Now the growth rate of capital per worker or output per worker when we are NOT in the steady state differs from g. How? Well, recall that if we are below the steady level of $\tilde{k}^*$ then $\tilde{k}$ is growing, and therefore $\tilde{y}$ is growing. Looking at the derivation above, we see that this means that output per person is then growing faster than g. The reverse also holds. If $\tilde{k} > \tilde{k}^*$, then output per person would be growing slower than g. But the steady state level of growth is always g.

The last thing we might consider is what the actual interest rate or wage rate is (as this may be useful when we consider individual optimization later). What is the interest rate in the situation with technological change? Recall that a firm in this economy is trying to maximize profits, which can be written as follows

$$\pi = F(K, EL) - (r+\delta)K - wL$$

where $r+\delta$ is the total rate that a firm has to pay for capital services. We can consider a firm with an arbitrary size, where size is measured as the number of effective units of labor employed. Now profits are

$$\pi = EL\left\{f\left(\tilde{k}\right) - (r+\delta)\tilde{K} - wE\right\}$$

which can be maximized, given r and w, to get the following

$$\begin{aligned} f'\left(\tilde{k}\right) - \delta &= r \\ \left[f\left(\tilde{k}\right) - \tilde{k}f'\left(\tilde{k}\right)\right]E &= w. \end{aligned}$$

This shows that the interest rate in a model with constant technological progress is constant. The wage rate, though, grows as E grows.

2.3 Human Capital

The dynamics of the Solow model come from the accumulation of capital. We generally think of capital as machines and buildings, but there are other forms of capital we want to expand out thinking to include.. Namely, human capital. We're going to add a new factor of production to our analysis that can be accumulated as well.

The way we'll do this makes human capital operate just like physical capital (it accumulates through savings and depreciates, etc..) which may not be exactly realistic. But the mechanics of how H will evolve allow us to see how a second accumulable factor changes the results of the Solow model. We won't be able to go into, in this course, other modifications we can make to the model to make human capital more realistic. This generally involves making the decision to obtain it endogenous.

So first we write our overall production function as follows

$$Y = K^{\alpha}H^{\beta}(EL)^{1-\alpha-\beta} \tag{2.19}$$

where H is the human capital stock at any given point in time. We'll do everything in per efficiency units (so that $h = H/EL$) and we get

$$y = k^{\alpha} h^{\beta}$$

and this intensive function looks a lot like the ones with only capital. In particular, it has the property that this intensive form of the production function is decreasing returns to scale. That is, if I doubled both k and h, I'd get less than 2 times the output. This property ensures that we'll have both $\dot{k}$ and $\dot{h}$ equal to zero in the steady state (we can't generate output fast enough to keep both stocks growing). We assume that H is accumulated in a pattern just like that of K, with it's own savings rate so that we have

$$\begin{aligned} \dot{k} &= s_k y - (n + \delta + g)\, k \\ \dot{h} &= s_h y - (n + \delta + g)\, h \end{aligned}$$

To find the steady state you set both of these to zero and solve together to find

$$\begin{aligned} k^* &= \left(\frac{s_k^{1-\beta} s_h^{\beta}}{n + \delta + g} \right)^{1/(1-\alpha-\beta)} \\ h^* &= \left(\frac{s_k^{\alpha} s_h^{1-\alpha}}{n + \delta + g} \right)^{1/(1-\alpha-\beta)} \end{aligned}$$

which you'll notice are very similar and have a form related to the k only steady state. What does this model tell us is happening to k and h in the steady state? They both are growing at the rate zero. So their ratio must be staying constant, or k/h is fixed in the steady state. What is this ratio? You can solve it out to see that

$$\frac{k^*}{h^*} = \frac{s_k}{s_h}$$

or their ratio just depends on the relative savings rates. But notice that this implies that the ratio of K/H must be constant as well, or the overall stocks of physical and human capital must both be growing at the same rate of $n + g$. This holds even though there are different savings rates.

The addition of another type of capital doesn't change the overall implications of the Solow model. Increases in savings rates or decreases in population growth rates both increase income. Exogenous changes in the capital stocks create changes in the growth rate of income based on whether you are above or below steady states. The additional factor to consider is that k and h bear some relationship to each other, and so a change in one will alter the growth rate of the other.

Chapter 3

Essential Models of Dynamic Optimization

Whether you believe it or not, you now have all the intution necessary to understand the full blown models of dynamic optimization that sit at the heart of almost every macroeconomic paper written today or in the last twenty years. The two models are the infinitely lived agent model and the overlapping generations model. This chapter is going to cover the mathematics of these models in some detail, as I want you to see how the discrete and continuous time models mimic each other. I also want to show how there are different mathematical methods of looking at these problems, but that ultimately the methods are all getting the same answers.

This analysis is fundamentally identical to the two period models we looked at so far, but simply extends the analysis out to T periods of life. Friedman anticipated all of this, but at the time of his writing the mathematics was not available to solve the problems he had in mind. It's important to remember that we should be impressed with Friedman's insight, not with the fancy math.

3.1 The T Period Fisher Model

The Fisher model easily extends to multiple periods. The person is now planning consumption over periods 0 to $T-1$, (which are labelled this way for convenience). The path of wages is $W_0...W_{T-1}$. The person gets utility from the felicity function of $U(C_t)$ which is discounted at rate θ. So utility is

$$V=\sum_{t=0}^{T-1}\frac{U(C_t)}{(1+\theta)^t} \tag{3.1}$$

The interest rate is r, is constant over time, and the person can borrow or save at this rate. Call A_t the assets the person has at the beginning of a period, then

$$A_t = (1+r) * (A_{t-1} + W_{t-1} - C_{t-1}) \tag{3.2}$$

and the person starts with $A_0 = 0$. We also impose that the person must have zero assets at the end of life, or $A_T = 0$. In other words, in period $T-1$ the person consumes everything they have left, and they cannot die in debt.

We can use this information to get the intertemporal budget constraint. Note that

$$\begin{aligned}
A_1 &= (1+r)(W_0 - C_0) \\
A_2 &= (1+r)(A_1 + W_1 - C_1) = (1+r)(W_1 - C_1) + (1+r)^2 (W_0 - C_0) \\
&\dots \\
A_T &= (1+r)(W_{T-1} - C_{T-1}) + (1+r)^2 (W_{T-2} - C_{T-2}) + \\
&\dots + (1+r)^T (W_0 - C_0)
\end{aligned}$$

Divide all the terms in A_T by $(1+r)^T$ and note that because $A_T = 0$ we can write

$$0 = \sum_{t=0}^{T-1} \frac{W_t - C_t}{(1+r)^t} \tag{3.3}$$

and notice that all the asset terms are gone. Rearrange this slightly and we can see that

$$\sum_{t=0}^{T-1} \frac{W_t}{(1+r)^t} = \sum_{t=0}^{T-1} \frac{C_t}{(1+r)^t} \tag{3.4}$$

which says that the present discounted value of wages has to be equal to the present discounted value of consumption. This is identical to the 2 period budget constraint, just extended to T periods.

Given both (3.1) and (3.3) we can set up a massive Lagrangian like this

$$L = \sum_{t=0}^{T-1} \frac{U(C_t)}{(1+\theta)^t} - \lambda \sum_{t=0}^{T-1} \frac{W_t - C_t}{(1+r)^t} \tag{3.5}$$

and to solve this we'd have $T+1$ first order conditions which we could solve together for λ and the T values of C_t. However, this is a mess, and we can get pretty far by just looking at the FOC of two adjacent periods, t and $t+1$.

$$\frac{\partial L}{\partial C_t} = \frac{U'(C_t)}{(1+\theta)^t} - \lambda \frac{1}{(1+r)^t} = 0 \tag{3.6}$$

$$\frac{\partial L}{\partial C_{t+1}} = \frac{U'(C_{t+1})}{(1+\theta)^{t+1}} - \lambda \frac{1}{(1+r)^{t+1}} = 0 \tag{3.7}$$

which can be combined to give the following

$$\frac{U'(C_t)}{U'(C_{t+1})} = \frac{1+r}{1+\theta} \tag{3.8}$$

which relates consumption in adjacent periods. Notice that just like in the 2 period Fisher problem, this condition doesn't depend on total income. The condition in (3.8) is of central importance to all models of dynamic optimization, and is referred to as an Euler equation. It doesn't allow us to understand the level of C_t or C_{t+1}, only their relative size.

To see the centrality of this result, let's think through this without doing the math. I have some set of total consumption I'd like to do, how will I divide it up between multiple periods? Let's take any given path of consumption I could choose $(C_0...C_{T-1})$. Is this path optimal? Let's perturb this path and see what I get. So let's imagine I consume one less unit in period zero, save it and then consume the $1+r$ extra units in period 1.

What would I lose? I would lose $U'(C_0)$ of utility. (We assume the one unit of consumption is small relative to C_0).

What would I gain? I would get an additional $1+r$ units of consumption in period 1, and this would give me $(1+r)\,U'(C_1)$ in utility, but I have to remember to discount this, so my total gain is $\beta\,(1+r)\,U'(C_1)$.

Now, if these are not equal to each other, then I could gain in total utility by moving one unit of consumption around. So if my path $(C_0...C_{T-1})$ is optimal, then it must be the case that $\beta\,(1+r)\,U'(C_1) = U'(C_0)$, and a similar logic must hold between any other two periods as well. This is just the Euler equation we got from (3.8).

So let's look at the Euler equation some more and see what it is telling us about how consumption changes over time. What happens if $r > \theta$, so that $\frac{U'(C_t)}{U'(C_{t+1})} < 1$. This means that marginal utility in period t must be greater than in period $t+1$, and this implies that $C_t < C_{t+1}$. So if $r > \theta$, it must be the case that consumption is rising over time. The return on savings is so high that it is worth having low initial consumption, saving a lot, and then consuming a lot in the future.

On the other hand, if $r < \theta$, the opposite holds and $C_t > C_{t+1}$, or consumption is falling over time. The return on savings is so low that it isn't worth forgoing consumption today. I prefer to consume as much as I can right now, and over time my consumption will decrease.

If $r = \theta$ then these two effects exactly balance out, and I want to consume the same exact amount in every period. Recall that this holds regardless of the actual path of my income.

Note that the speed with which consumption grows or shrinks over time depends on the actual form of the marginal utility. And the form of marginal utility is summarized by the measure of risk aversion. The more risk averse I am, the more concerned I am with having smooth consumption (even in every period). So if I am very risk averse, it will take a really big gap or $r > \theta$ for me to choose to have a steeply increasing consumption path. If I am close to risk neutral, then I'm very sensitive to any difference between the interest rate and discount rate.

To see this, consider the CRRA utility function, which has a marginal utility of $U'(C_t) = C_t^{-\sigma}$. Putting this in the Euler equation gives us

$$\frac{C_{t+1}}{C_t} = \left(\frac{1+r}{1+\theta}\right)^{1/\sigma} \tag{3.9}$$

and we have an expression for exactly the relative size of consumption in adjacent periods. The larger σ, the closer this ratio goes to one. So the more risk averse you are, the closer your consumption path is to equal consumption in every period.

3.1.1 Infinitely Lived Agents

This will be quick, because it is nothing more than the Fisher T period problem extended to $T = \infty$. The problem is then as follows

$$\max \sum_{t=0}^{\infty} \frac{U(c_t)}{(1+\theta)^t} \tag{3.10}$$

subject to

$$a_{t+1} = (a_t + w_t - c_t)(1+r) \tag{3.11}$$

where a_t are assets at time t, w_t is your income at time t, and r is fixed over all periods. Again we assume that $a_0 = \bar{a}_0$, a fixed amount that could be zero. To rule out the possibility that people have infinite consumption financed by infinite borrowing, we impose the following present value budget constraint

$$\bar{a}_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} \tag{3.12}$$

You can set up a Lagrangian again, get the FOC for two adjacent periods, solve them together and get the Euler equation that says

$$\frac{U'(c_t)}{U'(c_{t+1})} = \frac{1+r}{1+\theta} \tag{3.13}$$

and this has the same interpetation as in the multiple period Fisher model.

Given the FOC, we could conceivably solve this explicitly for a value of c_0 that satisfied the budget constraint as well. Without further specifications for the utility function, though, we don't have enough information. A lot of the time it will be enough to understand the Euler equation.

Example 14 *But let's not leave it at that. If we have CRRA utility, what can we say about the optimal consumption path? First, we know that our Euler equation implies that*

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r}{1+\theta}\right)^{1/\sigma}$$

and we can use this to tell us something about c_0 *and* c_1. *Specifically,*

$$c_1 = \left(\frac{1+r}{1+\theta}\right)^{1/\sigma} c_0$$

and by analogy

$$c_2 = \left(\frac{1+r}{1+\theta}\right)^{1/\sigma} c_1 = \left(\frac{1+r}{1+\theta}\right)^{1/\sigma} \left(\frac{1+r}{1+\theta}\right)^{1/\sigma} c_0$$

or

$$c_2 = \left[\left(\frac{1+r}{1+\theta}\right)^{1/\sigma}\right]^2 c_0$$

which we can logically then extend to

$$c_t = \left[\left(\frac{1+r}{1+\theta}\right)^{1/\sigma}\right]^t c_0$$

Now, we can use this last equality to plug into our intertemporal budget constraint from (3.12) *to give us*

$$\bar{a}_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t} = \sum_{t=0}^{\infty} \left[\left(\frac{1+r}{1+\theta}\right)^{1/\sigma}\right]^t c_0 \frac{1}{(1+r)^t}$$

which can be rewritten as

$$\bar{a}_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t} = c_0 \sum_{t=0}^{\infty} \left[\frac{(1+r)^{1/\sigma - 1}}{(1+\theta)^{1/\sigma}}\right]^t$$

For the summation on the right to converge, we need that

$$\frac{(1+r)^{1/\sigma-1}}{(1+\theta)^{1/\sigma}} < 1$$

or that

$$(1+r)^{1-\sigma} < 1+\theta$$

which tells us that to even have an answer, we have to have a discount rate that is "large enough". Large enough to what? Large enough to keep me from wanting to save too much and end up with infinite consumption later in life. Notice that an increase in σ *means that the left hand side goes towards zero. This means that you can have a smaller discount rate and still get a solution. Why? Because as sigma rises, this means that people prefer flat consumption, and are less likely to choose explosive consumption growth. Regardless, assuming that the above condition holds, we can get an answer for* c_0 *that looks like this*

$$c_0 = \left[\frac{(1+r)(1+\theta)^{1/\sigma} - (1+r)^{1/\sigma}}{(1+r)(1+\theta)^{1/\sigma}}\right] \left[\bar{a}_0 + \sum_{t=0}^{\infty} \frac{w_t}{(1+r)^t}\right]$$

and notice that the first term in this just defines a fraction between 0 and 1. What this says is that the interest rate, discount rate and coefficient of risk aversion determine what share of your total PDV of income that you consume in period zero. You can have some fun figuring out the derivatives of c_0 *with respect to the different parameters. To see some simple logic, assume we have log utility, so that* $\sigma = 1$. *This gives us*

$$c_0 = \left[\frac{\theta}{1+\theta}\right]\left[\bar{a}_0 + \sum_{t=0}^{\infty}\frac{w_t}{(1+r)^t}\right]$$

and the interest rate falls out. The higher the discount rate, the higher the initial consumption, as expected.

3.1.2 Dynamic Programming

Dynamic programming is a method of writing and solving dynamic optimization problems that differs from the Lagrangian. It uses recursive equations, also called Bellman equations, to break a T period problem up into a bunch of much smaller one period problems. The most important thing to note is that the problem itself is not any different, only the solution method has changed. So we still have individuals trying to maximize their lifetime utility subject to some constraints on their lifetime wages and assets. We should get out exactly the same Euler equation.

If you read a book like Stokey and Lucas (1989), or Sargent (1987), much of the text is spent proving to you that it is theoretically possible to solve the following problem

$$\max_{c_1..c_\infty} V = \sum_{t=0}^{\infty}\frac{U(c_t)}{(1+\theta)^t} \tag{3.14}$$

subject to

$$a_{t+1} = (a_t + w_t - c_t)(1+r) \tag{3.15}$$

by using the techniques of dynamic programming. We'll take that proof as a given, and just proceed to show you how the method actually works. The proof depends a lot on the presence of time-separable preferences and then requires U(c) to be concave, continuous, etc..

The first step is to write down the value function, which in these problems you can think of as being similar to an indirect utility function.

$$V_t(a_t) = \max_{c_t}\sum_{s=t}^{\infty}\frac{U(c_s)}{(1+\theta)^s} \text{ s.t. } a_{t+1} = (a_t + w_t - c_t)(1+r) \tag{3.16}$$

This value function tells us that V is the maximized value of utility I have, given an initial asset level of at, from time period t until infinity, along my optimal path. Bellman then used the insight that if you performed your optimization at time t+1, the path of consumption that you would choose must follow the exact same path that you would have chosen for periods t+1 to infinity if you had done your optimization at time t. (The crucial assumption for this is that preferences are time separable).

This means I can write the above value function recursively, or as follows:

$$V_t(a_t) = \max_{c_t}\left\{U(C_t) + \frac{V_{t+1}(a_{t+1})}{1+\theta}\right\} \text{ s.t. } a_{t+1} = (a_t + w_t - c_t)(1+r) \tag{3.17}$$

This recursive, or Bellman, equation tells us that the value of my lifetime utility from time t forward is equal to the utility of consumption at time t plus the value of my lifetime utility from time t+1 forward.

Suppose for the moment that I actually know what the V function looks like (and notice that the V function can possibly change over time). Then my problem is no longer a many period problem but only a one period problem. The question is trading off current utility at time t for more financial wealth at time t+1 (which I already know I will spread optimally among the remaining periods of my life).

So let's do the maximization in the Bellman equation. This gives us the first order condition that

$$U'(C_t) = \frac{1}{1+\theta}\left\{(1+r)V'_{t+1}(a_{t+1})\right\} \tag{3.18}$$

which is already starting to look a lot like the Euler equation we found before. This says that I should trade off the marginal utility of consumption today against the (suitably discounted) marginal value of remaining lifetime utility starting tomorrow. But we don't know what this V function looks like, so this equation doesn't help us a lot.

However, the next big insight in the dynamic programming method is that there is a simple envelope relationship between V' and U' along the optimal path.[1] That is, the derivative of V with respect to assets is just

$$V'_{t+1}(a_{t+1}) = \frac{1}{1+\theta}(1+r)V'_{t+2}(a_{t+2}). \tag{3.19}$$

But we know that since V' in t+1 is the maximized value of utility, then it must be the case from the t+1 FOC that the right hand side of the above equation is equal to the marginal utility of consumption in period t+1. This means that

$$V'_{t+1}(a_{t+1}) = U'(C_{t+1}).$$

[1] More formally, the envelope theorem says that if you have y=max f(x,c) w.r.t x, then the derivative dy/dc can be evaluated as follows. First, define x*=g(c) as the optimal value of x given a value of c. Write y=f(g(c),c). Now dy/dc = f1(g(c),c)g'(c)+f2(g(c),c). But we know that f1(g(c),c)=0 by the first order conditions that made x*=g(c) in the first place. So the first term drops out and dy/dc = f2(g(c),c). In other words, the derivative of y with respect to c is just the derivative of the original f(x,c) function with respect to c.

Plugging this into the original period t FOC gives us

$$U'(C_t) = \frac{1}{1+\theta}\{(1+r)\,U'(C_{t+1})\} \tag{3.20}$$

and this is obviously just the Euler equation.

Solving the model completely requires that you then solve the Euler equation for some consumption path and utilize the budget constraint. This is just the same as before. The dynamic programming method doesn't necessarily offer any extra help during these last steps. It's main value is that certain problems are easier to set up as Bellman equations in the first place. The recursive equations are also useful because they are easier to translate to computer code that can iterate through periods quickly to find the optimal path (which allows you to calibrate your model).

An additional mathematical result of this technique that can be useful involves the nature of V. Under a certain set of conditions (continuity, concavity, etc..) it can be shown that the Bellman equation is an example of a contraction mapping, and that this means the V functions (which were previously allowed to vary over time) will converge to a single functional form $V(a_t)$. In addition, this means that the control function, or the rule for setting consumption in time t as a function of assets at time t, will be time invariant as well.

Example 15 *To see what this means, consider a problem with log utility, so that*

$$C_{t+1} = \frac{1+r}{1+\theta}C_t$$

and therefore consumption in any period s>t can be written as

$$C_s = \left(\frac{1+r}{1+\theta}\right)^s C_t.$$

The budget constraint at time t is the following

$$\sum_{s=t}^{\infty}\frac{C_s}{(1+r)^{s-t}} = A_t + \sum_{s=t}^{\infty}\frac{W_s}{(1+r)^{s-t}}$$

And you can solve these together to get that

$$C_t = \frac{\theta}{1+\theta}\left[A_t + \sum_{s=t}^{\infty}\frac{W_s}{(1+r)^{s-t}}\right].$$

This rule holds for any period t, so the consumption rule (or control rule) is identical for all periods. This doesn't mean that consumption itself is necessarily identical every period, but the rule for setting it is. You may still have consumption rising or falling depending on the relationship of r and the discount rate.

3.1.3 Lifespan Uncertainly and Insurance

I can set this up more generally in a T period problem by assuming that I have a P_t chance of being alive in period t. Then my expected utility in my life is

$$EV = \sum_{t=0}^{T-1} \frac{P_t U(C_t)}{(1+\theta)^t} \tag{3.21}$$

and I have a similar budget constraint to the regular T period problem. Note that the budget constraint describes how my assets are related to wages and consumption assuming that I am alive. The only uncertainty in the model is when you die, and since when that happens you can't change anything, you might as well plan out your whole path of consumption from the beginning. No new information arises during your life that you can actually use.

We still hold that you cannot die with negative assets, so this means you have to have positive assets in every period. There is some unpleasant math involving this, but we can get around it by assuming we have someone with only an initial stock of wealth but no wage income. The Lagrangian looks like this

$$L = \sum_{t=0}^{T-1} \frac{P_t U(C_t)}{(1+\theta)^t} + \lambda \left(A_0 - \sum_{t=0}^{T-1} \frac{C_t}{(1+r)^t} \right) \tag{3.22}$$

and we get FOC from periods t and $t+1$ that show

$$\frac{U'(C_{t+1})}{U'(C_t)} = \frac{1+\theta}{1+r} \frac{P_t}{P_{t+1}} \tag{3.23}$$

and now we have some fun with math. First we'll rewrite the ratio of dying probabilities as

$$\frac{P_t}{P_{t+1}} = \frac{P_{t+1} + (P_t - P_{t+1})}{P_{t+1}} = 1 + \frac{(P_t - P_{t+1})}{P_{t+1}} \approx 1 + \rho_t$$

where ρ_t is the probability of dying in a given period conditional on having lived to that age in the first place. The Euler equation is then

$$\frac{U'(C_{t+1})}{U'(C_t)} = \frac{1+\theta}{1+r}(1+\rho_t) \approx \frac{1+\theta+\rho_t}{1+r}$$

which shows us that we've basically just modified the discount rate (by raising it) so that we'll get a consumption path tilted more towards current period consumption.

Now the individual in this problem will always die holding onto some assets unless she happens to get very lucky and live all the way to period T. Leaving assets on the table doesn't help her consumption very much, so could we make her better off? Yes, buy getting her an annuity.

So if we have a constant probability of dying each period, $\rho_t = \rho$ for all t, and some market interest rate of r, we could start a company that makes deal with each individual that says "Give me your assets, and I will pay you some rate of interest z, but if you die before next year I get to keep all the assets remaining." Now obviously z has to be above r, or the individuals wouldn't take the deal. So what should z be? Well, assume that the annuity market is competitive, so that there are no profits to be made. Then it must be that my annuity company equates

$$(1+z)(1-\rho) = (1+r) \tag{3.24}$$

which says that the amount I have to pay out to individuals is $(1+z)(1-\rho)$: I pay the rate z to everyone who is still alive, which happens with probability $1-\rho$. The amount I earn on the assets I'm holding from these people is $(1+r)$. I can solve this out to see that

$$z = (1+r)/(1-p) - 1 \approx (1+r+\rho) - 1 = r+\rho$$

Now the individuals have a new interest rate of z that they earn each period, so their Euler equation is

$$\frac{U'(C_{t+1})}{U'(C_t)} = \frac{1+\theta+\rho}{1+z} = \frac{1+\theta+\rho}{1+r+\rho}.$$

What this tells us is that if $r = \theta$, the person will still have flat consumption across periods, even though her probability of being dead was rising. We've gotten the person around the problem of having to have declining consumption in the face of death by paying her a higher interest rate than she otherwise would face. And that is how annuities generally work. You trade your lump sum of assets for a predictable stream of income, with the insurance company offering to pay you a higher rate of return in exchange for the chance to keep your money if you die.

3.1.4 Continuous Time Optimal Consumption

We can reconsider the whole question of dynamic optimization, except now we can look at a continuous time version. That is, people don't have discrete periods of life, but evolve, well, continuously. This is primarily just a change in notation and mathematical technique, but all the same intuitions still apply.

Now we have

$$\max_c \int_0^\infty e^{-\theta t} U(c)\, dt \tag{3.25}$$

subject to the constraint that

$$\dot{a} = ra + w - c \tag{3.26}$$

where a is now the instantaneous level of assets, w is the instantaneous wage rate, c is the instantaneous level of consumption, and r is the constant rate of interest. In the utility function, θ again represents the discount rate, only it is set up in continuous time. The level of a is the state variable, meaning that it does not jump around, while c is the control variable, meaning that it can.[2]. For all variables, I've dropped the time subscript for convenience.

We again want to eliminate the possibility of unlimited borrowing and infinite consumption, so we have the present value budget constraint of

$$a_0 + \int_0^\infty e^{-rt} w dt = \int_0^\infty e^{-rt} c dt \tag{3.27}$$

You could set up a Lagrangian, but this is a pain and some Russians in the 1950's found a much better way to go about this. We'll set up a Hamiltonian, which is written as

$$H = \max_c \left\{ e^{-\theta t} U(c) + \mu (ra + w - c) \right\} \tag{3.28}$$

which looks a lot like a Lagrangian. However, the multiplier μ is the instantaneous shadow value of assets, and is time varying. Now, to solve this problem you need to apply several conditions to the Hamiltonian.[3]

First, you maximize H with respect to c, as written

$$U'(c) e^{-\theta t} - \mu = 0 \tag{3.29}$$

and this gives you something that looks like the FOC from the Lagrangian. It's telling us that we have to balance out the marginal gain in utility from consumption against the marginal cost, which is given by μ and represents the shadow value of assets at any given point in time.

Next, you recover the constraint by taking $\partial H / \partial \mu = \dot{a}$ or

$$\dot{a} = ra + w - c \tag{3.30}$$

which is just ensuring that we meet the constraint on how assets accumulate. The rate of change of the shadow value of assets, $\dot{\mu} = -\partial H / \partial a$ or

$$\dot{\mu} = -\mu r \tag{3.31}$$

and this is the least obvious one. The insight of the people who invented this technique was proving that this condition held. Finally, we have a transversality condition, which keeps the problem from "blowing up" and having infinite consumption later in life. This condition is that

$$\lim_{t \to \infty} \mu a = 0 \tag{3.32}$$

You solve $(3.29), (3.30)$, and (3.31) together in order to find the solution. First, take (3.29) and take the derivative with respect to time

$$\dot{c} U''(c) e^{-\theta t} - \theta U'(c) e^{-\theta t} - \dot{\mu} = 0$$

[2] A state variable is like your weight, while a control variable is like your calorie intake for the day. You can vary your calorie intake daily, jumping from 100 to 1000 to 5000 calories or back. But your weight will only adjust slowly in reaction to changes in the control.

[3] I'm not going to cover the theory of optimal control, only utilize the results. You can see Chiang's book for an explanation of why this all works.

which gives us another expression for $\dot{\mu}$. Plug (3.31) into the above equation to get

$$\dot{c}U''(c)\,e^{-\theta t} - \theta U'(c)\,e^{-\theta t} = -\mu r$$

Now notice from (3.29) that $\mu = U'(c)\,e^{-\theta t}$ and plug that in to get

$$\dot{c}U''(c)\,e^{-\theta t} - \theta U'(c)\,e^{-\theta t} = -U'(c)\,e^{-\theta t} r$$

Start going crazy with the algebra and you can get the following statement

$$\frac{\dot{c}}{c} = (r-\theta)\,\frac{U'(c)}{cU''(c)} \tag{3.33}$$

which describes the growth of consumption over time. Assume for the moment that the $\frac{U'(c)}{cU''(c)}$ term is constant. Then whether consumption is growing or falling depends on the relative size of r and θ, or exactly what we saw in the Fisher model. If r is larger, then consumption is rising as people save their incomes, and if θ is larger then consumption is falling as people discount the future a lot.

The second term on the right hand side of (3.33) should be familiar. It's just the inverse of the coefficient of relative risk aversion. What it says is that your consumption growth will be slower if your risk aversion (smoothing preference) is higher.

Example 16 *In the CRRA case, we know exactly what the risk aversion is, σ. So that means that if preferenes are CRRA, the optimal consumption growth is*

$$\frac{\dot{c}}{c} = (r-\theta)\,\frac{1}{\sigma}$$

Again, solving explicitly for the level of consumption is possible. Equation (3.33) *is a first order differential equation with a simple form and has the solution that*

$$c_t = c_0 e^{\frac{1}{\sigma}(r-\theta)t}$$

which gives us a nice way to describe consumption in any period. Now we need the budget constraint, which was

$$a_0 + \int_0^\infty e^{-rt} w dt = \int_0^\infty e^{-rt} c dt$$

and we can plug in our formula for c_t, play with some algebra and get

$$a_0 + \int_0^\infty e^{-rt} w dt = c_0 \int_0^\infty e^{\frac{1}{\sigma}(r(1-\sigma)-\theta)t} dt$$

The integral on the right hand side can be evaluated to be a positive, finite number if

$$(1-\sigma)\,r < \theta$$

which is just like the condition we saw in discrete time. Now, evaluating the integral and rearranging we get that

$$c_0 = \frac{1}{\sigma}\,(\theta - (1-\sigma)\,r)\left[a_0 + \int_0^\infty e^{-rt} w dt\right]$$

3.1.5 The Ramsey Model

At this point you may be wondering exactly why we bothered to go over the growth material in Chapter 2. We'll address that concern right now. What we're going to do, essentially, is make the interest rate endogenous. The optimizing agent will know how interest rates are formed, so will take into account the effect of their actions on the interest rate in the future. The interest rate is simply the marginal product of capital, so we'll need to know the capital stock. The capital stock is just accumulated investment effort, which in turn depends on savings. So the savings decisions of individuals lead to the interest rate itself, which of course influences the savings decision in the first place.

So let's think about the agent doing the optimizing. This person is performing the identical problem from the previous section, only now they have to account for the fact that now their state variable is k, and it evolves according to

$$\dot{k} = f(k) - c - nk \tag{3.34}$$

where I've assumed that depreciation and technological growth are both zero to begin with. This is almost identical to the asset evolution equation in (3.26). In fact, if we assert that each person has one unit of labor to offer at any point, then we can divide up total output as

$$f(k) = k \cdot f'(k) + w$$

which we established in equation (2.10) in the growth chapter. Putting this into the budget constraint gives us

$$\begin{aligned} \dot{k} &= k \cdot f'(k) + w - c - nk \\ &= k\left(f'(k) - n\right) + w - c \end{aligned}$$

and this tells us that the effective interest rate for the agent is $f'(k) - n$.

So the optimization problem is the same.[4] We know how to do the Hamiltonian, which gives us the Euler equation. Assuming CRRA utility for simplicity, we know that

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left(f'(k) - n - \theta\right) \tag{3.35}$$

So the instantaneous growth rate of consumption depends on the interest rate at that moment. This depends in turn on the actual capital stock at that moment. Well, we know how the capital stock evolves over time from (3.34). So we have two differential equations in two unknowns, c and k. We can solve them together to try and find the steady state where both c and k are constant.

We'll do this by looking at a phase diagram that graphs c against k. We first think about all the points at which $\dot{c} = 0$. This holds only when $f'(k) = n + \theta$,

[4] Well, not quite. What we are assuming is that the maximization is over the utility of the average person in the future. That is, c is consumption per person. An alternative would be to modify the optimization to be over the total utility of the future generations, or $V = \int_0^\infty U(c)\, e^{(n-\theta)t} dt$. Assuming that $n < \theta$, this is just like modifying the discount rate. If $n > \theta$ this becomes a matematical nightmare, so we'll ignore it.

which we see from (3.35). Given the nature of the production function, there is a unique value of k that solves this. So on our diagram, the $\dot{c} = 0$ locus is a vertical line at the value of k that solves this condition. Notice that if n or θ goes up, then this means that the $\dot{c}$ locus moves left (to a lower value of k). Also, if k very low, then this means $f'(k)$ is very high, $\dot{c}/c > 0$, and so consumption grows at low levels of capital. Conversely, if k is very high, the opposite condition holds, and $\dot{c}/c < 0$.

Now, let's turn to the dynamics of k. What are the points such that $\dot{k} = 0$? Go back to (3.34). This is equal to zero when $c = f(k) - nk$. This should look familiar. This is just the condition we were looking at when we looked at the golden rule level of savings in the Solow model. We know that there is some level of k that maximizes c, and this value is where $f'(k) = n$. So the $\dot{k} = 0$ locus achieves a maximum at this point. Notice that this maximum is to the right of the $\dot{c} = 0$ locus. Why? Because $n < n + \theta$, so the marginal product of capital must be smaller at the maximum of the $\dot{k} = 0$ locus, which means that it occurs at a higher level of k.

How does k react when not on the $\dot{k} = 0$ locus? If c is below this line, then c is relatively small and so capital is increasing. If c is above the line, then consumption is relatively large and capital is decreasing. This now gives us the full dynamics of the Ramsey model.

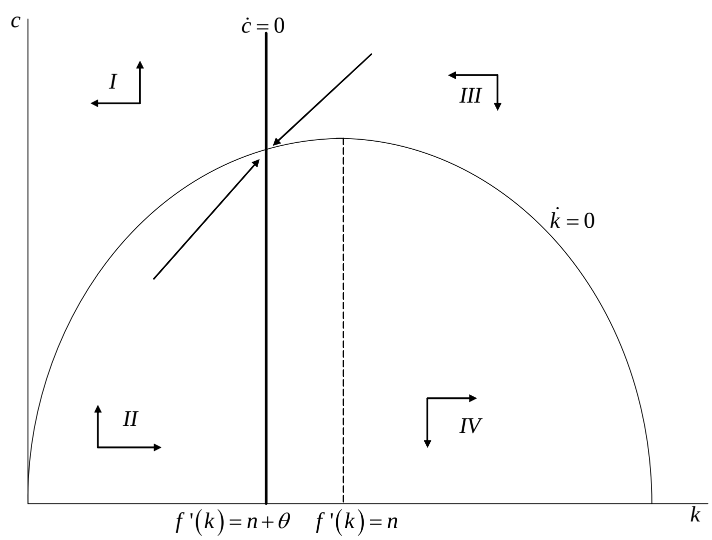

Figure 3.1:

The $\dot{c} = 0$ and $\dot{k} = 0$ loci divide the diagram into four regions for us. Each one has different implied dynamics for c and k. In area 1, c is increasing but k is decreasing. People are essentially eating up their assets and not saving enough to sustain the capital stock. So any point in area 1 is not feasible because it implies that consumption increases forever.

Area 2 has both increasing consumption and increasing capital, meaning people are saving enough to increase the capital stock and increase their consumption in the future. Points in this area are feasible to sustain, and notice that they tend to point you towards the point where $\dot{c} = 0$ and $\dot{k} = 0$ cross. Area 3 has both decreasing consumption and capital, which again is feasible and again implies some trend towards the point where the loci cross. Finally, area 4 has decreasing consumption and increasing capital, which I guess is feasible, but is certainly not optimal as you end up consuming nothing.

What is the steady state? Just the point where the $\dot{c} = 0$ and $\dot{k} = 0$ loci cross. At this point both consumption and capital are constant. So at this point, we have just enough capital that the interest rate exactly equals $n + \theta$, meaning that we want a flat consumption profile. In addition, the amount of saving we are doing is exactly offsetting the depcreciation that takes place, so we continue to have the same amount of capital. We can sustain this point forever.

What about our action away from this point? We are still optimizing subject to a budget constraint, so there are only a select set of points that will maximize our utility. It can be shown with lots of math that these points are described by stable arms - or the arrows I have drawn that point towards the steady state. If I have a (c, k) pair that is on the stable arm, then the dynamics of the system will carry me to the steady state. If I give you a value of k, you'll read up until you find the stable arm, and then choose the value of c from there and let the dynamics take over.

So we have an optimizing agent maximizing utility, why do we not have consumption right at the maximum achievable steady state? The answer is discounting. Recall in the Solow model that we could achieve the steady state by setting $f'(k) = n$. However, the agent in this problem has a discount rate as well, so the economy will not achieve the golden rule.

Finally, how does σ affect our choice? The value of sigma determines how smooth I want consumption. If sigma goes up, then $\dot{c}/c$ must be smaller at every point. The $\dot{c} = 0$ locus is the same, but the stable arms are flatter.

The Decentralized Economy

You may have noticed that I kept referring to "the agent" in the previous section. That is because I solved the model from a very specific perspective. I implicitly assumed that there was a "social planner" who was maximizing utility, taking into account how his actions change the rate of r over time. Do I get the same results if I allow for there to be multiple small households, each with the same maximization problem, who take the r as given at any point in time, as well as profit maximizing firms? The answer is yes, and we'll see quickly how this works.

So we consider each family, who have the same utility function as in (3.25), and knowing that their budget constraint is as follows

$$\dot{a} = \pi + w - c + (r - n)\, a$$

which says that their assets evolve as before, except we have an additional term, profits (π), which are added to the individuals income because they own the firms that operate. Notice that we have the family taking into account their own population growth rate of n.

Let's look now at the firms in the economy. Firms are competitive, and they rent capital on the market at the rate R and hire labor at the rate w_t. Their production function is the standard neoclassical function, $F(K, L)$. Firms maximize profits

$$\pi = F(K, L) - wL - rK$$

and maximization implies that

$$\begin{aligned} w &= F_L(K, L) \\ R &= F_K(K, L) = f'(k). \end{aligned}$$

The production function is assumed to be constant returns to scale, and given the competitive nature of the market, we will get that all output is used up in payments to the factors of production, or that profits are equal to zero. Without constant returns and competition, we'd have to deal with how the profits are handed out, and if they are equal, etc.. Zero profits make the problem easier to deal with. So $\pi = 0$.

So with profits equal to zero, the individuals problem is identical to a continuous time dynamic optimization problem. If you set up the Hamiltonian and solve it out taking r as a given you get the same conditions as you'd expect,

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(r - n - \theta).$$

Now we have the persons optimal decision, and we can plug in the values for r and w. First, the market interest rate is $r = R - \delta = f'(k) - \delta$ and $w = f(k) - k \cdot f'(k)$ (which is a result we saw in the growth chapter). Finally, how are a and k related? We assume this is a closed economy, so the total capital stock must be the same as the total stock of assets, so $a = k$. Substitute all this together and you get

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\sigma}(f'(k) - n - \theta) \\ \dot{k} &= f(k) - c - nk \end{aligned}$$

which are exactly the same dynamics as we found with the social planner. This is a very special result, it says that the market economy is just as efficient in maximizing total utility as if we had someone trying to do so directly. There are no inefficiencies from having decentralization. This won't hold for all the models we use, but it is a special feature of Ramsey-type models. It says that the optimal choices made by individuals, given r and w, are exactly the paths that make r and w follow that path in the first place. Take a while to turn that over in your head, it's not obvious, but it's an important thing to understand.

Fiscal Policy and Expectations

The Ramsey model allows us to play around with how people respond to unexpected shocks. Notice that in the models we've covered so far, the only thing that can make consumption jump discretely is uncertainty. Another way of saying this is that only new information can cause you to jump from one optimal path to a new optimal path. If we add government spending into the Ramsey model, we can see some of the subtleties of how this works.

So goverment spending, denoted G, is a per person amount of spending that the government does. We don't specify the nature of taxes or borrowing, because the Ramsey individual fulfills the conditions for Ricardian equivalence, and the financing of the spending doesn't matter. Government spending doesn't affect utility nor production, it only serves as a drain on output. The capital acccumulation equation is modified to be

$$\dot{k} = f(k) - c - G - nk$$

which means that the $\dot{k} = 0$ locus has shifted down.

Now consider that we are in a steady state and there is an increase in G. This shifts down the $\dot{k} = 0$ locus again. How do people respond? Well, we simply drop consumption by an equivalent amount and are right back at the new steady state. If the drop in G occurs outside of a steady state, we just drop down to the new stable arm and continue our path to the new steady state.

Now suppose that we are in a steady state, but that it is only announced that in the future, G would increase. What would we do? Well, we know that there cannot be an anticipated jump in consumption. So the reaction to this announcement must come immediately. We also know that at the date G actually drops, we want to be either at the new steady state, or on the stable arm approaching it. So what do we do? First, we drop our consumption, but not all the way down to the new steady state (because the change hasn't actually happened yet). What happens now? Well, the dynamics around the old steady state say that capital will increase, and then consumption will fall. This will send us to the right and down in the picture. If we timed it correctly, then on the day that G actually falls, we will find ourselves on the stable arm above the new steady state.

How about a temporary change in G? Now how do we accomodate this in our model? Well, let's say G goes up today, unannounced, but we know that it will go down again in the future. Today we lower our consumption, but not all the way down to the new steady state. Why? Because we know that we have to end up back on a path to the old steady state on the day that G goes back down. So we only drop consumption a little, and this puts us on the dynamic path of having decreasing capital stock and increasing consumption. This continues

until the day that G actually drops again, and we find ourselves on the lower stable arm for the old steady state again. Now consumption and capital both increase back to the old steady state.
In all these situations, we're trying to deal with some uncertainty by shocking a model that was built around certainty. We can get more sophisticated by adding stochastic shocks to our model instead.

The Ramsey Model in Discrete Time

So consider a model in which you want to optimize utility over an infinite lifetime

$$\max \sum_{t=0}^{\infty} \frac{U(c_t)}{(1+\theta)^t}$$

subject to the conditions that

$$\begin{aligned} k_{t+1} &= (1-\delta) k_t + f(k_t) - c_t \\ k_t &\geq 0 \end{aligned}$$

and given a value of k_0.

Rewriting the dynamic equation for capital we get that

$$\begin{aligned} k_{t+1} &= (1-\delta) k_t + w_t + k_t f'(k_t) - c_t \\ &= (1 + f'(k_t) - \delta) k_t + w_t - c_t \end{aligned}$$

or the return on a unit of capital is equal to $1 + f'(k_t) - \delta$.

We can set up a gigantic Lagrangian and take FOC, just as before. Now when we do that we're going to get FOC that look like this

$$U'(c_t) = \frac{1 + f'(k_t) - \delta}{1+\theta} U'(c_{t+1})$$

or a typical Euler equation. Rather than having $1 + r$, we have $1 + f'(k_t) - \delta$ in our Euler equation. So from the consumption side our optimal behavior is the same.

But now we have two difference equations (one for capital and one for consumption) that have to be solved together. The equations are

$$\begin{aligned} U'(c_t) &= \frac{1 + f'(k_t) - \delta}{1+\theta} U'(c_{t+1}) \\ k_{t+1} &= (1-\delta) k_t + f(k_t) - c_t \end{aligned}$$

and these are essentially the same as the differential equations we had in the continuous model. The phase diagram is identical, for all intents and purposes.

3.1.6 Stochastic Income Shocks

For stochastic income shocks we typically utilize discrete time models, because it is easier to conceive of discrete stochastic shocks than it is to think of a continually evolving random term. So we recast the model with a random shock to productivity in each period, and consider how people will respond to this. Notice that this is not quite the same as what we considered in the consumption section. There, we looked at stochastic wage income for the individual, but the interest rate remained constant. Now, total production is subject to shocks, and both the wage and interest rate are going to respond to this.

So consider a model in which you want to optimize utility over an infinite lifetime

$$\max \sum_{t=0}^{\infty} \frac{U(c_t)}{(1+\theta)^t}$$

subject to the condition that

$$k_{t+1} = A_t k_t^{\alpha} - c_t$$

which is essentially a discrete time version of the Ramsey model, and we solve it as if we were the social planner. Note that we've also implicitly assumed that depreciation is complete - or each period the capital stock is completely wiped out. The stochastic nature of income comes from the nature of A_t. It is assumed to be some random variable.

Just like the continuous time model, this set up gives us an endogenous interest rate In other words,

$$r_t = \alpha A_t k_t^{\alpha} - 1.$$

This means that r is endogenous (to our choices of k_t) and stochastic (because of A_t). Regardless, our FOC still holds regarding optimal consumption

$$U'(c_t) = \frac{1}{1+\theta} E\left\{(1+r_{t+1}) U'(c_{t+1})\right\}.$$

This says that marginal utility of consumption today must equal the discounted value of the expected value of marginal utility tomorrow, scaled by the expected interest rate. Nothing surprising here. If we consider a situation in which we have log utility, then we find that we have

$$\frac{1}{c_t} = \frac{1}{1+\theta} E\left[\frac{\alpha A_{t+1} k_{t+1}^{\alpha}}{c_{t+1}}\right] \tag{3.36}$$

which provides us with a first order difference equation in c_t that we can utilize along with our first order difference equation in k_t (the budget constraint) to solve for the steady state. They also allow us to draw a phase diagram as in the Ramsey model.

For the purposes of seeing what the path of consumption and income looks like, at this point it can be noted that with log utility, this kind of problem will end up with a straightforward answer for the path of c_t, and thus for k_{t+1} and for income. The answer (see the appendix) is that consumption will follow a rule like this

$$c_t = \left(1 - \frac{\alpha}{1+\theta}\right) A_t k_t^{\alpha}$$

which implies the following path for capital

$$k_{t+1} = \frac{\alpha}{1+\theta} A_t k_t^{\alpha} \tag{3.37}$$

and in logs this is

$$\ln k_{t+1} = \ln \frac{\alpha}{1+\theta} + \alpha \ln k_t + \ln A_t.$$

Knowing that output is defined as $\ln y_t = \ln A_t + \alpha \ln k_t$, we can substitute in the equation for capital and find that

$$\ln y_{t+1} = \alpha \ln \frac{\alpha}{1+\theta} + \alpha \ln y_t + \ln A_{t+1}$$

or output will follow an autoregressive process, and is thus serially correlated over time. The shocks to this economy persist, due to their effects on the accumulation of capital.

There is something else interesting we can consider. If we had a certain environment with $A_t = \bar{A}$ for all t, then in the steady state the interest rate would be $r = \alpha \bar{A} k_{ss}^{\alpha} - 1$. So $1 + r = \alpha \bar{A} k_{ss}^{\alpha}$. Since this is a steady state, it must be that $c_{t+1} = c_t$ as well. From (3.36) this implies that $1 + \theta = 1 + r = \alpha \bar{A} k_{ss}^{\alpha - 1}$.

What does the interest rate look like when we have stochastic productivity shocks? Let's think about this. We know that the uncertainty involved in the productivity shocks will make people want to do precautionary savings, meaning that their consumption will be lower in earlier periods than if there was certainty.

Example 17 *Let's take a look at a problem that involves stochastic income and explicity will involve precautionary saving because of this. We can't use the CRRA, because it just won't work, but we can use the CARA utility function defined as*

$$U(c) = -\frac{1}{\alpha} e^{-\alpha c}$$

which has the following derivatives

$$\begin{aligned} U' &= e^{-\alpha c} \\ U'' &= -\alpha e^{-\alpha c} \\ U''' &= \alpha^2 e^{-\alpha c} \end{aligned}$$

which shows that CARA functions have the requisite properties for risk aversion and precautionary savings.
For now we'll assume that $r = \theta = 0$ and that income follows a random walk

$$Y_t = Y_{t-1} + \varepsilon_t$$

where ε is distributed $N\left(0,\sigma^2\right)$. This means that income can become negative during the life of the individual. The problem is to

$$\max E_t\left[\sum_{t=0}^{T-1}-\frac{1}{\alpha}e^{-\alpha C_t}\right]$$

subject to

$$A_{t+1}=A_t+Y_t-C_t$$

and the stochastic process for income.
The FOC for two adjacent periods give us

$$U'\left(C_t\right)=EU'\left(C_{t+1}\right).$$

Now let's evaluate the expected value term using the rule that $E\left(e^X\right)=\exp\left(E\left(X\right)+\sigma^2/2\right)$ so that we get

$$EU'\left(C_{t+1}\right)=\exp\left[-\alpha E\left(C_{t+1}\right)+\alpha^2Var\left(C_{t+1}\right)/2\right].$$

The value of C_{t+1} is defined by the following

$$\begin{aligned}C_{t+1} &= Y_{t+1}+A_{t+1}-A_{t+2}\\ &= Y_t+\varepsilon_{t+1}+\left(A_{t+1}-A_{t+2}\right)\end{aligned}$$

so that the variance is

$$Var\left(C_{t+1}\right)=Var\left(\varepsilon_{t+1}\right)=\sigma^2.$$

Now, using our handy rule about expectations again, we can write

$$E\left(C_{t+1}\right)=C_{t+1}-\varepsilon_{t+1}$$

which we can cmobine together with the $EU'\left(C_{t+1}\right)$ expression to get

$$EU'\left(C_{t+1}\right)=\exp\left[-\alpha C_{t+1}-\alpha\varepsilon_{t+1}+\alpha^2\sigma^2/2\right].$$

Setting this back equal to $U'\left(C_t\right)$ we get

$$\exp\left[-\alpha C_t\right]=\exp\left[-\alpha C_{t+1}-\alpha\varepsilon_{t+1}+\alpha^2\sigma^2/2\right].$$

Solving this by cancelling the exponentials gives us that

$$C_{t+1}=C_t+\alpha\sigma^2/2+\varepsilon_{t+1}.$$

So consumption has not only a random element every period, but also a fixed growth term $\alpha\sigma^2/2$. This means that consumption is growing over time, and that it grows faster when there is more uncertainty. Why? Because with more uncertainty, I want to do more precautionary saving, so I lower my consumption in early periods a lot, and as my life progresses and the uncertainty gets less and less I increase my consumption (recall that in period 0 you have T periods of

uncertainty to deal with ahead of you, while in period $T-1$ you only have the final period of uncertainty to think about).
You can solve for the actual consumption in any period, which is

$$C_t = \frac{A_t}{T-t} + Y_t - \frac{(T-t-1)\,\alpha\sigma^2}{4}$$

and you can see the Appendix to see how to get this answer.
In the presence of stochastic income and precautionary savings, we get that consumption rises over time and it rises faster the higher the variance of income, but notice from our expression of C_t that the level of consumption of any specific period goes down with the variance. So what the uncertainty about income has done is to tilt our consumption path to a steeper slope, but lowered the whole path. That is, we lower our consumption a lot early on so that we can have some precautionary savings in hand to handle bad transitory shocks. This has implications for how we study aggregate consumption. If we have a large group of people, each with individual uncertainty, but on aggregate there is no uncertainty (the transitory shocks are not correlated), then we should observe rising consumption over time. Alternatively, if we compared people with uncertain incomes (construction workers or farmers) versus those with certain incomes (surgeons or professors) we should see the farmers and construction workers with faster consumption growth. We used to assume that any uncertainty washed out in the aggregate, but this model tells us that it affects behavior directly and so might be relevant at the aggregate level.

3.1.7 Fixed Technological Growth

The next thing we introduce is the prospect of productivity growth. The Ramsey model presented handled the accumulation of factors with diminishing marginal returns (capital and labor), and this resulted in a steady state that had constant consumption per person and constant capital stock per person (and hence a constant income per person. Just like in the Solow model, we need to introduce technological change in order to achieve an sustained growth in consumption or output per person.

Assume now that technology is such that we have Harrod neutral technological growth at rate g. We know from our work with the Solow model that we should expect a steady state in the variables $\tilde{c} = C/EL$ and $\tilde{k} = K/EL$. This implies that both consumption per person and output per person should be increasing in the steady state. How does this fit into the Ramsey model?

Note that all the individuals care about is consumption per person, so their optimal path will still hold that

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(r - n - \theta)$$

and the interest rate is now the return on each unit of capital per effective worker, so $r = f'\left(\tilde{k}\right)$. The path of consumption per effective worker is as follows

$$\frac{d\tilde{c}/dt}{\tilde{c}} = \frac{\dot{c}}{c} - \frac{\dot{E}}{E} = \frac{1}{\sigma}\left(f'\left(\tilde{k}\right) - n - \theta\right) - g.$$

This implies that the $\frac{d\tilde{c}/dt}{\tilde{c}} = 0$ line is at the level of capital per effective worker that solves $f'\left(\tilde{k}\right) = \sigma g + n + \theta$. Notice that the higher is g, the smaller will be the level of $\tilde{k}$ at which consumption per effective person is at a steady state. This condition basically says that the effective interest rate $\left(f'\left(\tilde{k}\right)\right)$ must equal the effective discount rate, $\sigma g + n + \theta$. The term σg is part of the effective discount rate because it picks up the diminishing marginal utility of consumption due to the growth of c at rate g. So people tend to discount the future more because they know they have a built in increase in consumption coming to them in the future. The more that people care about smoothing consumption (the higher is σ), the more they discount the future, meaning that they try to transfer more consumption into the present, smoothing out their consumption path. This means that the marginal product of capital has to remain high to ensure that people will still choose to have consumption growing at rate g, or that $\tilde{k}$ has to remain low.

For the capital per effective worker, the equation of motion is simply

$$d\tilde{k}/dt = f\left(\tilde{k}\right) - \tilde{c} - (n+g)\,\tilde{k}.$$

This gives us a similar shaped curve for the $d\tilde{k}/dt = 0$ locus. The analysis of the model is the same as in the original Ramsey model, only we are looking at the values of consumption and capital per effective worker. Once we've established the properties of these, we can back out the values of consumption and capital per person.

3.2 The Overlapping Generations Model

We're going to shift gears now to the second major model used in macro, OLG. This model essentially takes a bunch of 2 period Fisher models and staggers them. The idea is that individuals only live two periods. At any given time, two generations are alive, one old and one young. People in the young generation can work to earn income, consume some, and save. People in the old generation just consume the earnings they have from their savings. The tie between generations is that the capital stock in any period is made up of the savings of the old generation.

3.2.1 The Basic Model

Take $s_t = w_t - c_{1t}$ to be the savings of people in the young generation. The subscript on c says that this is the consumption of a young person (age 1) in period t. The consumption of an old person in period $t+1$ is $c_{2,t+1} =$

$(1 + r_{t+1})\, s_t$ where r_{t+1} is the interest rate earned by the old person, and s_t is the amount of savings the old person did when they were young (in period t).

The budget constraint is thus

$$w_t = c_{1t} + \frac{c_{2,t+1}}{1 + r_{t+1}}. \tag{3.38}$$

We'll also allow for the population to be growing in every period at the rate n. So that $L_{t+1} = (1 + n)\, L_t$. This allows us to describe the evolution of the capital stock per worker (notice this is per worker, not per person. We only care about the capital stock per young person). The capital stock per worker is equal to the savings of the people who are currently old, adjusted for the fact that there are more young people alive.

$$k_{t+1} = \frac{s_t}{(1 + n)} \tag{3.39}$$

Now we have to see how a person in this economy will optimally choose to consume. Set up their optimization as follows (the Fisher model, basically)

$$MaxV = U(c_1) + \frac{U(c_2)}{1 + \theta} \tag{3.40a}$$

$$s.t. w_t = c_{1t} + \frac{c_{2,t+1}}{1 + r_{t+1}} \tag{3.40b}$$

which you can solve by doing a Lagrangian and finding the familiar Euler equation.

For convenience we'll assume that the person has log utility, so that the optimization yields the following

$$\frac{c_2}{c_1} = \frac{1 + r}{1 + \theta} \tag{3.41}$$

and we can combine this with the budget constraint to get explicit expressions for consumption in each period.

$$c_{1t} = \frac{1 + \theta}{2 + \theta} w_t \tag{3.42}$$

$$s_t = \frac{1}{2 + \theta} w_t. \tag{3.43}$$

Note that these are not affected by the interest rate at all. This is only an artefact of the use of log utility.

Now we have to find out where r and w come from. We'll take a typical production function that says

$$f(k) = k^{\alpha} \tag{3.44}$$

so that a fraction α of output goes to capital and $(1 - \alpha)$ goes to wages. Therefore

$$w_t = (1 - \alpha)\, k_t^{\alpha} \tag{3.45}$$

$$r_t = \alpha k^{\alpha - 1} - \delta \tag{3.46}$$

where I've included a term to account for depreciation of capital each period.

Now, take the expression for the capital stock from (3.39) and use the expression for savings in (3.43), along with wages from (3.45) and you get that

$$k_{t+1} = \frac{1}{(1+n)}\left(\frac{1}{2+\theta}\right)(1-\alpha)\,k_t^{\alpha} \tag{3.47}$$

which is a simple difference equation in k. You can draw this out by mapping k_{t+1} against k_t. Draw in a 45 degree line as well, and where this curve crosses the 45 degree line is the steady state. The steady state is stable, meaning that if you have less capital than the steady state, you will grow capital until you reach it - and vice versa.

Mathematically, you can solve for the steady state by setting $k_{t+1} = k_t = k_{ss}$ and you get

$$k_{ss} = \left(\frac{(1-\alpha)}{(1+n)(2+\theta)}\right)^{1/(1-\alpha)} \tag{3.48}$$

and we can then solves for the interest rate and wage rate as well.

$$r_{ss} = \frac{\alpha}{1-\alpha}(1+n)(2+\theta) - \delta \tag{3.49}$$

$$w_{ss} = \left(\frac{(1-\alpha)^{1/\alpha}}{(1+n)(2+\theta)}\right)^{\alpha/(1-\alpha)} \tag{3.50}$$

Let's take a look at the golden rule level of capital per person. Recall from the Ramsey model that we didn't quite achieve this level, because of discounting. What is the result here? The golden rule says that $f'(k) = n + \delta$. Or alternatively, at the golden rule we should have that $r = n$. Clearly, from (3.49), this doesn't necessarily have to hold. The discount rate has to be a very specific value in order for this to hold. If $r > n$, then the capital stock must be less than the golden rule and if $r < n$, then the capital stock must be higher than the golden rule. In either case, consumption could be higher for every period and every generation of people if they could adjust the steady state of capital.

Clearly, being above the golden rule level of capital is really bad. Why? Because if everyone just saved less, consumption would go up immediately in every period following that one. There would be a Pareto improving rearrangement. Think about this, if we are above the golden rule, then $r < n$. So if we just implemented the following strategy, we could improve things for everyone. Have each young person take one dollar that they were going to save, and instead have them hand it over to the old generation. Because of the population growth, this means each old person would get $1+n$ extra units of consumption. This is clearly better than getting only $1+r$ units of consumption they could have gotten by investing.

The fact that the world could have a Pareto improving rearrangement of assets means that the economy is dynamically inefficient. This didn't happen in the Ramsey model because in that model, the decentralized result was identical to the result obtained when we assumed there was a single optimizing social planner in the world. In the OLG model, no one person is present across all the periods that the economy exists (people only live for 2 of an infinite number of periods), so no one has the ability to carry out this Pareto improving trade.

Example 18 *Let's consider an extension of the model to include social security. That is, the government is going to take some money from the younger generation and give it to the older generation. So each young person is taxed in the amount d. Each old person gets the amount $(1+n)\,d$, which holds because there are more young people than there are old people.*
What does the budget constraint look like now?

$$\begin{aligned} s_t &= w_t - c_{1t} - d \\ c_{2,t+1} &= (1+r_{t+1})\, s_t + (1+n)\, d \end{aligned}$$

which gives a lifetime budget constraint of

$$w_t + \frac{n-r}{1+r} d = c_{1t} + \frac{c_{2,t+1}}{1+r}.$$

If $n = r$, then social security has no effect on your budget constraint, as you can exactly offset its effects by borrowing or lending to get your desired consumption path. If $n > r$, then this expands the amount that a consumer can consume over their lifetime. If $r < n$, then this actually shrinks the consumption a person can do in their life, and raises the question of why we'd have social security in the first place. But there are other considerations (that people aren't forward looking, etc..) at play in the politics of social security. One big political consideration is that the first generation to receive it didn't have to pay the tax as youngsters, so they were happy to vote in the policy that lowered utility for every successive generation while raising their own utility.
If we use our new budget constraint and do the Lagrangian we get optimal consumption in period one of

$$c_{1t} = \frac{1+\theta}{2+\theta}\left[w_t + \frac{n - r_{t+1}}{1+r_{t+1}} d\right]$$

and savings of

$$s_t = \left(\frac{1}{2+\theta}\right)\left[w_t + \frac{(n - r_{t+1})(1+\theta)}{1+r_{t+1}} d\right] - d.$$

Combine the definitions of the capital stock, wages, and the interest rate and do a lot of algebra and you get

$$k_{t+1} = \frac{(1-\alpha)}{(1+n)(2+\theta)} k_t^{\alpha} - \frac{d}{1+n}\left[\frac{1+\theta}{2+\theta}\left[\frac{n+\delta-\alpha k_t^{\alpha-1}}{1+\alpha k_t^{\alpha-1}-\delta}\right]+1\right]$$

which looks like the old equation for the capital stock evolution in (3.47) *but has this extra term hanging on the end. It can be shown (see the Appendix) that* *this describes a relationship between k_{t+1} and k_t that lies below the non-social security relationship. Thus social security lowers the steady state level of capital in the economy. Now if we were oversaving before, then this could be a good thing. If we were undersaving, then probably not.*

3.2.2 Fixed Technological Growth

What happens if there is technological progress in an OLG economy? If we have our typical Harrod neutral technological change, then our production function is

$$f\left(\tilde{k}\right) = \tilde{k}^{\alpha}$$

and the associated interest rate and wages (per person) are

$$\begin{aligned} r_t &= \alpha \tilde{k}_t^{\alpha-1} - \delta \\ w_t &= E_t \left[(1-\alpha) k_t^{\alpha}\right]. \end{aligned}$$

The evoluation of capital per effective worker is

$$\tilde{k}_{t+1} = \frac{\tilde{s}_t}{(1+n)(1+g)}$$

and since savings will still just depend on wages as in (3.43), we get

$$\begin{aligned} \tilde{s}_t &= \frac{s_t}{E_t} = \frac{1}{2+\theta} w_t \\ &= \frac{1}{2+\theta}\left[(1-\alpha)\tilde{k}_t^{\alpha}\right]. \end{aligned}$$

This means that capital per effective worker evolves at

$$\tilde{k}_{t+1} = \frac{1}{2+\theta} \frac{\left[(1-\alpha)\tilde{k}_t^{\alpha}\right]}{(1+n)(1+g)}$$

and essentially looks identical to before except with the extra $(1+g)$ term floating around. So we get the similar result from the Solow model that as g goes up, the actual steady state value of $\tilde{k}_{ss}$ will fall. Income, on the other hand, is increasing in g. Income per person is

$$\begin{aligned} y_t &= E_t \tilde{y}_t = E_t \tilde{k}_{ss}^{\alpha} \\ &= E_t \left[\frac{(1-\alpha)}{(2+\theta)(1+n)(1+g)}\right]^{\alpha/1-\alpha} \end{aligned}$$

where I've plugged in the steady state value of capital per effective worker. Notice that y_t goes up every period because E_t goes up every period. So in the OLG model we get a very similar result to the Solow model.

3.2.3 Stochastic Income Shocks

Now we take the OLG model and consider how it responds to the presence of stochastic output shocks. Output is now assumed to Cobb-Douglas with a productivity term that is subject to random shocks, or

$$y_t = A_t k_t^{\alpha} \tag{3.51}$$

where the properties of A_t will be discussed later. Each individual is assumed to have log utility, and therefore is looking to maximize the following

$$V = \ln c_1 + \frac{E \ln c_2}{1+\theta} \tag{3.52}$$

where the E operator is the expectations term (not productivity as in prior examples). The budget constraint is typical with

$$c_2 = (1+r_t)(w_t - c_1). \tag{3.53}$$

The solution to the optimization gives us that

$$s_t = \frac{w_t}{2+\theta} \tag{3.54}$$

and by using the fact that wages are $w_t = (1-\alpha) A_t k_t^{\alpha}$ and that $k_{t+1} = s_t / (1+n)$ we get the evolution of the capital stock as

$$k_{t+1} = \frac{(1-\alpha) A_t k_t^{\alpha}}{(2+\theta)(1+n)}. \tag{3.55}$$

This is the same form as without income shocks, but notice that it doesn't necessarily let us solve directly for a steady state, because the A_t term is stochastic. So the actual evolution of capital from period t to period $t+1$ changes based on the shock received to productivity. At any given period, people act like they are heading towards a steady state in the model, but the steady state they are heading to keeps jumping around. To see the implications of the stochastic income further, consider a log linear version of (3.55)

$$\ln k_{t+1} = \ln B + \alpha \ln k_t + \ln A_t \tag{3.56}$$

where $B = (1-\alpha) / (1+n)(2+\theta)$. Now take logs of (3.51) and replace it in (3.56) and you get the following

$$\ln y_{t+1} = \alpha \ln B + \alpha \ln y_t + \ln A_{t+1}. \tag{3.57}$$

This is a first order difference equation for income, saying that income per person is autoregressive with a parameter equal to the share on capital in the economy. This is showing us that shocks to output have lingering effects on output in the future. How? By affecting the capital stock, the shocks affect the wage, which affects the savings done, and in turn affects the capital stock in the next period.

So the OLG with random shocks gives us an implication that output can follow an autoregressive pattern, implying serial correlation. (Note that if we extend the periods of the OLG beyond 2, we'll get more periods of propogation for the shocks).

This is based on the idea that $\ln A_{t+1}$ is a white noise error term. If it has some kind of trend to it, as in

$$\ln A_{t+1} = g + \ln A_t + \varepsilon_t \tag{3.58}$$

which says that productivity has a trend g and also has a unit root, meaning that productivity shocks have a permanent effect on output. In this case productivity is a random walk with drift. This gives us the following for output

$$\Delta \ln y_{t+1} = g + \alpha \Delta \ln y_t + \varepsilon_t \tag{3.59}$$

or that income *growth* is autoregressive and serially correlated, not the level. A shock to output growth lingers through the following periods, until eventually income growth settles back down to the rate g.

Chapter 4

Open Economy Macroeconomics

We now turn to a particular application of the models and intution so far, the open economy. To summarize how this will work, just imagine that a country is composed of a large number of identical individuals. With each of them solving an identical consumption/savings problem, the amount of savings per person in the economy is simply the same as the amount of savings derived from any individuals problem. Let's see how this builds up our intution about how the macroeconomics of an open economy work.

4.1 Open Economy Accounting

The first this we need to do is cover the preliminaries of national income accounts, so that we keep everything straight. Recall that the income identity for a country is

$$Y = C + I + G + NX \tag{4.1}$$

but this leaves us with a question. Is Y GDP or GNP? The answer is that it can be either one, as long as we define imports and exports correctly. For our purposes in this course, the distinction won't matter that much. We'll be thinking about capital flows, and we'll be thinking about them in terms only of debt (which we'll denote the net value of as B_t) and not about portfolio investment or foreign direct investment (FDI). With only debt, there is no foreign ownership of assets, and so GDP and GNP are equal.

Recall that total savings is

$$S = I + NX \tag{4.2}$$

in an open economy.

The current account is the change in foreign assets in an economy, so it is equal to NX plus interest on the assets we hold abroad, minus interest on the debt that we own to foreigners. So in discrete time

$$CA_t = B_{t+1} - B_t = rB_t + NX \tag{4.3}$$

where the r on borrowing and lending is presumed to be the same. In continuous time this would be

$$CA = \dot{B} = rB + NX. \tag{4.4}$$

Now, we need to be specific about what exactly we mean by openness. We'll only be considering openness to capital flows here, and won't be thinking about the nature of openness to goods trade (even though the economy can trade). So we define the following:

Definition 19 *A* ***closed economy*** *has $NX = 0$ and r is endogenous.*

Definition 20 *A* ***small open economy*** *has r is exogenous and fixed at some world rate r^*; NX is endogenous.*

Definition 21 *A* ***large open economy*** *is large enough to affect r^*, so r^* is endogenous. Also, the level of NX is then $NX = -NX^*$ or the trade balance for this large country is exactly offset by the trade balance of the rest of the world.*

4.2 Optimal Savings and the Interest Rate

For simplicity, we start again with a two period model of a country with defined levels of GDP in each period, Y_1 and Y_2. We assume everyone in the economy is identical, so that the solution to any one persons two-period optimization problem is the same as the solution to the country's optimization problem. So we can solve the optimization for this country as

$$\max V = U(C_1) + \frac{U(C_2)}{1+\theta}$$

subject to

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}.$$

This is just identical to our Fisher model problem, so the answer is for the country to consume such that

$$\frac{U'(C_1)}{U'(C_2)} = \frac{1+r}{1+\theta}.$$

How does the nature of openness affect the consumption decision? Let's take each in turn

1. **Closed economy**. In this case, what happens? Well, there is no ability for the country as a whole to borrow or lend, so the country must be constrained to consuming exactly $C_1 = Y_1$ and $C_2 = Y_2$. But within the country we have a large number of optimizing individuals, how do we ensure their aggregate choices for C_1 and C_2 don't violate this rule? First, recall that everyone is identical. So everyone will have the same answer to any optimization program. Second, what r could we set such that everyone chose to exactly consume so that $Y_1 = C_1$ and $Y_2 = C_2$? Well, that would the r that solves the following equation

$$\frac{U'(Y_1)}{U'(Y_2)} = \frac{1+r}{1+\theta}.$$

So even without any net borrowing or lending going on, there is still an interest rate. This interest rate is endogenous, and clears the loanable funds market so that people choose to consume exactly Y_1 and Y_2.

2. **Small open economy**. This is a little easier. We take r^* as given, and solve the Euler equation for the pattern of C_1 and C_2. Compare C_1 to Y_1 to find the savings of the individuals. If savings is negative, then the country has $NX < 0$ (to see this note that if $S = Y - C < 0$, and I and G are positive, then it must be that NX is negative). If $NX_1 < 0$, then with $B_1 = 0$, it must be that $CA_1 < 0$, or you run a current account deficit in the first period. In the second period, what happens? With $CA_1 < 0$, that means that $B_2 < 0$, or individuals are in debt to foreigners. In period 2, then, it must be that $CA_2 > 0$, as $B_3 - B_2 > 0$ given that $B_3 = 0$ by assumption. What does this imply about NX? It must be positive.

3. **Large open economy**. This is easiest to think of when you have 2 countries. You have to have both countries solving their Euler equations, but with a common r^*. If everyone in both countries is exactly identical, then there is zero savings in both. However, if they differ (which they can usually only through θ or the pattern of income growth) then there may be an r^* that makes the net savings across both countries equal to zero. That is, the borrowing of one country is offset by the lending of another. The pattern of the current account follows the small open economy pattern for the borrowing country, and is reversed for the lending country.

The upshot of all this is that the value of the autarky interest rate (that which solves the Euler equation with $C_1 = Y_1$ and $C_2 = Y_2$) relative to the world interest rate determines whether you run a current account deficit or surplus in any given period.

4.3 Growth in an Open Economy

4.3.1 Capital versus Assets

We'll set aside the optimizing consumers for a while, and focus instead on how openness to capital flows fits within the framework of a growing economy. We start with the standard Solow model. With an open capital account, capital must flow into or out of the country until the marginal product of capital (net of depreciation) equals the world interest rate, or

$$r^* = f'(k) - \delta. \tag{4.5}$$

Now we have the situation where some of the actual assets of the economy may be owned by foreigners, so we need to distinguish between GDP and GNP. Let k represent the quantity of capital operating in the economy and let a represent the quantity of assets owned by the residents of that country. Net foreign assets are

$$b_t = a_t - k_t.$$

Now GDP per person is

$$GDP/L = f(k)$$

while GNP per person is

$$GNP/L = (r^* + \delta)\, a + w$$

where the term $(r + \delta)$ is multiplied by a because we want to calculate *gross* national product, not net national product.

What is the wage rate? That is just what remains from GDP after payments to capital, or

$$w = f(k) - kf'(k) = f(k) - k(r^* + \delta).$$

If we have Cobb-Douglas production, then $f(k) = k^{\alpha}$ and we can solve for the equilibrium wage rate rate and level of capital per person by using (4.5). This gives us that

$$k = \left(\frac{\alpha}{r^* + \delta}\right)^{1/1-\alpha}$$

and wages are then equal to

$$w = (1-\alpha)\left(\frac{\alpha}{r^* + \delta}\right)^{\alpha/1-\alpha}.$$

Notice that both k and w are decreasing in the world interest rate. The more expensive capital is to rent, the less capital is rented and the lower wages are. Now, with capital openness the capital stock will jump to exactly the above level immediately, and stay constant thereafter. Wages will then be constant too, as we have no economic growth yet. But the stock of assets in the economy will be changing over time. Their evolution is described by

$$\dot{a} = s\left((r^* + \delta)\, a + w\right) - (n + \delta)\, a$$

which says that the change in domestic assets is equal to the level of GNP per person times the savings rate, minus a depreciation of assets per person over time based on population growth and actual depreciation. You can solve this for the steady state of assets per person by plugging in for wages and setting $\dot{a} = 0$. This gives you

$$a_{ss} = \frac{s(1-\alpha)}{n+\delta - s(r^* + \delta)} \left(\frac{\alpha}{r^* + \delta} \right)^{\alpha/1-\alpha}$$

which assumes that $n + \delta > s(r^* + \delta)$ or that the economy cannot accumulate assets without bound.

With this in hand we can think about the rest of the world too. Let k_w be the capital stock of the rest of the world. The rest of the world is large, so does not take the interest rate as exogenous, but rather this is determined by the world following the Solow model where the rest of the world saves at rate r_w. The steady state of the world's capital stock is

$$k_w = \left(\frac{s_w}{n+\delta} \right)^{1/1-\alpha}$$

where I'm assuming the world has the same n and δ as the home country. This implies that

$$r^* = \alpha \left(\frac{n+\delta}{s_w} \right) - \delta.$$

Given this information, we can solve for the actual ratio of a to k in our home country. Since $k_w = k$ because both the home country and the world have identical production funtions (and therefore if both have to have $MPK = r^* + \delta$ then they must have the same k level), this means that the ratio a/k is the same as the ratio of a/k_w. After a lot of algebra you get

$$\frac{a_{ss}}{k} = \frac{1-\alpha}{\frac{s_w}{s} - \alpha}.$$

So the relative savings rates of the home and rest of world determine how our domestic assets stand in relationship to our domestically installed capital stock per worker. If $s > s_w$ then $a > k$ and we own more assets than are installed in our domestic country - we are net exporters of capital and are running a trade surplus. And vice versa. The ratio of GNP to GDP is then given by

$$\frac{GNP}{GDP} = (1-\alpha) + \alpha \frac{a_{ss}}{k}.$$

Does this model of capital flows seem to work? Not really, as there are lots of puzzles that it leaves unanswered. For instance, Feldstein and Horioka showed that savings and investment are very highly correlated across countries, while if capital is flowing freely across borders than they should not be so correlated as investment should be similar across countries. The Lucas paradox says that the implied return on capital in developing countries is enourmous, and asks why capital doesn't flow to those poor countries to chase the high returns (although some recent works have effectively shot down a lot of that question by showing that if one actually calculates the MPK correctly then there aren't big discrepancies in MPK across countries).

4.3.2 Non-tradable Human Capital

How does this model change if we allow for human capital, which is nearly untradable across borders (the level of international migration is tiny compared to actual populations). So the model will now have two types of capital, but one can jump instantly to its steady state, while another must evolve as before. Specifically, let's let production be

$$y = k^{\alpha} h^{\beta}$$

and we know that

$$f'(k) - \delta = r^{*}.$$

This allows us to solve for the level of capital as a function of human capital, or

$$\begin{aligned} \alpha k^{\alpha-1} h^{\beta} - \delta &= r^{*} \\ k &= \left(\frac{\alpha}{r^{*}+\delta}\right)^{1/1-\alpha} h^{\beta/1-\alpha} \end{aligned}$$

We can solve for y as a function of just h then, and get

$$y = \left(\frac{\alpha}{r^{*}+\delta}\right)^{\alpha/1-\alpha} h^{\beta/1-\alpha}.$$

So how exactly does human capital accumulate? We might think that h is generated by savings out of GNP, but y is GDP. So which is the appropriate measure? One could make an argument for either. We'll assume that h is financed out of savings from GDP, or domestic output. So we get that

$$\dot{h} = s_h y - (n+\delta) h$$

or

$$\frac{\dot{h}}{h} = s_h \left(\frac{\alpha}{r^{*}+\delta}\right)^{\alpha/1-\alpha} h^{(\beta+\alpha-1)/1-\alpha} - (n+\delta)$$

which can be solved for a steady state value of

$$h_{ss} = \left(\frac{s_h \left(\frac{\alpha}{r^{*}+\delta}\right)^{\alpha/1-\alpha}}{n+\delta}\right)^{(1-\alpha)/1-\beta-\alpha}$$

from which can see that human capital is positively affected by the savings rate, as expected, but negatively affected by the world interest rate. Why? Because with a higher r^*, there will be less k, and that lowers total output, leading to lower overall savings and hence lower human capital per person. So this model predicts that human capital is dependent on the level of the world interest rate. Note, though, that h could really represent any domestic capital that is non-tradable (or immobile). This might include things like knowledge capital that is location specific.

Now, in the model with only one tradable capital good, we saw that GDP was unaffected by the domestic savings rate. Now what is the case? If we plug h_{ss} into the production function, you can see that

$$y = \left(\frac{\alpha}{r^* + \delta}\right)^{\alpha/1-\alpha-\beta} \left(\frac{s_h}{n+\delta}\right)^{\beta/1-\beta-\alpha}$$

abd the level of GDP depends on the human capital savings rate. This isn't too surprising, since the how stock of human capital has to be domestically financed. So this model gives us something that sits between a fully closed version of the Solow model and a fully open version of the Solow model.

4.4 Optimal Consumption and Growth in an Open Economy

We'll put these two views of the open economy together now. Start with the consumption side, and again we have that the current account is defined as

$$CA_t = B_{t+1} - B_t = rB_t + NX_t. \tag{4.6}$$

Now this says that to hold foreign debt constant (i.e. have a zero current account) we have to set our trade balance equal to the interest on our debt.

We can write the difference equation for the evoulation of B_t from (4.6), much like we'd solve for the budget constraint in a typical infinite horizon model. This gives us

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(C_s + I_s) = (1+r)B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - G_s) \tag{4.7}$$

which says that your total present discounted value of spending on consumption and investment has to equal the present discounted value of your income (minus what the government takes) and initial assets (B_t). This can be rearranged to show that

$$\begin{aligned} -(1+r)B_t &= \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - C_s - I_s - G_s) \\ &= \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} NX_s. \end{aligned}$$

This says that the present discounted value of the future trade balance has to equal the negative of the current net foreign assets. So if you have current foreign debt, meaning that $B_t < 0$, then you must have a PDV of your trade balances that is positive. In other words, you have to run some kind of trade surpluses eventually to pay off that debt.

Assume for the moment that $\theta = r$, then what is your optimal consumption path? Constant over time, or $C_t = C_{t+1}$. Knowing this, you can plug into the above equation and solve for first period consumption or

$$C_t = \frac{r}{1+r}\left[(1+r)\,B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - G_s - I_s)\right]$$

which shouldn't be surprising given our knowledge of optimal consumption. This just says you should consume a constant fraction of your wealth. Now suppose that $r \neq \theta$, and we have CRRA utility. Then we know that $C_{t+1} = C_t\left((1+r)\,/\,(1+\theta)\right)^{1/\sigma}$ and you can solve again for C_t using the budget constraint to find that

$$C_t = \left[\frac{r+\gamma}{1+r}\right]\left[(1+r)\,B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - G_s - I_s)\right]$$

where $\gamma = 1 - \left((1+r)\,/\,(1+\theta)\right)^{1/\sigma}$. Again, this is something familiar from our work on typical consumption models.

A quick aside for some notation. For any given variable X that has a path over time, we can define $\bar{X}$ as the constant level of X that has the same present discounted value as the actual path of X. In math

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}X_s = \bar{X}_t\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} = \frac{1+r}{r}\bar{X}_t$$

This allows us to write

$$C_t = \left[\frac{r+\gamma}{1+r}\right]\left[(1+r)\,B_t + \frac{1+r}{r}\left(\bar{Y}_t - \bar{G}_t - \bar{I}_t\right)\right] \qquad (4.8)$$

or that consumption today is just a function of the long run values of output, government, and investment, given some initial value of foreign debt. If $\theta = r$ this reduces to $C_t = rB_t + \bar{Y} - \bar{G} - \bar{I}$. Therefore NX_t must equal $-rB_t$. This is just a special case where the current trade balance must exactly offset the current payments on foreign debt holdings.

We can use the definition of C_t to plug into the CA_t equation to find that

$$CA_t = rB_t + Y_t - G_t - I_t - \left[\frac{r+\gamma}{1+r}\right]\left[(1+r)\,B_t + \frac{1+r}{r}\left(\bar{Y}_t - \bar{G}_t - \bar{I}_t\right)\right]$$

which seems like a big mess. But if we again assume that $\theta = r$ this reduces nicely to

$$CA_t = \left(Y_t - \bar{Y}_t\right) - \left(I_t - \bar{I}_t\right) - \left(G_t - \bar{G}_t\right)$$

and this tells us that the current account is just equal to the deviations of output, government and investment from their long run values. The current account is therefore only a way to temporarily smooth consumption when we have periods of shocks to our long run path. This comes our modelling the economy as similar to an individual who uses the financial markets to smooth their consumption. If $r \neq \theta$, then this is modified slightly, but the intuition remains that the current account is a smoothing mechanism.

Now we can consider adding economic growth into this mix. Define output more specifically as

$$\begin{aligned} Y_t &= A_t K_t^{\alpha} \\ A_{t+1} &= (1+g)^{1-\alpha} A_t \end{aligned}$$

so that in the steady state, output and capital per worker will grow at rate g. The adjustment $1-\alpha$ on the growth of A just makes sure this works out.

This is an open economy, so the interest rate will always be equal to the world interest rate, r^*. We will always assume that $r^* > g$, otherwise people would be able to grow completely out of debt. This pins down the marginal product of capital (assume no depreciation) and that pins down the capital stock.

$$r^* = \alpha A_t K_t^{\alpha-1} \tag{4.9}$$

$$K_t = \left(\frac{\alpha A_t}{r^*}\right)^{1/1-\alpha} \tag{4.10}$$

Since capital will grow at rate g (given the above equation), then investment must be equal to the following

$$I_t = K_{t+1} - K_t = g\left(\frac{\alpha A_t}{r^*}\right)^{1/1-\alpha}.$$

Substitute the equation for capital in (4.10) into production and you get

$$Y_t = A_t^{1/1-\alpha}\left(\frac{\alpha}{r^*}\right)^{\alpha/1-\alpha}$$

and using this we can write investment as a proportion of output as

$$I_t = \left(\frac{\alpha g}{r^*}\right) Y_t.$$

We'll just assume now that government spending is a constant fraction of total output, β (and that $\beta < 1 - \frac{\alpha g}{r^*}$. So output minus investment and government spending is

$$Y_s - I_s - G_s = (1+g)^{s-t}\left(1 - \frac{\alpha g}{r^*} - \beta\right) Y_t$$

which says that output minus investment and government at time s is a function of output at time t. Consumption at time t can be solved for now, using our expression from (4.8). This gives

$$C_t = \left[\frac{r^* + \gamma}{1 + r^*}\right]\left[(1 + r^*) B_t + \left(1 - \frac{\alpha g}{r^*} - \beta\right) Y_t \sum_{s=t}^{\infty} \left(\frac{(1 + g)}{(1 + r^*)}\right)^{s-t}\right]$$

and this can be evaluated to see that

$$\begin{aligned} C_t &= \left[\frac{r^* + \gamma}{1 + r^*}\right]\left[(1 + r^*) B_t + \frac{1 + r^*}{r^* - g}\left(1 - \frac{\alpha g}{r^*} - \beta\right) Y_t\right] \\ &= (r^* + \gamma) B_t + \frac{r^* + \gamma}{r^* - g}\left(1 - \frac{\alpha g}{r^*} - \beta\right) Y_t \end{aligned}$$

This and the definitions of investment and outpu will allow us to specify the current account now, which gives us

$$CA_t = -\gamma B_t - \frac{g + \gamma}{r^* - g}\left(1 - \frac{\alpha g}{r^*} - \beta\right) Y_t.$$

So the current account again depends on the state of foreign debt (B_t) and on the current level of income. To see more clearly how the economy responds to differences in g or r^*, it will be useful to consider the ratio of foreign debt to income, rather than the current account specifically. So

$$\begin{aligned} B_{s+1} &= B_s + CA_s \\ Y_{s+1} &= (1 + g) Y_s \end{aligned}$$

and their ratio is then

$$\frac{B_{s+1}}{Y_{s+1}} = \frac{1 - \gamma}{1 + g}\frac{B_s}{Y_s} - \frac{g + \gamma}{r^* - g}\left(1 - \frac{\alpha g}{r^*} - \beta\right)\frac{1}{1 + g}.$$

This perhaps doesn't look like it is helping. But notice that we have a difference equation in B/Y that is linear. What is the steady state?

$$\frac{B_s}{Y_s} = -\frac{1 - \frac{\alpha g}{r^*} - \beta}{r^* - g}$$

which you'll notice is always negative (recall that $r^* > g$ by assumption always). So if you're at this steady state, you've got a permanent level of foreign debt. However, will you end up at this steady state? Consider first the case when $g > -\gamma$, and this means that the slope of the difference equation is < 1 and the intercept is negative. In this case the steady state is stable, meaning that whatever level of B/Y I start with, I'll end up at this steady state with foreign debt. What is happening? The growth rate of output is greater than the growth rate of consumption (recall that γ measures the inverse of the growth rate of consumption). The growth rate of consumption is low because the country has high initial consumption, which is funded by borrowing from foreigners. Eventually, the country spends all of its income on paying back this debt. In fact the present value of consumption is exactly equal to the present value of the foreign debt.

Now, consider when $g < -\gamma$ or consumption growth is higher than economic growth. In this case the steady state is unstable (the intercept is greater than zero and the slope is greater than one). If the country starts with a small level of foreign debt, then it can pay this back quite easily given that it has low initial consumption and high consumption growth. The country pays off it's debt and starts to actually acquire positive foreign assets - lending money to other countries.

Thus the nature of foreign debt relative to output depends primarily on the preference parameters for consumption, and also on the exogenous growth rate.

Chapter 5

Endogenous Growth Models

In this chapter we return to the topic of economic growth, but we want to think again about where this growth comes from. In the Solow model, sustained economic growth (meaning increases in income per person) are only possible through purely exogenous growth - g. This has two primary problems, the first theoretical and the other empirical.

- Theoretical: We haven't actually explained anything if we just assert that growth is exogenous. Why is $g = 0.02$ in the U.S. and not higher or lower in the long run? Why is g low in Nigeria and high in South Korea? The Solow model, and models with exogenous technological progress in general, have the unappealling feature that some of the most important features of economies are actually not explained. So we want to solve that by trying to write down models that actually generate the growth rate of output per person internally, and ideally as the result of some optimizing decision by individuals.

- Empirical: The Solow model does have growth in it, only out of the steady state. If different countries were simply different distances away from their steady states, then differences in growth rates would be explained by this distance, and not by g. When we examine the data, though, this doesn't appear to be the case. Additionally, if we do some simple accounting for the sources of growth within countries or across countries, we see that the total factor productivity is the most important explanation for growth and development. So the Solow model, which is a model of capital accumulation, doesn't actually address the most important component of economic growth.

So we're going to examine this empirical evidence in a little more detail first, covering the convergence of countries to their steady states as well as development and growth accounting. This will show us that we need something more than just the Solow model. This will lead us to think about endogenous growth models, starting with mechanical ones without optimizing agents, and then considering models of endogenous technology creation and population growth.

5.1 Empirics of Economic Development

5.1.1 Growth and Development Accounting

The Solow model is a model of capital accumulation. But how important is capital exactly for growth. Alternately, how important is capital in explaining the differences in income across countries? A simple way to evaluate this is to perform what is called growth accounting (if we look at a country over time) or development accounting (if we compare countries). This is an a-theoretical exercise - all it requires is a production function. So take your typical production function of the form

$$Y = AK^{\alpha}L^{1-\alpha} \tag{5.1}$$

and in intensive form we get

$$y = Ak^{\alpha}. \tag{5.2}$$

Take logs and a time derivative and we get

$$\frac{\dot{y}}{y} = \frac{\dot{A}}{A} + \alpha\frac{\dot{k}}{k} \tag{5.3}$$

and this tells us that growth in income per capita is a combination of growth in TFP and growth in capital per worker. We have data on output per worker by country, and we have data on capital stocks by country. So we can solve for $\dot{A}/A$ as a residual of this equation (this is often why you'll hear me refer to A as the residual). Using data from 1960-1998, and assuming that $\alpha = 0.3$ (reasonable given the data we do have) we can do this for several countries and get

Country	$\dot{y}/y$	$\dot{K}/K$	$\dot{L}/L$	$\alpha\dot{k}/k$	$\dot{A}/A$
Brazil	0.028	0.054	0.022	0.010	0.015
China	0.041	0.068	0.016	0.016	0.020
India	0.026	0.055	0.021	0.010	0.012
Mexico	0.018	0.050	0.025	0.008	0.008
Nigeria	−0.001	0.003	0.028	−0.008	0.009
United States	0.024	0.038	0.011	0.008	0.013

and we see that growth in A is at least as important as growth in capital when it comes to explaining growth in income per capita. In most cases, its more important. If you look across a larger cross-section of countries you'll see that growth in TFP accounts for between 50-60% of growth in income per capita. So the Solow model, even if we didn't have theoretical issues with it, is only getting us halfway to an explanation, at best.

Alternatively, we can look at how countries relate to each other, and why country i is rich or poor relative to country j. Simply think about the ratio of their production functions

$$\frac{y_i}{y_j} = \frac{A_i}{A_j}\left(\frac{k_i}{k_j}\right)^{\alpha} \tag{5.4}$$

and we can decompose the relative incomes of i and j into two components: the ratio of TFP and the ratio of capital per person. Using data from 1998, we get the following when we compare each of these countries to the U.S.

Country	y_i/y_{US}	$(k_i/k_{US})^{\alpha}$	A_i/A_{US}
Brazil	0.225	0.653	0.399
China	0.104	0.479	0.276
India	0.078	0.381	0.282
Mexico	0.255	0.681	0.426
Nigeria	0.032	0.272	0.180
United States	1.000	1.000	1.000

and you can see that each country is proportionally much closer to the U.S. in terms of capital per worker than they are in TFP. So the source of the gap between countries is mainly a TFP issue, and capital accumulation looks less important. Now, we unfortunately don't actually know what TFP is, precisely. It's simply what's left over after we account for what we know of (capital, human capital, etc..). So at this point we're still scrambling around trying to be more specific about where these A_i/A_{US} differences come from. One of the goals of the endogenous technology models is to provide some explanation for differences that we see in TFP.

5.1.2 Growth Rates and Convergence

Another empirical question that we might look at is whether countries are converging to similar income levels. That is, are poor countries growing faster than rich ones? If they are, then they'll eventually be as rich as the U.S. or France. If not, then the gap in income levels will be sustained. Let's go back and think about the sources of growth in income per capita. We'll start with a production function which is written for convienence as

$$Y = K^{\alpha}(AL)^{1-\alpha} \tag{5.5}$$

so that we can write growth in income per capita as

$$\frac{\dot{y}}{y} = (1-\alpha)\frac{\dot{A}}{A} + \alpha\frac{\dot{k}}{k} \tag{5.6}$$

$$= \frac{\dot{A}}{A} + \alpha\frac{d\tilde{k}/dt}{\tilde{k}} \tag{5.7}$$

where the second line just follows from writing $\tilde{k} = K/AL$. If we now allow for some theory, as in the Solow model, we can be more specific about this. In the Solow model, $\dot{A}/A = g$, and the growth rate of efficiency units of capital is

$$\frac{d\tilde{k}/dt}{\tilde{k}} = s\tilde{k}^{\alpha-1} - (n+g+\delta). \tag{5.8}$$

So growth in income per capita can be written as

$$\frac{\dot{y}}{y} = g + \alpha \left[s\tilde{k}^{\alpha-1} - (n+g+\delta) \right] \tag{5.9}$$

and notice that the term in brackets is a non-linear function. We can evaluate this term in brackets by doing a first-order Taylor expansion and linearizing around the steady state value. So start by defining the bracketed term as

$$h\left(\ln \tilde{k}\right) = \alpha s e^{(\alpha-1)\ln \tilde{k}} - \alpha\,(n+g+\delta) \tag{5.10}$$

and the linear expansion around $\tilde{k}^*$ is defined as

$$h\left(\ln \tilde{k}^*\right) + h'\left(\ln \tilde{k}^*\right)\left(\ln \tilde{k} - \ln \tilde{k}^*\right). \tag{5.11}$$

Now, we know that in the steady state $h\left(\ln \tilde{k}^*\right) = 0$ (plug in the steady state value into $h\left(\ln \tilde{k}\right)$ and you'll see it comes out to zero). Next, note that

$$h'\left(\ln \tilde{k}^*\right) = (\alpha-1)\,\alpha s e^{(\alpha-1)\ln \tilde{k}^*} \tag{5.12}$$

and given the definition of the steady state value of $\tilde{k}^* = (s/\,(n+g+\delta))^{1-\alpha}$ we can plug in to this equation and get

$$h'\left(\ln \tilde{k}^*\right) = (\alpha-1)\,\alpha\,(n+g+\delta)\,. \tag{5.13}$$

Finally, note that $\ln \tilde{y} = \alpha \ln \tilde{k}$ and $\ln \tilde{y}^* = \alpha \ln \tilde{k}^*$. We can put all this information into (5.11) and (5.9) we get that

$$\frac{\dot{y}}{y} = g + (1-\alpha)\,(n+g+\delta)\,(\ln \tilde{y}^* - \ln \tilde{y})\,. \tag{5.14}$$

This equation tells us that growth in income per capita has two components. The first is regular technological growth (g) and the second is growth due to convergence to the steady state. The farther a country is from steady state ($\ln \tilde{y}^* - \ln \tilde{y}$ is larger), the faster a country grows. Why? Because of decreasing returns to capital. The less capital you have, the higher the marginal product of that capital, and the faster you grow when you add a unit of capital. If you are at your steady state, you'll just grow at the rate g. Note that the actual speed of convergence depends on your fertility rate, on g itself, and on the depreciation rate (but NOT on the savings rate).

What this tells us is that - holding everything else equal - poor countries should grow faster than rich countries. Does this hold up in the data? No, not directly. The growth rates of the developed world are generally as high or higher than the growth rates of developing countries. Aside from several countries that are making the "leap" (India, China, Asian Tigers). However, not everything is equal. Countries may have different steady states or g, and in that case we might get any kind of pattern to growth rates that we can think of. What seems to be true, given the data, is that once we control for factors affecting the steady state (savings, fertility, etc..), the YES, it is true that countries with lower income per capita do grow faster than countries with high income per capita. So we have what we call "conditional convergence". The structure of the Solow model does seem to hold up. But it leaves us wondering why the steady state's themselves differ in the first place, which the Solow model can't explain.

5.1.3 Interest Rate Differentials

The last empirical issue to consider is the nature of interest rates. These reflect the marginal product of capital in a country, and so any growth model that tells us the size of the capital stock should be able to tell us what interest rates are. If we have countries with identical production functions of

$$y = Ak^{\alpha} \tag{5.15}$$

then the interest rate is

$$r = \alpha A k^{\alpha - 1} - \delta. \tag{5.16}$$

If we take the U.S. as a baseline, we might think that real returns (r) are equal to 5%. With depreciation of around 5%, this would imply that the marginal product of capital must be about 10%.

If we look at a developing country with income per capita of only about 20% of the U.S., we have that

$$\frac{y_i}{y_{us}} = \frac{Ak_i^{\alpha}}{Ak_{US}^{\alpha}} = \frac{1}{5} \tag{5.17}$$

and for now let's assume that technology in the countries is the same. So that means that

$$\frac{k_i}{k_{us}} = \frac{1}{125} \tag{5.18}$$

if we assume that $\alpha = 1/3$. If this is true, then the marginal product of capital in country i must be

$$\frac{MPK_i}{MPK_{us}} = \left(\frac{k_i}{k_{us}}\right)^{\alpha - 1} = \left(\frac{1}{125}\right)^{-2/3} = 25 \tag{5.19}$$

and therefore the interest rate in country i must be

$$r_i = 25\,(10\%) - 5\% = 245\% \tag{5.20}$$

which seems aboslutely too incredible to be true. We should see capital flow from the U.S. to this country to reap the rewards of a 245% return on capital. And we don't see anything like these kinds of differences.

However, we could assume that interest rates ARE equalized between countries, and then back out what the implied differences in A are across countries that can account for this. In other words, we think the marginal product of capital should be equated across countries, and if so we get

$$A_i k_i^{\alpha-1} = A_{us} k_{us}^{\alpha-1} \quad (5.21)$$

$$\frac{A_i}{A_{us}} = \left(\frac{k_{us}}{k_i}\right)^{1-\alpha} \quad (5.22)$$

and in the worst cases we observe something like $\frac{k_{us}}{k_i} = 50$. This implies that $\frac{A_i}{A_{us}} = 13.6$. Is this realistic? Do we really think that one country is really fourteen times more production than another? This seems way too big to make sense.

So it would be nice if our models of growth were consistent with equalized rates of return across countries (because we think international financial markets work to some extent) but don't require crazy differences in TFP to generate them.

5.2 Basic Models

The endogenous growth literature is motivated by an attempt to explain where Solow's g comes from. We see sustained, stable growth in most developed economies, and these models are an attempt to show how this can arise through economic behavior, as opposed to the remarkable coincidence that technology grows at some constant rate per year. These models, though, are less useful in describing differences across countries, as they don't seem to capture the important factors separating developing countries from the rich world.

5.2.1 The AK Model

Think about the Solow model with $Y = Ak^\alpha$ and a growth rate of capital of

$$\frac{\dot{k}}{k} = sAk^{\alpha-1} - (n+\delta).$$

We know that the growth rate of capital (and therefore of output) increases as we get away from the steady state. Imagine that as we are approaching the steady state, we have a shift up in the savings rate. What happens? There is a temporary increase in growth again (as we're now far away from the steady state) but this growth will eventually die down again as well.

Now imagine this scenario again, but with α very close to one If this parameter is close to one, then the $sAk^{\alpha-1}$ is getting less sloped, and starting to look like a straight line. In this case, the increase in growth when s goes up is very large, and it takes a very long time for the growth effect to die out. In the limit, as $\alpha = 1$ what happens to the growth rate of capital?

$$\frac{\dot{k}}{k} = sA - (n+\delta)$$

What does this tell us? Notice that there is nothing in this equation that implies that $\frac{\dot{k}}{k}$ ever has to fall. Assuming $sA > (n + \delta)$, then the growth rate of capital is constant and positive. The growth rate of output in this case is then $\dot{y}/y = \dot{k}/k$, or output per person is growing constantly as well. The only thing that affects this growth rate is the preference parameters s, n, and δ. So we have sustained growth without relying on exogenous technology creation. This seems to be an improvement.

What does the AK model imply?

1. We have the marginal product of capital as $f'(k) = A$, so $r = A - \delta$. Since we presume that A is common across countries (sometimes) this implies that there should be no differences in interest rates across countries. This is a big improvement over the Solow model, and seems to match the data pretty well.

2. What does the AK model imply about the relationship of income level and income growth? The AK model says there is no relationship - a country has a constant growth rate regardless of its initial income level. This seems to be good, because in the data we don't see a clear relationship betwee income levels and growth.

3. Savings are positively related to growth, where in the Solow model savings are related to income levels.

4. Population growth is negatively related to growth, where in the Solow model population growth lowers the income level.

So does the AK model have any justification? All we've done is assert that $\alpha = 1$, and recall that α is roughly synonymous with capital's share in total output, and empirically this is closer to 1/3. So how does the AK model get $\alpha = 1$?

The basic story is that there are externalities to production. Assume the actual production function is $Y = AK^{\alpha}L^{\beta}$. But now, the level of technology A is a function of how much capital exists in the economy. Maybe more capital means more specialization and this has productivity gains due to comparative advantage. This could also be through learning by doing, in that we learn better techniques every time we install new capital.

So let's call the level of A

$$A = \left(\frac{K}{L}\right)^{\gamma}.$$

This makes the production function

$$Y = K^{\alpha}L^{\beta}k^{\gamma}.$$

Now, we assume that $\alpha + \beta = 1$, or there are constant returns to scale for each firm. This is because each firm takes k^{γ} as a given. Firms don't take into account that an increase in their own capital stock will actually increase everyone's A level. Also, because the k^γ term is just an externality, the firms don't have to make factor payments on this. This means that the production function looks like this for a given firm i

$$Y_i = K_i^\alpha L_i^{1-\alpha} k^\gamma$$

which implies that each firm pays out $1-\alpha$ of its output to labor. Setting $\alpha = 1/3$ we can match the aggregate data pretty well.

Looking again at aggregate output, if we consider the per person output, we get

$$y = k^\alpha k^\gamma$$

which gives us the AK model if $\alpha+\gamma = 1$. So the AK model can be justified if we assume that firms have some externality that they receive to their productivity based on the aggregate capital labor ratio. Notice that this also means that the economy does not optimally invest in capital. The marginal product of capital from the perspective of individual firms is only $\alpha k^{\alpha+\gamma-1}$ while from the aggregate perspective it is $(\alpha+\gamma)\, k^{\alpha+\gamma-1}$.

The biggest issue with the AK model is probably that it implies that the externalities are enourmous. The elasticity of output with respect to capital, based solely on the externality is

$$\frac{\partial y}{\partial k}\frac{k}{y} = \gamma$$

or a 1% increase in the capital stock leads to a 0.67% increase in output, just due to the externality. This seems awfully high.

5.2.2 Generalizing Endogenous Growth

So the AK model delivers endogenous growth by building in something to the Solow model that eliminates the decreasing returns to capital. In general, models that have decreasing returns to the accumulable factors (such as our basic Solow human capital model) all have zero steady state growth by themselves (that is, they require exogenous technological growth). Models that somehow generate constant returns (or increasing returns) to accumulable factors will deliver endogenous growth. Let's see how this works a little more generally.

Take a general production function of the form

$$Y = AX^\alpha L^\beta$$

where X represents accumulable factors of production (physical capital, human capital, etc). L is population, as normal. Also assume we have a constant savings rates (recall that this really doesn't matter, the endogenous savings rates in the Ramsey model give us the same steady state analysis). Then the change in the stock of factors can be written as

$$\dot{X} = sAX^{\alpha}L^{\beta} - \delta X$$

where there is some depreciation factor. If population grows at the rate n, and we rewrite everything in per capita terms using small letters, then we get

$$\begin{aligned} \dot{x} &= sAx^{\alpha}L^{\alpha+\beta-1} - (\delta + n)\,x \qquad (5.23) \\ \frac{\dot{x}}{x} &= sAx^{\alpha-1}L^{\alpha+\beta-1} - (\delta + n) \end{aligned}$$

which should look vaguely familiar at this point.

So now, if this model has a steady state, it must be that x is growing at a constant rate. We don't know what it is, but let's label this rate γ. And therefore in steady state

$$\frac{\dot{x}}{x} = \gamma.$$

The value of A isn't going to impact the conclusions at this point, so we can safely assume it to be equal to one (if you don't trust me, follow through on the math with A included and you'll see it always cancels). So we can use our equation of motion for x in (5.23) to see that

$$\gamma = sAX^{\alpha-1}L^{\alpha+\beta-1} - (\delta + n)\,.$$

What we're interested in is NOT the size of X and L that make this hold. What we want to know is what restrictions on the parameters of the model yield constant growth. In other words, the LHS of the above equation is not changing - the growth rate is constant. So what makes the RHS of the above equation stay constant as well? To see, let's take the time derivative of this equation on both sides.

$$\begin{aligned} 0 &= s\left((\alpha - 1)\,x^{\alpha-2}\dot{x}L^{\alpha+\beta-1} + x^{\alpha-1}\,(\alpha + \beta - 1)\,L^{\alpha+\beta-2}\dot{L}\right) \\ 0 &= sx^{\alpha-1}L^{\alpha+\beta-1}\left((\alpha - 1)\,\frac{\dot{x}}{x} + (\alpha + \beta - 1)\,\frac{\dot{L}}{L}\right) \\ 0 &= (\alpha - 1)\,\frac{\dot{x}}{x} + (\alpha + \beta - 1)\,\frac{\dot{L}}{L} \\ 0 &= (\alpha - 1)\,\gamma + (\alpha + \beta - 1)\,n \end{aligned}$$

and using this last expression we can characterize all the possible circumstances under which steady state growth at rate γ is possible.

1. $\alpha + \beta = 1$ (Constant Returns to Scale)

    (a) $\alpha < 1, \beta < 1 \Longrightarrow \gamma = 0$ (Solow Model)

   (b) $\beta = 0, \alpha = 1 \Longrightarrow \gamma > 0$ (AK Model)

2. $\alpha + \beta > 1$ (Increasing Returns to Scale)
   (a) $\alpha < 1, \beta < 1$
      i. $n > 0 \Longrightarrow \gamma = \frac{\alpha+\beta-1}{1-\alpha} n$
      ii. $n = 0 \Longrightarrow \gamma = 0$
   (b) $\beta > 0, \alpha = 1$
      i. $n > 0 \Longrightarrow$ no feasible solution
      ii. $n = 0 \Longrightarrow$ any value of γ is feasible

So whatever the details of your endogenous growth model, in order to generate perpetual growth without exogenous technological progress you must be able to reduce the model to a form similar to either 1.b, 2.a.i, or 2.b.ii.

The point is that any model that hopes to permit perpetual long-run growth has to boil down to a structure in which there are constant returns to scale for the set of factors that can be jointly accumulated. Another way of looking at this is to consider what your production function looks like in intensive form. That is, does per person output have constant or decreasing returns to scale? If you have decreasing returns in the intensive form, you cannot achieve endogenous growth. If you have constant returns, you can achieve endogenous growth.

Example 22 *Let's think about a production function that includes human capital and has different properties.*

$$\begin{aligned} Y &= K^{\alpha} h^{1-\alpha} (EL)^{1-\alpha} \\ &= K^{\alpha} H^{1-\alpha} \end{aligned}$$

where $H = hEL$ or H is the total amount of human effort that is brought to bear in production and h measures the per efficiency unit value of this. So we have a production function that satisfies the AK model conditions (1.b above) - it has constant returns to the accumulable factors (K and H), and a coefficient of zero on labor. The intensive production function can be written as

$$y = k^{\alpha} h^{1-\alpha}.$$

Notice that this intensive form has constant returns to scale, so that if we doubled k and h we'd double output. This changes our expectations for the steady state. Now, even as k and h increase, output is increasing just as fast, so that we'll be able to continue to grow both k and h indefinitely. The equations of motion are the similar to before, so that we get

$$\begin{aligned} \frac{\dot{k}}{k} &= s_k \frac{k^{\alpha} h^{1-\alpha}}{k} - (n + \delta + g) \\ \frac{\dot{h}}{h} &= s_h \frac{k^{\alpha} h^{1-\alpha}}{h} - (n + \delta + g) \end{aligned}$$

which you can write in terms of the ratio k/h

$$\begin{aligned}\frac{\dot{k}}{k} &= s_k\left(\frac{k}{h}\right)^{-(1-\alpha)} - (n+\delta+g)\\ \frac{\dot{h}}{h} &= s_h\left(\frac{k}{h}\right)^{\alpha} - (n+\delta+g)\,.\end{aligned}$$

So now the change in each capital stock is dependent on the ratio of the two stocks. We can look at a diagram of this situation to see how the system evolves.Note first that the steady state value of k/h is actually identical to the

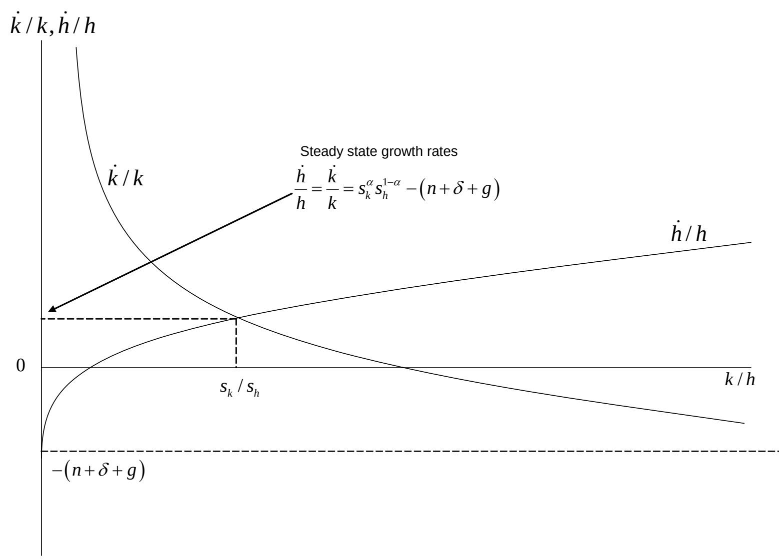

Figure 5.1:

original problem presented in this section. It's just the ratio of the savings rates. But now we have that both k and h grow even in the steady state at a rate that depends on the savings rates. Why does this system continue to grow? Because of the constant returns in the intensive form - it allows output to continue to grow fast enough that total savings keep up with the depreciation terms. (This would be like a single k Solow model having a linear $sf(k)$ line that lay above the δk line).

So what does this tell us? Well, it tells us that even without technological change (imagine setting $g=0$), we could have sustained growth if we were accumulating two separate factors of production that had the right properties in the aggregate. The growth rate of output per person recall, is the growth rate of output per *effective unit plus the growth rate of E. The growth rate of output per effective unit is*

$$\begin{aligned} \dot{y}/y &= \alpha \dot{k}/k + (1-\alpha)\,\dot{h}/h \\ &= s_k^{\alpha} s_h^{1-\alpha} - (n+\delta+g) \end{aligned}$$

so the growth rate of output per person is

$$\frac{(\dot{Y/L})}{Y/L} = s_k^{\alpha} s_h^{1-\alpha} - (n+\delta)\,.$$

And our standard of living now doesn't depend at all on the rate of g, only on the savings rates relative to the depreciation rates. This is a simple example of an endogenous growth model in which the actual growth of output per person depends only on preference parameters and not on an exogenous rate g.
You can also use the diagram to analyze what happens to k/h when there is a change in the savings rates. One of the curves (or both) may shift, creating some transitional growth to a new steady state value of k/h, and there will be plenty of chances on homeworks to see how this operates.

5.2.3 Lucas' Human Capital Model

Lucas provided a slightly different outlook on the accumulation of human capital, incorporating the fact that it involves a time commitment, as opposed to simply financial backing. The production function is

$$Y = K^{\alpha} (uhL)^{1-\alpha} \tag{5.24}$$

where L is labor, h is human capital per person, and u is the fraction of time spent working (so it lies between zero and one). In per capita terms this becomes

$$y = k^{\alpha} (uh)^{1-\alpha} \tag{5.25}$$

and so production, in the intensive form, is constant returns to scale, and we will have endogenous growth. Note that nothing about the mechanics of u we describe can change this, it's a property of the production function alone.

Capital accumulates as expected

$$\dot{k} = y - c - (n+\delta)\,k \tag{5.26}$$

and human capital accumulates as follows

$$\dot{h} = \phi h (1-u) \tag{5.27}$$

which says that human capital is accumulated in a constant returns to scale manner. If we double the time spent on education, we double the growth rate of human capital.

One can solve this problem in a Ramsey setting with optimizing agents, but we know that in the steady state this isn't any different from the Solow model.

So we're going to do the Solow version of this, which just assumes a constant savings rate of s, and therefore $c = (1-s)\,y$ and the capital accumulation equation is just what you'd expect.

Look at the growth rate of capital

$$\frac{\dot{k}}{k} = s\left(\frac{k}{h}\right)^{1-\alpha} u^{1-\alpha} - (n+\delta) \tag{5.28}$$

and map this against the growth rate of human capital

$$\frac{\dot{h}}{h} = \phi\,(1-u) \tag{5.29}$$

on a graph with k/h on the x-axis. The human capital growth rate is constant, and $\dot{k}/k$ will, in steady state, just be equal to the growth rate of human capital. One of the things to take out of this model is that the factor that depends on itself (human capital) will drive the growth rate. Physical capital growth depends on both (because it depends on total output) and therefore it will have to fall in line with human capital.

Growth in output per person is

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} + (1-\alpha)\,\frac{\dot{h}}{h} \tag{5.30}$$

and therefore the growth rate of output per person equals $\dot{h}/h$ as well.

There are transitional dynamics, which can be parsed out of the diagram when the initial k/h ratio isn't exactly equal to the steady state value. One interesting experiment is to ask what would happen if you lower the value of u (thus raising the growth rate of human capital). Note that the level of y has to fall initially, but then will grow faster, and eventually income will be higher overall. However, the choice of u can then be seen to depend on your discounting of the future path of income.

5.3 Endogenous Technology Creation

The models so far have developed endogenous growth through very specific choices about the parameters of the production function. Note that even if the production does allow for growth, this doesn't mesh with our observations about increasing technological sophistication over time. In addition, it doesn't mesh with our empirical observations about how important TFP is. So another line of research on endogenous growth looks to explain the increase in technology economically. That is, what are the incentives to invent better ways of doing things?

These models will depart from the standard constant returns to scale, perfect competition set up we've used so far. Why? With CRS and perfect competition, all output is used up in paying for the factors of production. There is nothing left over, and so there is no output left to reward inventive activity. What the models in this section will do is introduce some kind of market imperfections that allow for profits to accrue to those who know better technology. This creates incentives to generate new technology, and we'll get growth.

5.3.1 Increasing Variety of Intermediate Goods

The first major family of models thinks of technology as the introduction of new goods. This is easy to conceive of mentally, and it must matter to some extent. This model will introduce the idea of monopolistic competition, which will look complicated, but is a very common way to generate profits for firms within a model.

There is no capital (but we could add it if we wanted to). There is a constant supply of labor, L. Y is output, which is produced using labor and intermediate goods (denoted X). N is the number of these intermediate goods in the economy. Intermediate goods are not capital, they are flows of inputs into final production.

Let's think about firms in the final goods sector (those that use labor and intermediate goods). Output for any final goods firm is as follows

$$Y_i = L_i^{1-\alpha} \sum_{j=1}^{N} X_{ij}^{\alpha} \tag{5.31}$$

where i indexes the firm, and j indexes the intermediate goods. Note the following: 1) the use of a single intermediate good does NOT affect the productivity of the others, 2) there is decreasing marginal productivity of each intermediate good, and 3) this means that all intermediate goods will be used in the same quantity. The way to see this is to imagine that it wasn't true. Then it would be possible to use a little less of one intermediate good, lowering output a little, and use a little more of another good, raising output - but by more than the loss. This is due to the diminishing marginal product assumption.

If X_i is the quantity of each intermediate good used by the firm, then the production function for the final goods firm can be written as

$$\begin{aligned} Y_i &= L_i^{1-\alpha} N X_i^{\alpha} && (5.32) \\ &= L_i^{1-\alpha} \left(N X_i\right)^{\alpha} N^{1-\alpha} && (5.33) \end{aligned}$$

where NX_i is the total amount of intermediate goods employed. Notice that total output is increasing in the number of goods, even holding constant the total intermediate goods employed. Why? More N means that you spread your NX_i across more goods, which raises the average product of each intermediate good.

Now, for the FINAL GOODS FIRMS (and only these firms), there is perfect competition (an assumption), which means that all of output has to be divided up as payments to the two inputs, labor and intermediate goods. What is their actual demand for each intermediate good? Look at each firms optimal profit conditions (just because we have zero profits in equilibrium doesn't change that everyone is optimizing profits).

$$\pi_i = Y_i - wL_i - \sum_{j=1}^{N} P_j X_{ij} \tag{5.34}$$

and the FOC on X_{ij} is

$$\alpha L_i^{1-\alpha} X_{ij}^{\alpha-1} = P_j. \tag{5.35}$$

The firm's demand for good X_{ij} is

$$X_{ij} = L_i \left(\frac{\alpha}{P_j}\right)^{1/1-\alpha} \tag{5.36}$$

and therefore the economy-wide demand for good j is

$$X_j = \sum_i X_{ij} = \left(\frac{\alpha}{P_j}\right)^{1/1-\alpha} \sum_i L_i = L\left(\frac{\alpha}{P_j}\right)^{1/1-\alpha} \tag{5.37}$$

which says that demand for good X_j depends on the price of the good, as well as the total size of the economy (in terms of population).

Now let's turn away from the final goods firms and look instead at the firms that will supply the intermediate goods. They will produce their goods and sell them to the final goods firms for the price P_j. The big question, though, is whether they should bother to operate at all. There is a fixed cost to operating - one has to invent the new product (it doesn't have to be pure invention - it might just be considered a fixed cost of opening a new firm).

The important part here will be that whoever operates to produce good X_j is a monopolist - they will earn monopoly profits forever on this good, and this is balanced against the cost of starting up in the first place. If these goods are newly invented, then essentially we are giving the inventor an infinite patent. If these just represent new firms, then we assume that products are sufficiently differentiated that it's impossible to replicate them. Either way, the owner has a monopoly on his kind of intermediate good.

We assume that the cost of inventing the new good (or of opening the factory), is a constant d. So what is the value of having invented some new good? You'll earn P_j for every unit you sell, and we simply assume that the cost of actually producing one of these units is equal to one. So the present discounted value of your earnings is

$$V(t) = \int_t^\infty (P_j - 1) X_j e^{-r(v-t)} dv \tag{5.38}$$

and notice that this is the value of earnings from time t to infinity. The question is whether you'll open the firm today (time t) and run it forever. The interest rate is assumed to be constant, and one can prove that in fact it is, but we'll leave that aside for now.

So now the intermediate good producer, if they produce, choose X and P to maximize profits (recall they are a monopolist, so they choose the price), knowing the demand curve we derived above for the final goods firms.

$$\begin{aligned} \pi &= (P_j - 1) X_j && (5.39) \\ &= (P_j - 1) L \left(\frac{\alpha}{P_j}\right)^{1/1-\alpha} && (5.40) \end{aligned}$$

and notice we have two effects. First, profits rise with increases in price, but second, the amount demanded falls as well. Find the FOC to get the optimal price level. You can do this easily, and the answer is very nicely

$$P_j^* = \frac{1}{\alpha}. \tag{5.41}$$

Therefore the total amount of X_j that is demanded and used in final goods production is

$$X_j = L\alpha^{2/(1-\alpha)} \tag{5.42}$$

and again notice that total demand depends on the size of the population.

Now, notice that the price level of each X_j is identical (because the intermediate goods firms are identical), and therefore X_j will be identical for all goods (and this jives with our finding that each firm will use identical amounts of each good). So, now we can reevaluate the $V(t)$ function to find

$$\begin{aligned} V(t) &= \int_t^{\infty} (P_j - 1) X_j e^{-r(v-t)} dv && (5.43) \\ &= \left(\frac{1}{\alpha} - 1\right) L\alpha^{2/(1-\alpha)} \left(\frac{1}{r}\right). && (5.44) \end{aligned}$$

Now, further conditions on the intermediate goods sector. We assume that there is free entry into the sector, so that if there are profits over and above the cost of opening (d), they will be driven to zero. How does this make sense? Recall that each firm is still earning monopoly rents. But if new people open firms, they provide more intermediate goods, and each final goods firm then demands less of each intermediate good. So entry of firms lowers the profits. Ultimately it must be that

$$V(t) = d \tag{5.45}$$

which can be solved for

$$r = \left(\frac{L}{d}\right)\left(\frac{1-\alpha}{\alpha}\right)\alpha^{2/(1-\alpha)} \tag{5.46}$$

and we have pinned down the interest rate. Notice that it is higher the large the population is, which seems like a strange result, and it is. We'll get back to this kind of thing later.

Let's think about aggregate output now

$$\begin{aligned} Y &= \sum_i Y_i = \sum_i L_i^{1-\alpha} N X_i^{\alpha} && (5.47) \\ &= N \sum_i L_i^{1-\alpha} \left(L\alpha^{2/(1-\alpha)} \right)^{\alpha} && (5.48) \\ &= N L\alpha^{2/(1-\alpha)} && (5.49) \end{aligned}$$

which tells us that

$$\frac{Y}{L} = N\alpha^{2/(1-\alpha)} \quad (5.50)$$

and notice that income per person in this economy is proportional to the number of varieties. Therefore

$$\frac{\dot{y}}{y} = \frac{\dot{N}}{N} \quad (5.51)$$

So how do we pin down the values of both of these? Start by looking at consumption. Consumption in total is equal to the following

$$C = Y - NX - d\dot{N} \quad (5.52)$$

which says that consumption is what's left over of total output after we pay for producing our intermediate goods (NX) and pay the costs of setting up new factories. Divide this through by Y and you get

$$\begin{aligned} \frac{C}{Y} &= 1 - X\frac{N}{Y} - d\frac{\dot{N}}{N}\frac{N}{Y} && (5.53) \\ &= 1 - X\frac{N}{Y} - d\frac{\dot{y}}{y}\frac{N}{Y}. && (5.54) \end{aligned}$$

Now, we know that N/Y has to be constant, given that their growth rates are identical. We also know that X_j (the quantity of each intermediate good used) is constant. (recall its equal to $L\alpha^{2/(1-\alpha)}$). This means that we can solve this for

$$\frac{\dot{y}}{y} = \frac{-\frac{C}{Y} - X\frac{N}{Y} + 1}{d\frac{N}{Y}} \quad (5.55)$$

and this expression will only make sense if C and Y are growing at the same rate at all times. If they were not, then C and Y would have to diverge permanently. So what is the growth rate of consumption?

Well, we haven't mentioned the people in this economy at all, who are consuming the output of the final goods sector. They will have to make an optimal consumption decision that leads to some growth rate of consumption. This growth rate, in the steady state, has to be equal to $\dot{y}/y$ because there is NO capital.

We assume they are CRRA with the normal parameters. So we know that

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(r - \theta) \quad (5.56)$$

which solves to

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left[\left(\frac{L}{d}\right)\left(\frac{1-\alpha}{\alpha}\right)\alpha^{2/(1-\alpha)} - \theta\right]. \tag{5.57}$$

Now the question is whether the values of d, L, and *alpha* are such that this is positive. It doesn't make sense to have negative growth in consumption in this model, because we have no capital that can be stored up and then run down. So we have to assume that the parameters are such that growth in consumption is positive.

So this gives us that

$$\frac{\dot{y}}{y} = \frac{\dot{N}}{N} = \frac{\dot{c}}{c} = \frac{1}{\sigma}\left[\left(\frac{L}{d}\right)\left(\frac{1-\alpha}{\alpha}\right)\alpha^{2/(1-\alpha)} - \theta\right] \tag{5.58}$$

and therefore the growth rate of varieties, income per capita, and consumption are all dependent on the size of the economy (L). This is an issue, because there is no evidence that how big an economy is is correlated at all with growth (i.e. why doesn't China grow even faster? why isn't Lichtenstein the poorest country in the world?). It's a common issue with models of varieties, because of the assumptions that more varieties increase productivity. The more labor, the higher the marginal product of each good, and so more get produced. There are ways around this that require a variety of contortions to the model.

5.3.2 Increasing Quality of Intermediate Goods

The other aspect of technological change that is apparent is that goods get better. Cars are faster, safer, etc.. than they were thirty years ago. Computers get faster year by year. So some element of the improvement in technology over time is due to increasing quality of goods. So now we consider a model in which new inventions replace old ones - "creative destruction" in the words of Schumpeter and the work of Aghion and Howitt (1992) who reintroduced this to economics.

Again, there will be no capital. There are L individuals and they have typical intertemporal preferences of

$$U = \int_0^\infty y_t e^{-rt} dt \tag{5.59}$$

where r is the discount rate (and hence also the interest rate given that there is no capital). Final output depends on ONE intermediate good, X, so that

$$Y = AX^\alpha \tag{5.60}$$

and notice that labor does not enter this equation. Labor, on the other hand, is divided up into two uses, producing X (denoted d) and working to invent new goods (denoted n).

$$L = d + n \tag{5.61}$$

or in other words people spend their time either producing the intermediate good (d) or working at R&D (n).

Now, if the amount of labor employed in aggregate in R&D is n, then a new innovation arrives randomly with a Poisson arrival rate of $n\lambda$ where $\lambda > 0$ is a parameter that indicates the productivity of R&D. Recall that a Poisson process describes the chance that a random event will occur at any given point in time, given that the random event is described by an exponential distribution. An innovation raises A by the factor γ, meaning that intermediate goods get more productive. Thus a new innovation has $A_{k+1} = \gamma A_k$ where k indexes the innovations (which occur in a "quality ladder" - or in other words they are always productivity improving).

If you DO make an innovation (you get lucky), then you have invented a better version of the intermediate good X, and you become a monopolist selling this intermediate good to the final good sector. You'll retain this monopoly until the next person invents something better than you and takes your business.

From an individuals perspective, they are indifferent between R&D and work, and they can move back and forth between sectors freely. Therefore they equate the value of work (wages) with the value of R&D

$$w_k = \lambda V_{k+1} \tag{5.62}$$

where w_k is the wage rate and λV_{k+1} is the expected value of doing research. You'll invent something with probability λ and this invention will yield you some permanent value V_{k+1}. The subscript k is the number of innovations that have occurred, so w_k is the wage earned when there have been k innovations, and V_{k+1} is the value of inventing the next innovation.

The value of an innovation is evaluated as any other asset as

$$rV_{k+1} = \pi_{k+1} - \lambda n_{k+1} V_{k+1} \tag{5.63}$$

which says that the expected income earned by the innovator during a unit of time (rV_{k+1}) is equal to the proft flow minus the expected loss of "capital" that will occur if someone replaces the innovator (which happens with the probability λn_{k+1}). Another way to see this is that the value V_{k+1} is just the net present value of profits minus the NPV of the loss of monopoly rents.

Solve this for V_{k+1} and we get

$$V_{k+1} = \frac{\pi_{k+1}}{r + \lambda n_{k+1}}. \tag{5.64}$$

So now we need to figure out what the profit rate is in this model. The innovator, when they are the monopolist, will maximize

$$\pi_k = p_k(d)\, d_k - w_k d_k \tag{5.65}$$

where $p_t(d)$ is the price they charge, and this depends on the amount of intermediate goods they supply to the final goods producers.

So what about those final goods producers? They operate in a perfectly competitive environment, so they have zero profits and their FOC indicates that

$$p_k(X) = A_k \alpha X^{\alpha-1}. \tag{5.66}$$

Now recall that one unit of labor produces exactly one unit of intermediate good, or $X = d$. So now we know that $p_k(d) = A_k \alpha d^{\alpha-1}$. Plug this into the monopolists maximization problem and get

$$\pi_k = A_k \alpha d_k^{\alpha} - w_k d_k \tag{5.67}$$

and maximize with respect to d gives you

$$d_k = \left(\frac{A_k \alpha^2}{w_k}\right)^{1/1-\alpha} \tag{5.68}$$

$$\pi_k = \left(\frac{1}{\alpha} - 1\right) w_k \left(\frac{A_k \alpha^2}{w_k}\right)^{1/1-\alpha} = \left(\frac{1}{\alpha} - 1\right) \alpha^{2/1-\alpha} A_k \left(\frac{A_k}{w_k}\right)^{\alpha/1-\alpha} \tag{5.69}$$

Now notice that both the amount of d_k employed and the profit are decreasing in the wage rate. Recall that $n_k = L - d_k$ so that the amount of research done is increasing in the wage rate. Because if the wage rate is high, less people will want to work in research, which means that the value of being the monopolist is very high because you're less likely to be replaced. So you'll be more inclined to do research.

So now let's characterize the two things we need to look at to solve this. First, the arbitrage equation in (5.64), along with (5.62) with the other values plugged in gives us

$$w_k = \lambda V_{k+1} = \frac{\lambda \pi_{k+1}}{r + \lambda n_{k+1}} \tag{5.70}$$

$$\frac{w_k}{A_k} = \frac{\lambda \gamma \left(\pi_{k+1}/A_{k+1}\right)}{r + \lambda n_{k+1}} \tag{5.71}$$

which relates the productivity adjusted wage today (w_k/A_k) to productivity adjusted profits at the time of the next innovation (π_{k+1}/A_{k+1}). It also relates the current adjusted wage inversely to the number of people who will do research next period (n_{t+1}).

The second equilibrium condition is simply the labor market clearing, or

$$L = n_k + \left(\frac{A_k \alpha^2}{w_k}\right)^{1/1-\alpha}$$

which relates the productivity adjusted wage again to the share of people doing research today. But now notice that there is a positive relationship.

These two relationships show the distinct tradeoffs present for people within the economy. On the one hand, if wages increase, then people will move out of research and into the labor force. On the other hand, people understand this movement and realize that if less people do research, then the value of doing research yourself goes up.

Now, in steady state we can write the arbitrage equation as

$$\left(\frac{w}{A}\right)_{ss} = \frac{\lambda\gamma\left(\pi/A\right)_{ss}}{r+\lambda n_{ss}}$$

and we know that we can express profits as

$$\begin{aligned}\left(\frac{\pi}{A}\right)_{ss} &= \left(\frac{1}{\alpha}-1\right)w_k X \\ &= \left(\frac{1}{\alpha}-1\right)\left(\frac{w}{A}\right)_{ss}\left(L-n_{ss}\right)\end{aligned}$$

and then we can rewrite the arbitrage equation as

$$1 = \frac{\lambda\gamma\left(\frac{1}{\alpha}-1\right)\left(L-n_{ss}\right)}{r+\lambda n_{ss}}$$

and solve this for n_{ss} as

$$n_{ss} = \frac{\gamma\left(\frac{1}{\alpha}-1\right)L - r/\lambda}{1+\gamma\left(\frac{1}{\alpha}-1\right)}.$$

So we can see that steady state research effort depends positively on L, negatively on r, and positively on λ.

Plotting (w_k/A_k) against n_k, one can find the equilibrium level of both. In steady state, adjusted wages are equal across innovations (meaning wages are rising at the rate γ every time an innovation occurs. But notice that this advancement of wages is random.

In steady state the flow of final output produced during the time period between k and $k+1$ innovations is

$$y_k = A_k\left(L-n_{ss}\right)^{\alpha}$$

which since n_{ss} is fixed means that $y_{k+1} = \gamma y_k$, or that income per person goes up by γ every time there is an innovation. However, this is not the same as the growth rate over time. But the amount of time that actually passes between innovations is random. Looking at income over time we can write

$$y_{t+1} = \gamma^{\varepsilon} y_t$$

where ε is the number of innovations that occur between periods t and $t+1$. This parameter is distributed as a Poisson with a rate of λn_{ss}. Then we have that

$$E\left(\ln y_{t+1} - \ln y_t\right) = \lambda n_{ss} \ln\gamma$$

or that the expected growth rate is proportional to the arrival rate of innovations (λ) and the steady-state labor spent doing research. So any changes to n_{ss} will affect the long-run growth rate.

	Annual Growth Rate of: Population	GDP	GDP per capita
Agrarianism 500-1500	0.1	0.1	0.0
Advanced Agrarianism 1500-1700	0.2	0.3	0.1
Merchant Capitalism 1700-1820	0.4	0.6	0.2
Capitalism 1820-present	0.9	2.5	1.6

Source: Angus Maddison, *Phases of Capitalist Development*

Table 5.1: Phases of Economic Growth in Europe

One last thing to notice is that this model suffers from the scale problem as well. That is, the growth rate depends on n_{ss}, and this depends on L. So growth should increase with the size of the population. Again, one can get around this with some machinations, but most endogenous growth problems tend to result in this kind of scale issue.

5.4 Endogenous Population Growth

To this point we've taken population as fixed, or we've had a constant growth rate of population. However, this is certainly an endogenous process, as people's fertility responds to their economic conditions. We'll start with some mechanical models of how population is related to the resources available, and then think more clearly about the optimal choice of fertility.

A few quick facts about broad population trends before we start. If we look at the growth rate of population, the growth rate of GDP, and the growth rate of GDP per capita in Europe over the long run we see several broad eras.

Over much of history (and even prior to the year 500) population and GDP grew very slowly, resulting in no net gain in GDP per capita. By 1500 there was some creep in GDP per capita as GDP started to accelerate slightly faster than population, but not by much. By 1700 this gap had increased, but still not by a significant amount. Only after capitalism develops in the 1800's do we see sustained large scale growth in income per capita. Notice that over the whole of history the growth in population is increasing. So increases in GDP per capita are not because population growth slows down (as in the Solow model), rather GDP starts to grow at a much faster rate.

The initial periods are known as Malthusian because they fit within the paradigm of Thomas Malthus, who predicted that any increase in income would be met with a matching increase in population, and hence no one would be better off. Malthus wrote at around 1820, or just when this relationship was breaking down, interestingly enough. So in thinking about population growth, we want to be able to account for both the initial Malthusian period, as well as the more modern period in which GDP grows faster than population.

5.4.1 Lucas' Malthusian Model

Let's start with some terminology. First, c is consumption per person and n is children per person. Production takes place using land (X) and people (N) and there is no capital. Production is CRS and defined as

$$Y = AX^{\alpha}N^{1-\alpha} \tag{5.72}$$

and in intensive form

$$y = Ax^{\alpha} \tag{5.73}$$

where $x = X/N$ is land per person. So income per person depend on technology and on the amount of land per person. We cannot accumulate land, so this will be the limiting aspect of income per capita.

Assume that land is not privately owned. That is, there are no enforceable property rights, and so individuals earn all the output - y. This kind of assumption can be justified in a number of ways, but is consistent with a lot of evidence from poor agrarian areas in developing countries - similar, we assume, to agrarian areas in the past.

Each child costs k units of output to raise and so the budget constraint is

$$y = c + nk. \tag{5.74}$$

Individuals utility has a log form over their own consumption, their number of kids, and their kids utility

$$U_t = (1-\beta)\ln c_t + \theta \ln n_t + \beta U_{t+1}. \tag{5.75}$$

Notice that there is nothing in the model that allows a person to affect their kids utility directly, and since it enters in a separable manner, it has really not effect on any decisions. A more advanced treatment would allow this to be involved more in the optimization.

Maximizing utility subject to the budget constraint we get the following FOC

$$\frac{kn_t}{c_t} = \frac{\theta}{(1-\beta)} \tag{5.76}$$

or the ratio of spending on children to spending on consumption is just the ratio of the parameters from the utility function, as we'd expect with log utility. This FOC does actually tell us alot. In steady state, we know that x will be constant. Since X is constant, that means that N must be constant as well, and if N is constant then $n = 1$. Knowing this we can solve for steady state consumption per capita as

$$c_{ss} = \frac{(1-\beta)\,k}{\theta} \tag{5.77}$$

and notice that steady state consumption does not depend on technology (A) or the amount of land (X) at all. So no matter how advanced our technology or how much land we have, our consumption will be the same. Consumption only depends on our preferences for kids and their cost.

Now let's go back to the FOC and combine this with the budget constraint and the definition of the production function. This gives us

$$n_t = \frac{y_t}{k}\left(\frac{\theta}{1-\beta+\theta}\right) \tag{5.78}$$

$$= \frac{Ax_t^{\alpha}}{k}\left(\frac{\theta}{1-\beta+\theta}\right) \tag{5.79}$$

or fertility depends on the amount of land per person and on technology. If resources are high, we produce lots of kids, if they are low, less kids.

So what about dynamics? The evoluation of land per worker looks like this

$$x_{t+1} = \frac{X}{N_{t+1}} = \frac{X}{n_t N_t} = \frac{x_t}{n_t} \tag{5.80}$$

Now substitute the expression for n_t in (5.78) to this and we get

$$x_{t+1} = x_t^{1-\alpha}\frac{k}{A}\left(\frac{1-\beta+\theta}{\theta}\right) \tag{5.81}$$

and we have a dynamic response of future land per worker to current land per worker. What is happening? Well, current land per worker tells us how high fertility is, and this in turn will determine the land per worker that obtains in the following period when our kids are old. This equation also allows us to solve for the steady state level of land per worker and income per worker

$$x_{ss} = \left[\frac{k}{A}\left(\frac{1-\beta+\theta}{\theta}\right)\right]^{1/\alpha} \tag{5.82}$$

$$y_{ss} = k\left(\frac{1-\beta+\theta}{\theta}\right) \tag{5.83}$$

and again notice that income per person does not depend at all on A or X. That is, every economy should end up with the same output per worker. So does this mean that resources and technology don't matter? No, but they only matter for population size. Solve for N_{ss} by noting that $x_{ss} = X/N_{ss}$ and we get

$$N_{ss} = XA^{1/\alpha}\left[\frac{1}{k}\left(\frac{\theta}{1-\beta+\theta}\right)\right]^{1/\alpha} \tag{5.84}$$

and population size is directly affected by both technology and resources. This is the Malthusian effect - population size responds completely to any increase in X or A that no increases in income per capita are possible. So this is a nice description of most of history. But the real question is how come this kind of model breaks down around 1800. Why changes about fertility behavior at this point that breaks this relationship? Well, we'll have to get that n_t is no longer directly related to land per person or A, or that it's elasticity with respect to these is changed.

5.4.2 Kremer's Model of Long Run Population Growth

Output is produced with only labor and land (land, a fixed factor for the most part, is usually necessary to generate the kind of Malthusian results we expect). Production is

$$Y = X^{\alpha}\left[AN\right]^{1-\alpha} \tag{5.85}$$

where A is TFP, X is land, and N is population. Kremer's idea was that the growth rate of A is actually proportional to the size of the population. In other words, more people mean more ideas and therefore growth in TFP should be faster.

$$\frac{\dot{A}}{A} = BN \tag{5.86}$$

where B is just some parameter of the model. In addition, we'll include a Malthusian type constraint on the economy. We won't model it directly (but you could), but rather we'll just say that income per person is always at the subsistence level (as in the Lucas model)

$$y = \bar{y}. \tag{5.87}$$

Given this, we can solve for the level of production as a function of technology and land

$$\bar{y}N = X^{\alpha}\left[AN\right]^{1-\alpha} \tag{5.88}$$

which solves to

$$N = \left(\frac{1}{\bar{y}}\right)^{1/\alpha} A^{(1-\alpha)/\alpha}X \tag{5.89}$$

or again we see that population is proportional to resources. This is just a function of assuming that $y = \bar{y}$ and is the same result we get out of the Lucas model, but without a bunch of optimization.

The interesting part comes now as we look at population growth

$$\frac{\dot{N}}{N} = \frac{1-\alpha}{\alpha}\frac{\dot{A}}{A} = \frac{1-\alpha}{\alpha}BN \tag{5.90}$$

and this says that the growth rate of population is proportional to the size of the population. And this would mean that population is explosive in growth. Now this sounds kind of crazy, but it fits the data rather well when you look at really long run population growth rates, as from 1 million BC to the present. The only time the model doesn't work is in the last few decades when industrialization has changed the relationship of fertility to economics.

5.4.3 Optimal Fertility Choice

So how does fertility respond to income going up? Becker, Murphy, and Tamura (1990) look at higher returns to child education as inducing a substitution of quality of children for quantity. In general, models of fertility tend to identify competing income and substitution effects for fertility. If income goes up, then you are richer and will "buy" more children. However, since we believe strongly that child-raising takes time, if your wage is higher, this raises the marginal cost of having another child. Overall, it appears that the substitution effect dominates. But this research really only looks at modern era growth, and doesn't consider any transitions.

Following the model in Galor and Weil (2000) pretty closely, describe utility for an individual as

$$u_t = (1-\gamma)\ln c_t + \gamma \ln n_t h_t \tag{5.91}$$

where c_t is their consumption, n_t is the number of children they have and h_t is the human capital per child (or could alternatively be the wage the child would earn - it doesn't matter). This appears to be a pretty reasonable description of how people operate, and there is evidence that most animals operate on this kind of utility structure as well.

Now your budget constraint involves the following. There is a fixed cost τ of time per child you raise, regardless of their education. In addition, you spend an amount of time e_t educating each child. This gets translated into their human capital by some concave function $h_t = h(e_t)$. Your budget constraint thus looks like

$$w_t n_t (\tau + e_t) + c_t \leq w_t \tag{5.92}$$

or in other words you "spend" your wage by buying children of a certain education level.

If we maximize this over n_t and e_t we get the following FOC

$$\frac{1-\gamma}{1-n_t(\tau+e_t)} - (\tau+e_t) + \frac{\gamma}{n_t} = 0 \tag{5.93}$$

$$\frac{1-\gamma}{1-n_t(\tau+e_t)} - n_t + \frac{\gamma}{n_t}\frac{n_t h'(e_t)}{h(e_t)} = 0 \tag{5.94}$$

which can be solved together for

$$\frac{n_t}{(\tau+e_t)} = \frac{n_t h'(e_t)}{h(e_t)}. \tag{5.95}$$

What does this condition tell us? Well, the LHS is the ratio of marginal costs. If we want to increase e_t by a little, then it costs us n_t of time because we want to educate each child equally. If we want to increase the number of children, then we pay $(\tau + e_t)$ for that child to raise and educate them. On the RHS, we have the ratio of marginal benefits. If we increase education, then we get an additional $n_t h'(e_t)$ in utility. If we increase the number of kids, we get an additional $h(e_t)$. So the FOC are typical optimization results. [Note that they don't depend on γ either, this is just the decision about how to allocate our child-raising time to quantity and quality].

Let's think about what happens when the returns to education change. That is, $h(e_t)$ shifts up, but the shape of it $(h'(e_t))$ doesn't change. (All years of schooling now give you more human capital, but the amount you gain from one additional year hasn't changed). Then the marginal benefit of all children has gone up relative to the marginal benefit of education. So the answer to equating this situation is to lower the number of kids, and raise the amount of education. So to account for the intertwined drop in fertility and rise in education, we need something to push up $h(e_t)$.

This FOC, though, doesn't offer us any situation in which the wage matters. That is, there is no effect of parent wage on fertility, and in particular we cannot see how fertility would go up with income, as in a Malthusian world. So this model is "modern" in that it captures the quantity quality tradeoff, but not Malthusian. To add a Malthusian element, we need simply establish a subsistence constraint. That is, there is a minimum level of consumption $\bar{c}$ that people must consume at all times.

$$c_t = w_t [1 - n_t (\tau + e_t)] \geq \bar{c} \tag{5.96}$$

so we need to evaluate this somehow. First, go back to the utility function. Note that we have CD preferences, and we know that with these, we spend a constant fraction of our wealth on each "good". In our case, we would like to spend the fraction $(1-\gamma)$ of our wealth, w_t, on consumption. Since we have one unit of time, that means that we must spend $(1-\gamma)$ of our time working, leaving γ of our time for raising kids. So we know that $n_t(\tau + e_t) = \gamma$, when we are unconstrained. Plug this in above and we get that

$$c_t = w_t [1 - \gamma] \geq \bar{c} \tag{5.97}$$

So if $w_t > \bar{c}/(1-\gamma)$, then we are unconstrained by subsistence. However, if $w_t < \bar{c}/(1-\gamma)$ then we are constrained. Now, $w_t [1 - n_t(\tau + e_t)] = \bar{c}$ binds, and this dictates that share of our time spent raising children $n_t(\tau + e_t) = 1 - \bar{c}/w_t$. If w_t is very low, then this number goes to zero. In other words, if we are very poor, then we spend very little time on having kids (regardless of their education level). To see this, imagine that $e_t = 0$. Then $n_t = (1 - \bar{c}/w_t)/\tau$. So if w_t were to rise, so would fertility. The subsistence constraint delivers the result that at low incomes, rising income leads to higher fertility. Once we get to a cutoff of $w_t = \bar{c}/(1-\gamma)$, then the time spent no longer increases (but might be allocated differently).

Draw the diagram showing time on children on y-axis and c on the x-axis. At subsistence, vertical until γ and then horizontal. As wages increase, the budget line rotates out from a point on the y-axis equal to one.

So with these mechanisms in place, what can we say about how fertility evolves with income? Well, if wages increase, at first the time spent on children increases and then flattens out. So we can get a Malthusian result, but nothing that necessarily drives a quality quantity tradeoff. For that we have to make further assumptions about how wages and human capital evolve. Using our FOC we will have

$$h(e_t) - (\tau + e_t) h'(e_t) = 0 \text{ if } e_t > 0 \tag{5.98}$$

$$h(e_t) - (\tau + e_t) h'(e_t) > 0 \text{ if } e_t = 0 \tag{5.99}$$

which just involves the condition that $e_t \geq 0$ (i.e. you can't uneducate your children). So if it turns out that you'll fail to educate your children if the return on education isn't high enough to overcome the cost of adding additional education.

What does our fertility regime tell us about education? Well, let's look back at the FOC for education, which if rearranged says that

$$\frac{1-\gamma}{\gamma} \frac{n_t(\tau + e_t)}{1 - n_t(\tau + e_t)} = \frac{h'(e_t)(\tau + e_t)}{h(e_t)}. \tag{5.100}$$

Look back at the fertility decision, which if we are not at subsistence requires that $n_t(\tau + e_t) = \gamma$ and so this whole equation balances nicely. In other words when the subsistence constraint does not bite, all our FOC conditions hold, and we have positive education (as $h'(e_t)(\tau + e_t) = h(e_t)$). However, when the subsistence constraint does bite with low wages, we have that $n_t(\tau + e_t) < \gamma$ and this then implies that $h'(e_t)(\tau + e_t) < h(e_t)$ and this is what we already know implies that $e_t = 0$. So what we have found is that when the subsistence constraints bind, we fail to educate our children. Why? We have very little time to devote to kids, and this is taken up with the fixed time cost of children, so to maximize utility we have lots of uneducated kids. Once we have wages high enough, we actually achieve positive e_t.

Appendix A

Solutions

A.1 Solving for actual consumption (Example 4)

Using the expression for the path of consumption, we can solve for actual consumption by period by using the expected value budget constraint that holds at time t

$$E\left[\sum_{s=t}^{T-1} C_s\right] = E\left[\sum_{s=t}^{T-1} Y_s\right] + A_t.$$

and notice the subscripts. What we are saying here is that the expected value of future consumption (from today until period $T-1$) must equal the expected value of income over that same period plus the level of assets you have on hand at period t. So is there some way to evaluate the expectations? There is, and let's start with the expectation of income first.

$$\begin{aligned}
E\left[\sum_{s=t}^{T-1} Y_s\right] &= Y_t + E\left[\sum_{s=t}^{T-1} Y_{s+1}\right] \\
&= Y_t + E\left[\sum_{s=t}^{T-1} Y_s + \varepsilon_{s+1}\right] \\
&= Y_t + \sum_{s=t}^{T-1} E\left(Y_s\right) + \sum_{s=t}^{T-1} E\left(\varepsilon_{s+1}\right) \\
&= Y_t + (T-1-t)\, Y_t + 0 \\
&= Y_t\,(T-t)
\end{aligned}$$

where the final line follows because our expectation of all the transitory shocks to income is just zero. Now we can look at the expectation of consumption, where we want to try and get C_s in terms of C_t. Look first at the following

$$E\left(C_{t+1}\right) = C_t + \alpha\sigma^2/2$$

which follows from our expression for the optimal consumption path. Iterate this forward and you'll find

$$\begin{aligned} E(C_s) &= C_t + \left(\alpha\sigma^2/2\right)(s-t) + \sum_{s=t}^{T-1} E(\varepsilon_{s+1}) \\ &= C_t + \left(\alpha\sigma^2/2\right)(s-t) \end{aligned}$$

where again we've used the fact that the expected value of the transitory shocks is equal to zero. Now, we can plug this into the summation of expected consumption

$$E\left[\sum_{s=t}^{T-1} C_s\right] = \sum_{s=t}^{T-1} C_t + \left(\alpha\sigma^2/2\right)(s-t)$$

which can be evaluated as

$$E\left[\sum_{s=t}^{T-1} C_s\right] = (T-t)\,C_t + \frac{(T-t-1)(T-t)}{2}\frac{\alpha\sigma^2}{2}.$$

We now can fill in all the pieces of our expected budget constraint, which is now

$$(T-t)\,C_t + \frac{(T-t-1)(T-t)}{2}\frac{\alpha\sigma^2}{2} = A_t + Y_t\,(T-t)$$

and can be rearranged to be

$$C_t = \frac{A_t}{T-t} + Y_t - \frac{(T-t-1)\,\alpha\sigma^2}{4}$$

and this gives us exactly what consumption will be in any period t. Note that we can't actually specify what A_t is because it depends on the actual realizations of the transitory shocks. But given some level of A_t, this is how we should act.

A.2 Consumption Path in the Stochastic Ramsey Model

The answer to the following first order condition

$$\frac{1}{c_t} = \frac{1}{1+\theta} E\left[\frac{1 + \alpha A_{t+1} k_{t+1}^{\alpha} - \delta}{c_{t+1}}\right]$$

involves using the inelegantly named "guess and verify" method. It's know that problems that have log utility tend to have linear relationships of consumption and income. So we guess that consumption is of the form

$$c_t = \beta A_t k_t^{\alpha}$$

but we don't know what β is, or whether in fact this really will work. So we plug this in to the first order condition to see if we can make it work.

$$\frac{1}{\beta A_t k_t^\alpha} = \frac{1}{1+\theta} E\left[\frac{\alpha}{\beta k_{t+1}}\right].$$

We know how k_{t+1} evolves under our suggested rule of consumption, or $k_{t+1} = (1-\delta) k_t + (1-\beta) A_t k_t^\alpha$ so we have

$$\frac{1}{\beta A_t k_t^\alpha} = \frac{1}{1+\theta} E\left[\frac{\alpha}{\beta (1-\beta) A_t k_t^\alpha}\right]$$

and we can do some algebra, and then evaluate the expectation which is now all in terms of stuff we know.

$$1-\beta = \frac{\alpha}{1+\theta}$$

This means that if we set $\beta = 1 - \frac{\alpha}{1+\theta}$, then it must be the case that our FOC holds. So we say that

$$c_t = \left(1 - \frac{\alpha}{1+\theta}\right) A_t k_t^\alpha.$$

A.3 Social Security in the OLG Model (Example 17)

We showed that the evolution of the capital stock with social security was

$$k_{t+1} = \frac{(1-\alpha)}{(1+n)(2+\theta)} k_t^\alpha - \frac{d}{1+n}\left[\frac{1+\theta}{2+\theta}\left[\frac{n+\delta-\alpha k_t^{\alpha-1}}{1+\alpha k_t^{\alpha-1}-\delta}\right]+1\right]$$

and then asserted that this had k_{t+1} less than it would be in the non-social security world. This is the same as saying that the second term above is positive. When is

$$\frac{d}{1+n}\left[\frac{1+\theta}{2+\theta}\left[\frac{n+\delta-\alpha k_t^{\alpha-1}}{1+\alpha k_t^{\alpha-1}-\delta}\right]+1\right] > 0.$$

This is the same as asking if

$$\frac{1+\theta}{2+\theta}\left[\frac{n-r}{1+r}\right] > -1$$

which can be solved to show that

$$\begin{aligned} n &> -\frac{2+\theta}{1+\theta}(1+r)+r \\ &> r\left(\frac{-1}{1+\theta}\right) - \frac{2+\theta}{1+\theta} \end{aligned}$$

or that n is non-negative, and if it is negative, is can only be slightly negative. Since $n > 0$ almost everywhere and at all times, this seems like it is satisfied.

Appendix B

Problems

B.1 Consumption/Savings

1. Gertrude lives for two periods. In the first period she earns a wage of 100. In the second period her wage is zero. She earns interest on her savings of $r > 0$. Her utility is $U = c^{1-\sigma}/1 - \sigma$ and she has a zero discount rate. For what values of σ will her first period consumption be equal to 50? For what values will it be less than 50? For what values will it be exactly equal to 50?

2. Let's say your utility function is $U = x^3$. If you live for two periods and have \$300 to allocative across those two periods, what is your optimal strategy? Now, let's say you are offered either \$150 or a chance to flip a coin for either \$100 or \$200 - which do you choose? What property of your utility function determines these choices?

3. Calculate the intertemporal elasticity of substitution for a CRRA utility function. The intertemporal elasticity is defined as

$$\frac{\partial \left(c_2/c_1\right)}{\partial \left(1+r\right)} \frac{\left(1+r\right)}{c_2/c_1}. \tag{B.1}$$

4. Consider a two-period Fisher model in which utility in each period is CRRA. Wages are equal to W in each period. Solve for the derivative of first period savings with respect to the interest rate: $\partial S/\partial r$. For what values of σ is the sign of $\partial S/\partial r$ always negative? What are the conditions under which $\partial S/\partial r$ is positive?

5. Set up the Lagrangian for a two period Fisher model with wages equal to W_1 and W_2, a discount rate of θ, and an interest rate of r. Solve for the Euler equation relating consumption in period 1 and period 2. Does the relationship of consumption between period 1 and period 2 depend on a) the size of wages or b) the distribution of wages? Now imagine

that person lives in a world with no financial system, so that they cannot save or borrow at all. They have to consume exactly their wages in each period. Solve for the interest rate that makes this the optimal outcome in the Fisher model. How does this interest rate you just solved for change with the ratio of W_2/W_1?

6. This is a two period model. Anyone can borrow at the rate $r_b = 1$ (100% interest) and anyone can save at the rate $r_s = 0$. Their preferences are given by $V = \ln c_1 + \ln c_2$. Find optimal consumption for the following individuals who have different wage profiles in their life. A) Mr. Pink has $w_1 = 32$ and $w_2 = 32$, B) Mr. White has $w_1 = 0$ and $w_2 = 64$, C) Mr. Orange has $w_1 = 24$ and $w_2 = 40$. Who has the highest lifetime utility?

7. Danny lives for two periods. In the first period he has an income of X. In the second period he has an income of zero. He can save money at a real interest rate of zero, but he cannot borrow. He is born with zero assets and dies with zero assets. He has log utility and his time discount rate is zero. There is a government program with the following rules: if the sum of your income and your savings in any period is less than some level M, then the government will give you enough money so that you can consume M. If the sum of your income and savings in a period is greater than M, then the government gives you nothing. Describe the relationship between Danny's first period savings rate (S/X) and X. If you can't get an exact solution, draw a picture and discuss what the solution might look like.

8. Rusty will live for two periods with certainty, and he'll live for a third period with probability 50%. He'll find out at the end of period one (that is, after he's done his first period consumption) whether he'll die at the end of period two or period three. He has earning of one in the first period, and no income after that. Rusty has CRRA utility, a time discount rate of zero, and faces an interest rate of zero. He can't die in debt. A) Derive the equation you'd solve to get Rusty's optimal first period consumption - you don't actually have to solve it out. B) A government program is introduced that takes away T in takes from every worker in the first period and pays $2T$ to all people who survive into the third period of life. What is the value of T that maximizes Rusty's utility? (You don't have to use algebra for this, you can actually just reason out the answer).

9. Tess lives for two periods and in the first period she has income of 8 dollars. In the second period her income is either 0 or 8, each with probability of 50%. The interest rate is zero. Her utility is quadratic, $U = c - 0.05c^2$ and she has a discount rate of zero. Tess can take a test that will tell her what her second period income will be, and she can take this test prior to consuming in the first period. If she does not take the test, she will not know her second period income until after the first period is over. The test costs 2 dollars. Calculate whether Tess should take the test or not.

10. This is a two period model in which people work in both periods. They start with no assets and die with no assets. They can borrow and lend at $r = 0$, and their $\theta = 0$. Everyone has log utility. In the first period, everyone earns \$10. In the second period half the people earn \$15 and half make \$5. In the first period, no one knows which group they belong to. A) The government is considering cutting taxes by \$1 per person in the first period (raising income to \$11) and increasing taxes in the second period to pay back the debt (which amounts to \$1 per person). Taxes in the second period can either be lump sum of \$1 per person or can be proportional to 10% of each persons income in the second period. How does each of these tax plans affect utility and the national savings rate (that is, total savings divided by total income). B) A test is invented that will tell people at the beginning of period 1 what their income in period 2 will be. How would the availability of this test affect utility? How would it affect savings?

11. This problem is about certainty equivalence. Ty has quadratic utility of $U = \beta_0 + \beta_1 c - \beta_2 c^2$. He is a simple man. He lives only one period and so he consumes everything he earns in that one period. A) First, assume Ty has certain income of \$15 and calculate his total utility and marginal utility of consumption. B) Now assume Ty receives either \$10 or \$20 with probability 50%. Again calculate his total utility and marginal utility. C) In which situation is total utility higher? In which situation is marginal utility higher? D) Now Ty lives for two periods. In the first period he gets \$15 with certainty. Will his savings behavior in the first period depend on whether his second period income is \$15 with certainty or is \$10/\$20 with 50%? Explain.

12. Consider a one-period model in which individuals make a labor/leisure decision. Individuals have one unit of time. Utility is $U = \ln n + c^{1-\sigma}/1-\sigma$ where n is leisure and c is consumption. People take the wage w as given. They live only one period. They earn $(1-n)\,w$ and consume all their earnings. Show how the effect of the wage on labor supply depends on the value of σ. For what values will an increase in the wage have a positive effect on labor supply, and for what values will the effect be negative?

13. Take the usual intertemporal optimization problem over consumption and leisure. Variation in consumption and leisure can be caused by two things: variation in the interest rate or variation in the real wage. For which of these two causes will consumption and leisure be positively correlated and for which will they be negatively correlated? Explain.

14. Take a model where individuals live for two periods. Utility is $U = \ln c_1 + \ln c_2 + \ln n_1 + \ln n_2$. Eac individual has one unit of time. There are two types of individuals. Some earn a wage of 1 in the first period and 2 in the second. The others earn 2 in the first period and 1 in the second. Individuals cannot borrow, but that can save at an interest rate of zero.

Individuals are born with zero assets and die with zero assets. A) Without doing any math, which type of person is likely to have more variation in actual earnings ($w(1-n)$) between periods? B) Calculate total earnings ($w(1-n)$) in each period for each type of person. Does this match your intuition from part A?

15. A person may live for one, two, or three periods. At the end of period one, there is a 50% chance he will die, and a 50% chance he lives in the second period. If he does live to the second period, then at the end of period 2 there is a 50% chance of living into the 3rd period and a 50% chance of dying. His wealth at the beginning of period 1 is A, and he earns no wage income. The interest rate and discount rate are both equal to zero. He has log utility. Solve for his optimal consumption plan.

16. People live for two periods and wish to maximize their lifetime utility of $U = \ln c_1 + \ln c_2$. They have a zero discount rate and face a positive interest rate r. They have two options for their time. They can work or they can go to school. The time spent in school is n and the time spent working is $(1-n)$. Time spent in school creates human capital next period according to the formula $h_{t+1} = (n_t h_t)^{\alpha} x^{1-\alpha}$ where x represents public education spending. Note that human capital in period $t+1$ depends on your human capital in period t. Individuals are endowed with a value of h_1 which can be thought of as innate ability. Income depends on the base wage, human capital, and time working as $y_t = wh_t(1-n_t)$. The base wage rate w is fixed. A) Assuming individuals can borrow and lend freely, what is the optimal choice of education over time (n_1, n_2)? B) Now assume that individuals are financially constrained and can neither borrow nor lend. Now what is the optimal choice of education over time? C) The government is planning on raising education spending (increasing x). Compare the effect of this plan in both financial situations on time spent on education and on second period income. D) Do high ability individuals (those with large h_1) gain more second period income from the increase in x in either financial situation? Does this tell you anything about how education spending influences inequality within the economy?

17. Consider an economy that lasts for two periods, and is made up of two generations. The partents live only in the first period and the children live only in the second period. A family is composed of one parent and one child (and don't worry that they aren't alive at the same time). There is also a government, which exists for both periods. The interest rate is exogenous and equal to zero. People in the first period receive a before tax income of y_1 and people in the second period earn a before tax income of y_2. The government can impose lump-sum taxes (or transfers) of t_1 and t_2. The government could also levy a proportional inheritance tax, at rate t_b: if the size of bequests left is b, then the tax collected will be $t_b b$. The government faces the same interest rate (*zero*) as the people. It does not do any spending other than on transfers. The government must have

a balanced budget over the two periods, so $t_1 + t_x + t_b b = 0$. A) Consider the case where a familiy has the utility function $U = \ln c_1 + \ln c_2$ where the subscripts refer to the generations. The tax on bequests, t_b, is equal to zero. Calculate the size of bequests left as a function of y_1, y_2 and t. B) what happens to the size of bequests left if the government raises first period taxes by some amount d? C) Does Ricardian equivalence hold? (That is, does the pattern of consumption change due to this tax increase). D) Now suppose everything is the same, except that t_b is positive and less than one. Assume that the government sets taxes in period two so that is budget is balanced given taxes in period one aside plus any inheritance taxes collected. But not that second period taxes are exogenous from the perspective of families. Now, what is the size of bequests, and what happens if first period taxes go up by d? Now, does Ricardian equivalence hold?

18. A man has a potential lifespan of 100 years. For the first 50, he lives with certainty. After the 50th year, he will enter a battle, and there is a 50% chance that he will survive. If he survives the he will live with certainty for another 50 years. At birth he has \$100, and does not earn any other wages. He cannot die in debt. He has log utility, and the interest rate and discount rate are both zero. Solve for initial consumption.

19. Madeline lives for two periods. In each period she earns income of 1. Her utility is $U = \ln c_1 + \ln c_2$. She can borrow or lend some asset between periods 1 and 2 that has a real interest rate of zero. However, there is a 50% chancee that between periods 1 and 2 all debts and financial wealth will be wiped out. That is, if she borrows, there is a 50% chance that she will not have to pay back the loan, and if she saves, there is a 50% chance that all her savings will disappear. Solve for her optimal first period consumption.

20. Nicole lives for two periods. In the first period she has income of 3. In the second period she will have income of zero. The interest rate and the time discount rate are both zero. She cannot borrow and her utility function is $U = \ln c_1 + \ln c_2$. There is a government welfare program that provides a consumption floor of $c_{\min}$ in the second period. In other words: if she does not have enough money left over to afford to consume $c_{\min}$, then the government will give her enough money so that she can afford it. A) Obviously, for high enough values of $c_{\min}$ Nicole will spend all her income in the first period and just consume $c_{\min}$ in the second. For low enough levels of $c_{\min}$ she will act as if the program did not exist. Calculate the critical level of $c_{\min}$ at which she is indifferent between these two strategies. B) Now suppose there is only a 50% chance she'll be alive in the second period. Now calculate the critical value of $c_{\min}$.

21. Assume that the following facts are true. 1) OVer the long term ,the real wage has risen dramatically. 2) Over the long term, the fraction of

their time that people spend working has remained constant, and 3) over the long term the ratio of per capita consumption to the real wage has remained constant. Assuming that utility is given by

$$U = \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \frac{n_t^{1-\gamma}}{1-\gamma}. \tag{B.2}$$

Based on the three facts, what can you conclude about the value of σ?

B.2 Mechanical Growth

1. Output per worker is 5 times higher in country A as in country B. Assuming that the countries have identical Cobb-Douglas production functions, and that capital's share is 1/3, by what factor does MPK differe between the countries. What if capital's share is 2/3?

2. The production function is $y = k^{1/2}$. Suppose that capital per worker is 400. The savings rate is 50%. The depreciation rate is 5%. There is no population growth or technical change. A) Is the country at its steady state of output per capita, above its steady state, or below its steady state? B) Now suppose there is a large immigration into the country. The population quadruples. The new immigrants do not bring any new capital with them. Following the immigration is output per capita above, below, or at the steady state? C) What is the growth rate of income per capita immediately after the immigration?

3. There are two countries, A and B, each described by the Solow model. In both, the production function is $k^{1/2}$, and the depreciation rate is 0.02. There is no technological change. In A, the savings rates i 0.2, and population growth is zero. In country B the savings rate is 0.4 and population growth is 0.02. Both countries are observed to have income per capita of 5 at time zero. Draw a graph with time on the horizontal axis and income per capita on the vertical axis showing how the level of income per capita in the two countries will evolve over time.

4. Consider a Solow model with positive rates of population growth, depreciation, and technological change. Imagine a country is in steady state, and suddenly its rate of technological change increases. Describe how output per efficiency unit evolves over time. Describe how output per person evolves over time. If you have trouble with the math, draw the graphs.

5. Two countries are described by the Solow model with $y = k^{1/2}$. In both, $n+\delta = 0.1$. In country A, $s = 0.1$ while in country B, savings are a function of the capital stock, $s = 0.2\left(\frac{1}{1+k}\right)$. A) Show that the two countries have the same steady state, B) Solve for the growth rate of income per person. If both countries start with the same stock of capital per person, which country will grow faster? Will this country always grow faster?

6. Consider a Solow model in which the growth rate of population is endogenous. Specifically
$$n = \begin{cases} n_h & \text{if } y < \bar{y} \\ \\ n_l & \text{if } y \geq \bar{y} \end{cases} \tag{B.3}$$
where $n_h > n_l$. Production is $y = k^\alpha$. Capital depreciates at δ and there is no technological change. The savings rate is exogenous. A) Draw graphs showing the different possible configurations of the steady state values of capital per person (i.e. single steady states, multiple steady states) and lable each steady state as stable or unstable. B) For what values of the savings rate with the model display multiple steady states? For what values will there be a single steady state?

7. Consider an economy with two kinds of capital. k is private capital, and z is government capital (highways, ports, dams, etc.). The production function is $y = k^{1/3}z^{1/3}$. Both types of capital depreciate at the rate δ. There is no population growth or technological change. The government collects a fraction, τ, of output that it uses to build new government capital. Of the remaining output, s is saved to invest in building new private capital. A) Write down the equations of motion for k and z. B) Solve for the steady state values of k, z, y. C) What rate of taxation maximizes income per capita in the steady state?

8. Consider a country with a production function of $y = k^\alpha$. Population grows at the rate n and capital depreciates at the rate δ. There is no technological change. Consumption is each to a constant fraction of output, denoted $\bar{c}$. In addition, every period a payment in the amount of p per capita must be made to the foreign power which provides protection for this country. All output that is not consumed or paid to the foreign power is invested. A) Write down the differential equation governing the evolution of the per capita stock of capital in this country. B) Draw the Solow diagram for this country. Is there more than one equilibrium level of the capital stock and of output? Is so, identify all the equilibria. Indicate the dynamics on the diagram - that is, show to which equilibria an economy will move given it's initial stock of capital per person.

9. An economy is described by the Solow model with $y = k^{1/2}$, and some values of n and δ. Currently the savings rate is 0.6, and the country is at its steady state. Two course of action are proposed. One is to lower savings to 0.5, and the other is to lower savings to 0.4. Graphs the paths of consumption per capita over time following the introduction of each policy, and show how these consumption levels compare to consumption in the old steady state. Explain.

10. Two countries are described by the Solow model and have the same savings rates, the same values of n,δ, and g but differ in their technology. $A_1 \neq A_2$ and their production functions are $y_1 = A_1 k_1^\alpha$ and $y_2 = A_2 k_2^\alpha$. Both

countries are in steady state. A) By what factor do the two countries capital stocks differ? B) By what factor do the two countries values of A differ? C) By what factors do the two countries MPKs differ?e

11. A country has the production function $Y = A(bK)^{\alpha}(cL)^{1-\alpha}$ and the growth rate of A is g_A, the growth rate of b is g_b and the growth rate of c is g_c. Population growth is zero. Calculate the growth rate of output per capita in the steady state of a Solow model.

12. There are two countries described by the Solow model. They both have the same savings rates, population growth rates, and depreciation rates. There is no technological change. The production function in both is $y = k^{\alpha}$. In country 1 there is a proportional tax rate of τ on income. Savings is done from post-tax income. In country 2 there is constant per capita tax of Ψ per person. Savings is done out of post-tax income. The size of the per person tax Ψ is set so that taxes per person are equal to those collected in steady state by the country with the proportional tax. A) Solve for Ψ. B) Describe the dynamics of the two economies. Show that there are different possible configurations of steady states in country 2 relative to country 1. C) Show how the different configurations of steady states in 2 relative to 1 depend on the parameters. Derive the cases in which each possible configuratino will occur.

13. Two countries have production functions of $y = k^{1/3}h^{1/3}$ and physical and human capital are accumulated according to

$$\begin{aligned}\dot{k} &= s_k y - (n+\delta)k \\ \dot{h} &= s_h y - (n+\delta)h.\end{aligned}$$

Two countries have the same levels of s_k, but differ by a factor of two in the size of s_h. By what factor will their steady state levels of output per worker differ?

B.3 Dynamic Optimization

1. Rodney lives for 40 periods. He has CARA utility, or $U = -(1/\alpha)e^{-\alpha c}$. He earns 100 dollars in each period, and the interest rate is r and the discount rate is θ. He has zero assets when he is born and will die with zero assets, but Rodney can borrow and save during his lifetime. A) Derive the FOC relating consumption in adjacent periods of life. B) Assume that $\alpha = 1, \theta = 0.05, r = 0$ and derive optimal first period consumption. You'll have to use the approximation that $\ln(1+x) = x$ which holds for values of x close to zero.

2. Consider the optimal consumption path of Abigail, who will live exactly T periods. In the first period, she has earnings of one dollar. Subsequently, her earnings grows at the rate g in each period, so that second period

wages are $(1+g)$, third period wages are $(1+g)^2$, and so on. Abigail can borrow and lend freely at an interest rate of zero, and she has a zero discount rate. Her utility is CRRA. She starts life with zero assets and dies with zero assets. Calculate her optimal savings in the first period of life. What is the effect of increasing g on her first period saving. Explain.

3. Crockett and Tubbs have the same preferences (they have identical utility functions and identical discount rates). Both are born at time zero and die (with certainty) at time T. Both face the same interest rate. Each is born with zero assets and dies with zero assets. Both are liquidity constrained: they are never allowed to have negative assets. They have different lifetime wage paths. There is no uncertainty, and both Crockett and Tubbs know exactly, in advance, their entire path of wages over their life. You observe Crockett's consumption growing at a constant, positive rate over his life, while Tubbs' consumption declines at a constant rate over his life. The rate at which Tubbs' consumption declines is smaller than the rate at which Crockett's consumption rises. However, the present discounted value of their lifetime consumption is identical. Which man has higher lifetime utility? Explain.

4. Lacy has wealth of A at time zero, and she will live infinitely, but she will not earn any income in her life. The interest rate is zero. She is unable to borrow. She has log utility with some discount rate $\theta > 0$. Time is continuous. A) What is her optimal growth rate of consumption? Solve for her optimal initial consumption level. B) Now there is some government program that works as follows: if Lacy has any wealth at all (positive A), then they receive nothing. If Lacy has zero assets, then the goverment will provide her with some consumption level of M. Now solve for her optimal consumption strategy and intitial consumption level. Two hints - try thinking first about when $\theta = 0$ and don't do this in discrete time.

5. Bobby is trying to decide how to consume for the rest of his life. The interest rate is r and his discount rate is zero. Bobby has no income, but does have a stock of wealth, A_0. He faces a constant probability of dying, p each period. So the probability he is alive in period t is $(1-p)^t$. His utility is

$$E(U) = \sum_{t=0}^{\infty} (1-p)^t \frac{c_t^{1-\sigma}}{1-\sigma} \tag{B.4}$$

A) What is the relationship of his risk aversion (σ) and the expected size of Bobby's bequest (the amount of money left over when he dies). Be sure to distinguish between when $p > r$ and $p < r$. Explain the intution behind this. B) Suppose that Bobby has log utility. Solve for his assets at the beginning of a generic period t, A_t as a function of p, r, t, and A_0. C) Now presume that Bobby can buy an annuity which pays at the rate z. Given a perfectly competitive insurance market, what is the relationship of z to r and p. D) Suppose again that Bobby has log utility. Solve for his first period consumption if he buys the annuity. Is this higher or lower than if he doesn't buy the annuity?

6. Carl is born at time zero and will live forever. He is born with assets of \$10 and has income of \$1 per period. His discount rate is 5%. He has CARA utility of $U = -e^{-C}$. There are two interest rates. Carl can borrow at the rate of 5% and save only at 0%. Solve for Carl's optimal path of consumption. What is consumption at time zero? Draw a picture of the time path of consumption, labeling any inflection points. Note: to solve this in continuous time requires math we didn't cover. It's easier to do in discrete time, but you'll need that $\ln(1+x) = x$ and $\ln(1/(1+x)) = -x$.

7. Consider a Ramsey model with depreciation of δ, population growth of n, a time discount rate of θ and production function of $y = k^{1/2}$. Solve for the steady state level of consumption per capita in terms of the three parameters.

8. Consider two Ramsey economies which are the same in every respect except for their time discount rates, θ. People in country A discount the future more than those in country B. Assume both countries start off at the same initial capital stock, k_0, which is below both of their steady states. Which country will have higher initial consumption? Is it possible for the two countries stable arms to cross?

9. This is a variation of the OLG model presented in class. Individuals live for two periods. In the first period they work, supplying one unit of labor. They do not consume anything in the first period of life. In the second period, they do not work, and they consume their savings (along with any accrued interest). The economy is closed. The production function is $y = k^{\alpha}$. The depreciation rate is zero. The population growth rate is n. A) Solve for the steady state level of capital per worker. B) Assume $\alpha = 1/3$. For what value of n will the economy be at the golden rule level of capital per worker?

10. This is a variation of the OLG model in which people may die at the end of their first period of life. The is a p probability that people will live in the second period, and they do not know whether they will live or die until it happens. There is no time discounting and expected utility is $EU = \ln c_1 + p \ln c_2$. The production function is $y = k^{\alpha}$ and there is no population growth. Assume that when people die, their wealth is distributed among the remaining members of their generation. Assume also that there are enough people in each generation so that there is no uncertainty about the size of bequests (the law of large numbers holds). Also assume that interest is earned by the capital that belongs to the deceased prior to it being divided up amongst the remaining population. A) Write down the single period and lifetime budget constraint of an individual. Call the amount received as a bequest, b. B) Solve for the individuals optimal savings in peirod 1 as a function of b, r, and w. C)

Solve for b as a function of the amount of capital in the second period. D) Put everything together into a difference equation for k. E) How does an decrease in p affect the steady state capital stock?

11. Consider the optimal consumption problem for a person who will live infintely. TIme is discrete. The person earns a wage of w in each period. There is a positive interest rate r and a positive discount rate of θ. The person, though, is responsible for the consumption of her entire family, not just her own. Her family size in each period is N_t and increases at the rate n so that $N_{t+1} = (1+n) N_t$. These extra family members do not work and do not have nay assets of their own. THe per period utility function of the woman is $U = \ln C_t/N_t$ so that what she cares about is the consumption per person in her family. A) Set up the Lagrangian and write down the FOC, solve for the Euler equation. B) Solve for consumption in period t as a function of consumption in period zero and then use this to solve for initial consumption. C) Now presume that utility is $U = N_t \ln C_t/N_t$ and solve the problem again for initial consumption.

12. The economy is described by the Ramsey model. Individuals have a discount rate of θ and CRRA utility with an intertemporal elasticity of substitution of σ. Production is $y = k^\alpha$. Depreciation is equal to zero. The economy is subject to capital taxation at the rate τ. That is, the return to capital from an individuals perspective is $(1-\tau) f'(k)$. Government revenue from this tax is $G = \tau f'(k) k$. The government tax does nothing to enhance utility or productivity. A) Describe the steady state values of k and c. B) The government is going to lower the tax rate, and are concerned how this will affect their revenue. Calculate the "static scoring" $\partial G/\partial \tau$, or the immediate change in G following the tax decrease. C) Now calculate the "dynamic scoring" $\partial G/\partial \tau$ that takes into account the long run effect of the tax increase. Under what conditions is $\partial G/\partial \tau < 0$? D) Draw the Ramsey diagram for this economy. On the same graph show how a drop in τ affects the dyanmics of the system. On a separate graph show the paths of c, k, y due to the drop in taxes. E) Let's say the government is going to abolish the capital tax altogether and replace it with a tax on labor. Take home wages will be $(1-\tau_w) w$. Government revenue is now $G = \tau_w w$. Do the static and dynamic tax scoring methods produce different results under this tax? Why or why not?

13. Utility is defined over both consumption and leisure, so that

$$V = \sum_{t=0}^{\infty} \frac{\ln c_t + \ln n_t}{(1+\theta)^t}. \tag{B.5}$$

Wages are given by $w_t = w_{t+1} = ... = \bar{w}$ (they are constant over time). There is no capital, and therefore no means to transfer output from one period to another. Consumption in any period is exactly equal to what you earn. The government is considering imposing a tax on this economy, but only for one period, (call this period k). It has two options. The first is a lump sum tax of the amount g, so that consumption is $c_k = (1 - n_k)\,\bar{w} - g$. The second is a proportional tax charged on income so that consumption is $c_k = (1 - \tau)\,(1 - n_k)\,\bar{w}$. A) Write down all the FOC that hold in period k. B) For the lump sum tax, solve for the optimal leisure time in periods k and $k + 1$. Solve for the interest rate that holds between k and $k+1$. C) For the proportional tax, solve for optimal leisure time in periods k and $k + 1$ as well as the interest rate between k and $k + 1$. D) The government wants to collect the same amount of revneue regardless of the tax plan chosen. Solve for the tax rate τ that yields the same amount of tax revenue as the lump sum g, explicitly taking into account people's optimal leisure choices. E) Using the answer to D, solve for which tax plan will cause a bigger change in interest rates. Which tax plan causes a bigger change in consumption? F) In words, why is there a difference between lump sum and proportional taxes?

14. There is a Ramsey economy with a government sector, so that capital accumulation is defined by

$$\dot{k} = f\,(k) - c - nk - G.$$

There are two levels of government spending, $G_h > G_l$. Spending alternates between these two levels in a fixed cycle. Use the phase diagram to describe how consumption and capital will behave in this economy once the pattern has been in place for a long time. Draw times series pictures of how the interest rate behaves over this "political cycle". Consider the limiting case as the period of alternation becomes very short or as it becomes very long.

15. Take a OLG economy in which people work in the first period of life, but do not consume in the first period at all. In the second period they do not work, but do consume their savings plus interest. The economy is closed. THe production function is $y = k^{\alpha}$. The depreciation rate is zero and population grows at the rate n. That is, each generation is $N_{t+1} = (1 + n)\,N_t$. A) Solve for steady state capital per worker. B) Assume that $\alpha = 1/3$. For what value of n will this economy be at the Golden Rule level of the capital stock?

B.4 Open Economy

1. There is a country described by the Solow model, with no technological growth. The country is open to capital flows, and takes the world interest rate of r^* as given, implying some level of capital per person of k^*. Initially the country begins with assets per person less than k^*. A) write down the equation for teh evolution of assets in the economy. Assuming that $s\,(r + \delta) < (n + \delta)$, solve for the steady state of a^*. B) Assuming that

the domestic savings rate is s, and this is less than the world savings rate, solve for the ratio of a^*/k^*. C) What is the ratio of GNP/GDP for this economy at the steady state? D) What is the ratio of wages to GNP in this economy? E) Now assume that the domestic savings rate jumps above the world savings rate. Draw a graph showing how both GDP and GNP evolve following this.

2. A country is initially closed to the international capital market. Production and consumption are as described by the Ramsey model. The autarky interest rate is r^a. The world interest rate is r^* and is greater than the autarky rate. The economy is small. Assume that opening to the world economy is a surprise. Prior to this, the economy was in a steady state. Describe the time paths of consumption, output, and net foreign assets at the time of the opening and after. Be sure to indicate which variables jump and which do not jump at the time the economy opens up. [Figuring out what happens to consumption initially may not be possible, but say what you can about it].

3. There are two countries with equal populations. Income is exogenous and identical in the two countries. There is no means of storing output, so income in a given period has to be consumed in that period. There is no population growth. Both countries have CRRA utility functions, but country A is more risk averse: $\sigma_A > \sigma_B$. The time discount rate in both countries is zero. Income per capita at time zero is equal to one. Following this it grows at the rate g. A) Suppose there was no trade between the countries, what would the interest have to be in each country? B) Now suppose that both countries can trade with each other. Sketch out what would happen to the path of consumption per capita in each country. Also sketch the path of world interest rates. Toward what level with the world rate asymptote? What will the asymptotic growth rates of consumption be in the two countries, and what is the asymptotic ratio of consumption between the two countries? How will consumption at time zero in both countries compare? You don't have to do this with fancy math.

4. Consider an open economy OLG model. There is no depreciation, no time discount rate, no technological change, and no populatino growth. Utilty is $U = \ln c_1 + \ln c_2$. The world interest rate is r^* and is taken as exogenous. People can borrow or save at this rate. The production function is $y = k^\alpha$. Calculate the steady state levels of capital, wages, savings, and wealth in the country as a function of r^*. What determines in the steady state whether the country has positive or negative net exports?

5. Take an OLG model with capital mobility. There are two countries in the world with equal populations. There is no population growth, no technological change, and no depreciation. The production function is identical in both countries: $y = k^{1/2}$. In each country, people work in the first period of life and consume in both the first and second periods of life.

Their utility functions are

$$\begin{aligned} \text{Country 1:} \quad & U = (1-\gamma)\ln c_1 + \gamma \ln c_2 \\ \text{Country 2} \quad : \quad & U = (1-\beta)\ln c_1 + \beta \ln c_2 \end{aligned}$$

and $1 > \gamma > \beta > 0$. A) Solve for steady state GDP in each country. B) Solve for steady state GNP in each country.

6. A small open OLG economy faces a world interest rate of r^*. Production is $Y_t = K_t^{1/2}(e_t L_t)^{1/2}$ where e measures efficiency of labor and increases according to $e_{t+1} = (1+g)e_t$. Every period, a new generation is born and lives for two periods. In the first period of life people work and consume. In the second period they consume, but do not work. There are no bequests or intergenerational transfers. People have utility of $V = \ln c_1 + \ln c_2$. Derive an expression showing whether, in steady state, this country has positive, negative, or zero net foreign assets.

B.5 Endogenous Growth

1. This is an OLG problem. Individuals live for two periods. In the first period they work, supplying one unit of labor. They do not consume in the first period. In the second period they do not work, but consume their savings with interest. Individuals are born with zero assets and die with zero assets. There is no population growth. Production occurs in many small identical firms with the production function $Y_i = BK_i^{\alpha}L_i^{1-\alpha}$ where K and L are the capital and labor of firm i. Firms pay factors their marginal products. The aggregate productivity term B is determined by the aggregate capital stock, $B = AK^{1-\alpha}$. Firms do not take into account the effect of their own capital stock on B. Solve for the steady state of this model. If there is a steady state of output or a steady state growth rate, solve for it. If no steady state exists, explain why.

2. This is a Lucas-style model of growth with human capital. Output is produced according to $y = k^{\alpha}[(1-u)h]^{1-\alpha}$ where $(1-u)$ is the fraction of time spent working. There is no population growth. Physical capital accumulates according to $\dot{k} = sy - \delta k$ where the savings rate is exogenous. When workers are not working, they are acquiring human capital. The only input to this is time, and the evolution is $\dot{h} = u - \delta h$. Assume that $s = \delta$ for simplicity. A) Describe the steady state of the model. Is it one with constant level of output or a constant growth rate of output per person? B) Solve for the level of u that maximizes steady state income or growth rate (depending on which one is constant in steady state). C) Let u^* be the maximizing value from B. Suppose the economy is at $u < u^*$ and then jumps to u^*. Use a phase diagram in h, k to analyze the behavior of h and k in response to this jump. Draw time series pictures showing how both respond in their transition to the new steady state.

3. Consider a model of endogenous technology and population. Population growth depends on the difference between current income and subsistence income, or $n = B(y - \bar{y})$ where $\bar{y}$ is the subsistence level. Output is produced with labor L and land R, according to $Y = (AR)^{\alpha} L^{1-\alpha}$. Technology growth depends on output per capita, or $\dot{A}/A = gy$ and $0 < g < B$. Analyze the model's dynamics and steady states. Solve for any steady state values of output per capita.

4. Consier a model with two factors of production, physical and human capital. Production is $y = k^{1/2}h^{1/2}$. Equations of motion are $\dot{k} = s_k y - (n + \delta) k$ and $\dot{h} = s_h y - (n + \delta) h$. A) Will this model have steady state growth or steady state output per capita? B) Solve for the appropriate steady state. C) Suppose that initially $s_k = 0.2$ and $s_h = 0.3$. Now s_h drops to 0.25 and s_k rises to 0.25. Use a graph to show what will happen to the ratio of physical to human capital. Graph what happens to the growth rates of human capital and physical capital over time following this change in the savings rate. D) What is the effect of this change in the allocation of savings on the long run growth rate of k, h, y?

5. Consider the following variation on the Solow model. Suppose that the true production function is $y = A_1 k^{\alpha} + A_2 k$. There is no exogenous technological change. Population grows at n and capital depreciates at δ. Assume all countries in the world have identical values for A_1 and A_2 and that they all have the same savings rate. Countries differ only in their initial level of capital per person. Discuss the extent to which countries with different initial levels of capital per person will or will not converge over time. Distinguish, if appropriate, between different special cases based on the values of the parameters.

6. Consider an augmented Solow model which includes human capital. Let k be physical capital and h be human capital. Both capital stocks depreciate at the rate δ. There is no population growth. Let s_k be the fraction of output invested in physical capital and s_h be the fraction of output invested in human capital. The production function is $y = A_k k + A_h h$, where A_k, A_h are constants. Assume that $s_k A_k < \delta$ and $s_h A_h < \delta$. Analyze the dynamics of this model using a phase diagram in h/k space. Describe the paths of h and k from different initial positions. Under what conditions does the model produce "endogenous growth" and what happens when this condition is not met?

7. Consider the following model. Output is produced according to the production function $y = Ak$. There is no technological progress. Population grows at the rate n. A constant fraction, s, of output is saved. Capital depreciates in an unusual fashion: every period, d units of capital per person depreciates. Analyze the dyanmics of this economy. Describe the conditions under which there will be steady states, endogenous growth, etc.. Calculate the long-run rate of growth, if there is one.

8. A country has production of $y = Ak$. Capital depreciates at the rate δ. There is no population growth. Consumption is chosen by a social planner who maximizes
$$U = \int_0^\infty e^{-\theta t} \frac{c(t)^{1-\sigma}}{1-\sigma} dt \tag{B.6}$$
where $c(t)$ is consumption at time t, θ is the discount rate and $\sigma > 0$. The capital stock at time 0 is $k(0)$. Solve for the optimal value of initial consumption, $c(0)$.

9. Consider a two-country version of the Lucas human capital model. Production in both countries is $y_i = k_i^\alpha (u_i h_i)^{1-\alpha}$ and capital accumulation is $\dot{k}_i = s y_i - (n+\delta) k_i$. So far all of this is like the model presented in class. For human capital accumulation, we add in a new assumption. Let h_l be the level of human catpial per capita in the country that has a higher level of human capital ("the leader"), and h_f be the level in a country that has less human capital ("the follower"). We assume that human capital in the leader country works just like Lucas' model: $\dot{h}_l = \phi(1-u_l) h_l$. Human capital in the following country is produced by two methods. First there is production through basic accumulation, as before. But also, there is a spillover of human capital from the leader country. The amount of the spillover depends on the difference in their human capital stocks. So $\dot{h}_f = \phi(1-u_f) h_f + \beta(h_f - h_l)$ and $\beta > 0$. Assume the two countries have the same savings rate, but $u_1 < u_2$. Notice that this does not necessarily identify the leader and follower. A) Describe the steady state of the moel and solve for each country's growth in steady state. B) Which country is the leader and which is the follower? C) Solve for the relative level of human capital per person in the two countries in steady state. How do the parameters ϕ and β affect the ratio h_1/h_2? D) Solve for the relative consumption in the two countries in steady state. How (and why) do ϕ and β affect the relative consumption ratio c_1/c_2?

10. Consider a modification of the expanding quantity of goods model. Production is still $Y_i = L_i^{1-\alpha} \sum_{j=1}^N X_{ij}^\alpha$ and intermediate goods producers are still monopolists in their good. Final goods producers are perfectly competitive. The difference is that the cost of producing a new invention now depends upon how much labor is used, and is not just fixed at d. The quantity of labor required to invent a new product is η/N, and so the cost of producing a new product is $w\eta/N$. By having to use labor to produce new goods, less labor is available to work in the final goods sector. Therefore aggregate final goods production will actually be $Y = (L - L_R)^{1-\alpha} \sum_{j=1}^N X_j^\alpha$. Solve for the steady state growth rate of output, the steady state level of the interest rate, and the steady state number of researchers (L_R).

11. Imagine you are the dictator of an economy in which growth occurs through quality-enhancing innovations. Your goal is to mazimize the expected

present value of consumption y_t or

$$\max U = \int_0^\infty e^{-rt} E(y_t)\, dt \tag{B.7}$$

and given the quality model the income in period t is described by

$$E(y_t) = \sum_{k=0}^{\infty} \Pi(k,t)\, A_k x^\alpha \tag{B.8}$$

where $\Pi(k,t)$ is the probability that exactly k innovations will occur up to time t. With a constant choice of n, and the Poisson process for innovations, statistics tell us that

$$\Pi(k,t) = \frac{(\lambda n t)^k}{k!} e^{-\lambda n t}. \tag{B.9}$$

A) So you, as the social planner, need to choose x and n to maximize U subject to the constraint that $L = x + n$. You'll need to use the fact that $A_k = A_0 \gamma^k$ as well. B) Compare the optimal level of n you found as a social planner with the optimal n that we found in the decentralized economy - under what conditions on the parameters will growth in the decentralized economy be too fast?

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