The Role of Intermediate Goods in the Costs and Benefits of Trade*

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PRELIMINARY AND INCOMPLETE

Abstract

The Stolper-Samuelson theorem famously shows that abundant factors gain from trade while scarce factors lose. The supply-side so determines the outcome due to what Ronald Jones (1965) called the "magnification effect" of commodity prices on factor prices, that little emphasis has been placed on the demand side of the question. How does the demand side increase or decrease the gains to the abundant factor or losses to the scarce factors? Is it important whether countries export goods that loom large or small on the world stage? Does it matter whether the exported goods are intermediate or final? We try to show in this paper that the more important the good, final or intermediate, is on the world stage, the smaller the gains to the abundant factor and the larger the losses to the scarce factor. The fact that it appears to make no difference whether exported goods are used directly or indirectly for consumption purposes appears to justify the usual neglect of intermediate goods in the classical expositions of trade theory.

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1 Introduction

The benefits and costs of international trade are perhaps best exhibited by the famous Stolper-Samuelson theorem that trade benefits the abundant factor and hurts the scarce factor. The question, however, never seems to have been raised: is it better for the abundant or scarce factor to be intensively used in a good for which world demand is high or low? Ironically, this paper shows that it is better for the abundant factor to be used in a good for which world demand is low, and better for the scarce factor to be used in a good for which world demand is high. In other words, a country that exports goods that are in high demand, will find its abundant factors benefiting less and its scarce factors suffering more than in the opposite situation. Therefore, there is a larger down-side to exporting goods in high demand.

Why is this? The reason is straightforward in simple trade models. Trade raises (relative) export prices and lowers (relative) import prices. Accordingly, if the export good is in low demand, the consumption cost is low, raising the real wage even more for the abundant factor; and if the import good is in high demand, the consumption benefit is high, softening the blow to the real wages of the scarce factor from the lower import prices. Since calculations are difficult in a standard Heckscher-Ohlin model, to illustrate the proposition we use the Ricardian approach to the factor endowment theory of trade (Ruffin, 1988). It is hoped that by having a demonstration of the proposition will stimulate research to find out if there is any systematic evidence that supports it. We know for example that free trade is usually opposed for its distributional effects: are the negative effects really concentrated in the countries exporting goods in relatively high world demand? Mexico is an exporter of oil, for example, and faces wrenching debates over free trade.

We further investigate the role of intermediate goods in the costs and benefits of trade. Does it really matter whether the exported goods are intermediate or final? We show that the gains from trade would be the same regardless of what type of good the abundant factor exports. The fact that it appears to make no difference whether exported goods are used directly or indirectly for consumption purposes appears to justify the usual neglect of intermediate goods in the classical expositions of trade theory. This does not mean that intermediate goods should be neglected, as the literature on effective protection testifies, but that in empirical work on the importance of various goods on the world stage we need not worry about whether the good is final or intermediate—an important simplification that does no harm.

We explore this issue using the simplest possible factor endowment model, namely, the Ricardian factor endowment model spelled out by Ruffin (1988). This model has the distinct advantages of (1) universal factor price equalization and (2) ease of calculation.

2 The model

We explore this issue using the simplest possible factor endowment model, namely, the Ricardian factor endowment model spelled out by Ruffin (1988) where he combines the concept of comparative advantage resulting from the productivity differences of factors of production and the factor endowment theory of Hechker-Ohlin model. The model assumes that both goods and labor market are competitive and that individuals have identical homothetic utility functions represented by the same Cobb-Douglas preferences. Moreover technology is the same in both countries. The original model further assumes that there is no intermediate good which will be dropped in our model later in this paper. This model has the distinct advantages of (1) universal factor price equalization and (2) ease of calculation.

To see the role of demand and relative sizes of the countries on the gains from trade, let's start with the simplest framework: 2 countries 2 goods, and 2 factors of production.

Consider 2 types of labor (L_1 and L_2).

Let;

a_{ij} = The amount of type i labor that will produce 1 unit of good j.

b_1 = Share of income devoted for good 1

b_2 = Share of income devoted for good 2

Where $b_1 + b_2 = 1$

Assume without loss of generality;

$\frac{a_{21}}{a_{11}} > \frac{a_{22}}{a_{12}}$ (L_1 has a comparative advantage in Good 1).

L_1 will produce good 1 if and only if the price of good 1 when produced by L_1 is less than or equal to the price when produced by L_2.

$$w_1 a_{11} \leq w_2 a_{21}. \tag{1}$$

The same logic applies to good 2 as well. L_2 will produce good 2 if and only if the price of good 2 when produced by L_2 is less than or equal to the price when produced by L_1.

$$w_2 a_{22} \leq w_1 a_{12}. \tag{2}$$

These 2 conditions together define there regions where we see different productions patterns depending on where the relative wage ratio falls in.

$$\frac{a_{11}}{a_{21}} \leq \frac{w_2}{w_1} \leq \frac{a_{12}}{a_{22}} \tag{3}$$

If the wage ratio is strictly inside this interval then there there will be perfect specialization and each labor type will produce the good on which it has comparative advantage.In our case it is L_1 that will produce only good 1, and L_2 that will produce only good 2. We will refer to this perfect specialization case as *disjoint production* later in the paper. In case wage ratio is equal to $\frac{a_{11}}{a_{21}}$, then good 1 will be produced in common by both labor types such that L_1 will specialize in good , additionally some of L_2 will also produce good 1. Good 2 , however, will only be produced by the rest of L_2. Similar reasoning can be made for common production of good 2 when wage ratio is equal to $\frac{a_{12}}{a_{22}}$ which will be referred as *common production*. Relative wage ratio is endogenously determined in the model by the relative labor endowments, productivities, and demands for goods. Therefore , in equilibrium, inequality (3) translates to an expression of exogenously given parameters.

So, in order for production to be disjoint with L_1 producing good 1 and L_2 producing good 2. $\frac{L_2}{L_1}$ must be in the interval $\left[\frac{a_{22}}{a_{12}}\frac{b_2}{b_1}, \frac{a_{21}}{a_{11}}\frac{b_2}{b_1}\right]$.

Assume both $\frac{L_2}{L_1}$ and $\frac{L_2^*}{L_1^*}$ are in this interval which means that type1 will produce good 1, type 2 labor will produce good 2 in each country.

3 Equilibrium Under Autarky

Equlibrium conditions for disjoint and common production are different. When there is common production, say good 1 is produced in common, price of good 1 has to be the same no matter who produces it. Therefore the equilibrium condition is $w_1 a_{11} = w_2 a_{21}$, and inequality (3) converts to an equality where $\frac{w_2}{w_1} = \frac{a_{11}}{a_{21}}$. This can only happen when the country is endowed with too little L_1 and too much L_2, and neither the country nor the factors of production can gain from trade until and unless they switch to the disjoint production.[1] Thus, to see the role of demand on gains, disjoint production is analysed in detail in this paper.[2]

When there is disjoint production, L_1 spends b_2 share of its total income for good 2 which is prduced by L_2, and similarly L_2 spends b_1 share of its total income for good 1. Therefore an equilibrium condition under autarky is $w_2 L_2 b_1 = w_1 L_1 b_2$. Let w_1 be 1 and take it as a numeraire and write w_2 in terms of w_1.

$$w_1 = 1 \quad (4)$$

$$w_2 = \frac{L_1}{L_2}\frac{b_2}{b_1} \quad (5)$$

Therefore, inequality (3) can be rewritten as;

$$\frac{a_{11}}{a_{21}}\frac{b_1}{b_2} \leq \frac{L_1}{L_2} \leq \frac{a_{12}}{a_{22}}\frac{b_1}{b_2}$$

This condition reveals who will produce what , once the relative labor endowments is given.

Equilibrium prices will be;

$$p_1 = a_{11} w_1 \quad (6)$$

$$p_2 = a_{22} w_2 = \frac{L_1}{L_2}\frac{b_2}{b_1} a_{22} \quad (7)$$

[1] If a good is produced in common by both labor types before and after trade, gain to the abundant factor and the country will be 0. The country can only gain if it switches from common production to disjoint production which implies perfect specialization as in the classical Recardian Model.

[2] The analysis is the same if the country switches from common to disjoint prooduction except for the magnititudes of the gains. The same results hold for that case as well.

After getting the equilibrium wages and prices, one can compute the utility of a representative L_1 or L_2 using the Cobb-Douglass utility function. Let C_{ij} be the amount of good i counsumed by labor type j.

$$C_{ij} = \frac{w_j b_i}{p_i} \tag{8}$$

Inserting (4) and (6) into (8) yields;

$$C_{11} = \frac{b_1}{a_{11}}$$

and (5) and (7) into (8) yields;

$$C_{21} = \frac{L_2}{L_1}\frac{b_1}{a_{22}}$$

Then the utility of type 1 labor is $U^A_{L_1} = C^{b_1}_{11} C^{b_2}_{21}$ where the superscript "A" represents the autarky level.

$$U^A_{L_1} = \left(\frac{b_1}{a_{11}}\right)^{b_1} \left(\frac{L_2}{L_1}\frac{b_1}{a_{22}}\right)^{b_2} \tag{9}$$

Similarly, utility of type 2 labor, $U^A_{L_2} = C^{b_1}_{12} C^{b_2}_{22}$, is;

$$U^A_{L_2} = \left(\frac{b_1}{a_{11}}\right)^{b_1} \left(\frac{L_2}{L_1}\frac{b_1}{a_{22}}\right)^{b_2} \tag{10}$$

Notice that utility depends on the factor productivies, factor endowment and the demand for goods. The same calculations applies to the foreign country as well. The expressions are exactly the same except for the superscripts of factor endowments, (L^*_1, L^*_2).

4 Equilibrium Under Trade

At this point, let two countries start trading without any trade costs and distortions.[3]

The ratio of factor prices must be in the interval...... coming from the fact that $P_1 < P_{12}$ and $P_2 < P_{21}$

General setup of the model

Talk about the FPE

$w_i = \max\left(\frac{P_i}{a_{ij}}\right)$ and this holds for any number of goods.

Comparative advantage says that

In equilibrium; there can be three different equilibrium, disjoint

Type 1 labor will sell $w_2 L_2 b_1$; and purchase $w_1 L_1 b_2$. There is equilibrium when $w_2 L_2 b_1 = w_1 L_1 b_2$

$w_2 L_2 b_1 = w_1 L_1 b_2$

So;

$w_2 = w_1 \frac{L_1}{L_2}\frac{b_2}{b_1}$

[3] Our main results are robust to changes in trade costs.For simplicity we assume no trade costs.

Equlibrium Prices:

$p_1 = a_{11}w_1$

$p_2 = a_{22}w_2 = w_1 \frac{L_1}{L_2}\frac{b_2}{b_1}a_{22}$

Both types have the same Cobb-Douglas Utility Function.

$U = C_1^{b_1}C_2^{b_2}$

Where

c_{ij} = amount of good i counsumed by type j.

$C_1 = \frac{w_1 b_1}{p_1} = \frac{w_1 b_1}{a_{11}w_1} = \frac{b_1}{a_{11}}$

$C_2 = \frac{w_1 b_2}{p_2} = \frac{w_1 b_2}{w_1 \frac{L_1}{L_2}\frac{b_2}{b_1}a_{22}} = \frac{b_2}{\frac{L_1}{L_2}\frac{b_2}{b_1}a_{22}} = \frac{L_2}{L_1}\frac{b_1}{a_{22}}$

After rearranging all the terms , the utility of type 1 labor in home country under autarky will be;

$U_{L1} = \left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_1}{a_{22}}\right)^{b_2}\left(\frac{L_2}{L_1}\right)^{b_2}$

Similarly, Utility of type 2 labor will be;

$C_1 = \frac{w_2 b_1}{p_1} = \frac{w_2 b_1}{a_{11}w_1} = \frac{w_1 \frac{L_1}{L_2}\frac{b_2}{b_1}b_1}{a_{11}w_1} = \frac{\frac{L_1}{L_2}\frac{b_2}{b_1}b_1}{a_{11}} = \frac{L_1}{L_2}\frac{b_2}{a_{11}}$

$C_2 = \frac{w_2 b_2}{p_2} = \frac{w_2 b_2}{a_{22}w_2} = \frac{b_2}{a_{22}}$

$U_{L2} = \left(\frac{b_2}{a_{11}}\right)^{b_1}\left(\frac{L_1}{L_2}\right)^{b_1}\left(\frac{b_2}{a_{22}}\right)^{b_2}$

Total Utility of the home country will be:

$\sum U = L_1 U_{L_1} + L_2 U_{L_2}$

$\sum U = L_1\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_1}{a_{22}}\right)^{b_2}\left(\frac{L_2}{L_1}\right)^{b_2} + L_2\left(\frac{b_2}{a_{11}}\right)^{b_1}\left(\frac{L_1}{L_2}\right)^{b_1}\left(\frac{b_2}{a_{22}}\right)^{b_2}$

The same calculations would apply for the foreign country and the utilities would be:

$U_{L_1^*} = \left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_1}{a_{22}}\right)^{b_2}\left(\frac{L_2^*}{L_1^*}\right)^{b_2}$

$U_{L_2^*} = \left(\frac{b_2}{a_{11}}\right)^{b_1}\left(\frac{L_1^*}{L_2^*}\right)^{b_1}\left(\frac{b_2}{a_{22}}\right)^{b_2}$

4.1 Equilibrium under Free Trade

Develop the equilibrium condition and refer to the technology technology paradox. Explain these intervals insimple words and give the intuition.

Since both $\frac{L_2}{L_1}$ and $\frac{L_2^*}{L_1^*}$ are in the interval $\left[\frac{a_{22}}{a_{12}}\frac{b_2}{b_1}, \frac{a_{21}}{a_{11}}\frac{b_2}{b_1}\right]$, $\frac{L_2+L_2^*}{L_1+L_1^*}$ will also be in this interval.

Due to factor price equilization, utility of the same type of labor in each country will be the same.$(U_{L_1} = U_{L_1^*}, U_{L_2} = U_{L_2^*})$

Utility of type 1 labor will be;

$U_{L_1}^{AT} = \left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_1}{a_{22}}\right)^{b_2}\left(\frac{L_2+L_2^*}{L_1+L_1^*}\right)^{b_2}$

$U_{L_2}^{AT} = \left(\frac{b_2}{a_{11}}\right)^{b_1}\left(\frac{L_1+L_1^*}{L_2+L_2^*}\right)^{b_1}\left(\frac{b_2}{a_{22}}\right)^{b_2}$

Utility of the home country

$\sum U = L_1 U_{L_1} + L_2 U_{L_2}$

$\sum U^{AT} = L_1\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_1}{a_{22}}\right)^{b_2}\left(\frac{L_2+L_2^*}{L_1+L_1^*}\right)^{b_2} + L_2\left(\frac{b_2}{a_{11}}\right)^{b_1}\left(\frac{L_1+L_1^*}{L_2+L_2^*}\right)^{b_1}\left(\frac{b_2}{a_{22}}\right)^{b_2}$

4.2 Gains from Trade

The gain for type 1 labor will be;

$$\frac{U_{L1}^{AT}-U_{L1}}{U_{L1}} = \frac{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_1}{a_{22}}\right)^{b_2}\left(\frac{L_2+L_2^*}{L_1+L_1^*}\right)^{b_2}-\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_1}{a_{22}}\right)^{b_2}\left(\frac{L_2}{L_1}\right)^{b_2}}{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_1}{a_{22}}\right)^{b_2}\left(\frac{L_2}{L_1}\right)^{b_2}}$$

$$\frac{U_{L1}^{AT}-U_{L1}}{U_{L1}} = \frac{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_1}{a_{22}}\right)^{b_2}\left(\frac{L_2+L_2^*}{L_1+L_1^*}\right)^{b_2}}{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_1}{a_{22}}\right)^{b_2}\left(\frac{L_2}{L_1}\right)^{b_2}} - 1$$

$$\frac{U_{L1}^{AT}-U_{L1}}{U_{L1}} = \frac{\left(\frac{L_2+L_2^*}{L_1+L_1^*}\right)^{b_2}}{\left(\frac{L_2}{L_1}\right)^{b_2}} - 1$$

After simplification one can get;

$$\frac{U_{L1}^{AT}-U_{L1}}{U_{L1}} = \left(\frac{\frac{L_2+L_2^*}{L_1+L_1^*}}{\frac{L_2}{L_1}}\right)^{b_2} - 1$$

For type 2 labor;

$$\frac{U_{L2}^{AT}-U_{L2}}{U_{L2}} = \frac{\left(\frac{b_2}{a_{11}}\right)^{b_1}\left(\frac{L_1+L_1^*}{L_2+L_2^*}\right)^{b_1}\left(\frac{b_2}{a_{22}}\right)^{b_2}-\left(\frac{b_2}{a_{11}}\right)^{b_1}\left(\frac{L_1}{L_2}\right)^{b_1}\left(\frac{b_2}{a_{22}}\right)^{b_2}}{\left(\frac{b_2}{a_{11}}\right)^{b_1}\left(\frac{L_1}{L_2}\right)^{b_1}\left(\frac{b_2}{a_{22}}\right)^{b_2}}$$

$$\frac{U_{L2}^{AT}-U_{L2}}{U_{L2}} = \frac{\left(\frac{L_1+L_1^*}{L_2+L_2^*}\right)^{b_1}}{\left(\frac{L_1}{L_2}\right)^{b_1}} - 1$$

Taking the natural log of both sides will give us;

$$\ln(Gain-For-Type1) = b_2\left[\ln\left(\frac{L_2+L_2^*}{L_1+L_1^*}\right) - \ln\left(\frac{L_2}{L_1}\right)\right]$$

And for Type 2 labor;

$$\ln(Gain-For-Type2) = b_1\left[\ln\left(\frac{L_1+L_1^*}{L_2+L_2^*}\right) - \ln\left(\frac{L_1}{L_2}\right)\right]$$

4.3 Example

The easiest way to proceed is to use a numerical example.

Let $b_1 = 0.25$

$b_2 = 0.75$

$L_1 = 100$

$L_2 = 670$

$L_1^* = 50$

$L_2^* = 830$

$a_{11} = a_{12} = 1$

$a_{21} = 6, a_{22} = 2$

These endowments assure us that after trade both countries have about the same real income so that our results are not affected by different relative country sizes.

The intervals would be such that;

$$\frac{a_{22}b_2}{b_1} = 6 < \frac{L_2}{L_1} < \frac{L_2^*}{L_1^*} < \frac{a_{21}b_2}{b_1} = 18$$

So, Type 1 labor has a comparative advantage in commodity 1.

	Home Country			Foreign Country		
Inputs of 1	a_{11}= 1	a_{12}= 1	L_1= 100	a_{11}= 1	a_{12}= 1	L_1^*= 50
Inputs of 2	a_{21}= 2	a_{22}= 6	L_2= 670	a_{21}= 2	a_{22}= 6	L_2^*= 830
Autarky Wages	w_1= 1	w_2= 0.447		w_1^*= 1	w_2^*= 0.180	
Free Trade Wages	w_1= 1	w_2= 0.3		w_1^*= 1	w_2^*= 0.3	
Autarky Utilities	U_{L_1}= 0.619	U_{L_2}= 0.277	U = 247.6	$U_{L_1^*}$= 1.222	$U_{L_2^*}$= 0.221	U^*= 244.5
Free Trade Utilities	U_{L_1}= 0.837	U_{L_1}= 0.251	U = 251.9	$U_{L_1^*}$= 0.837	$U_{L_2^*}$= 0.251	U^*= 250.1
Gains	+35.1%	−9.4%		−31.5%	+13.6%	

What is remarkable about this example is that the gains from trade are modest, around 2%, but the distributional impacts are quite severe. The Home country exports commodity 1, so type 1 labor is the abundant factor; but commodity 1 is in low world demand. Notice, however, that the abundant factor therefore gains 35.2% from those modest trade gains while the scarce factor loses the relatively modest 9.4% of real income. In the Foreign country, where type 2 labor is abundant, the gain to the abundant factor is relatively modest 13.6% while the scarce factor loses 31.6% of its real income from trade. The reason for the tremendous losses to the scarce factor is that the Foreign relative wage for the scarce factor w1*/w2* falls from 1/.18 or 5.53 to 1/.3 or 3.33—a fall of 58% and little is offset by the fall in the cost of an imported commodity that is only 25% of the budget. The Home country's scarce factor loses comparatively little because while w2/w1 falls from .448 to .3, a 33% fall, this is largely offset by the fall in the price of commodity 2 that looms so large in the budget—hence, a relatively modest 9.4% loss.

5 General Model

Let's generalize our model to multi commodity case.

Again consider 2 types of labor (L_1 and L_2). Both has the the same Cobb-Douglas preferences.
Let;
a_{ij} = The amount of type i labor that will produce 1 unit of good j.
b_i = Share of income devoted for good i
θ_i = Total income share devoted for good 1 to i
$\theta_i = \sum_1^i b_i$
Where $\sum_1^N b_i = \theta_N = 1$
Assume;
$\frac{a_{21}}{a_{11}} > \frac{a_{22}}{a_{12}} > ... > \frac{a_{2k}}{ak} > ... > \frac{a_{2n}}{a_{1n}}$ (L_1 has a comparative advantage in Good 1, L_2 in good n).

Goods 1,...,k will be produced by L_1, and goods (k+1),....,N will be produced by L_2 (that is to say, for disjoint production)if $w_1a_{1k} < w_2a_{2k}$ and $w_1a_{1(k+1)} > w_2a_{2(k+1)}$Type 1 labor will sell $w_2L_2\theta_k$; and purchase $w_1L_1(1-\theta_k)$. There is equilibrium when $w_2L_2\theta_k = w_1L_1(1-\theta_k)$.

If there is a good produced by both labor types (say good k) then, it must be true that $w_1a_{1k} = w_2a_{2k}$

So, The question is to locate $\frac{L_2}{L_1}$ in the sequence;

$\frac{a_{2(k+1)}}{a_{1(k+1)}}\frac{1-\theta_k}{\theta_k} < I_1 < \frac{a_{2k}}{a_{1k}}\frac{1-\theta_k}{\theta_k} < I_2 < \frac{a_{2k}}{a_{1k}}\frac{1-\theta_{k-1}}{\theta_{k-1}} < I_3 < \frac{a_{2(k-1)}}{a_{1(k-1)}}\frac{1-\theta_{k-1}}{\theta_{k-1}}$

Where we see;

I_1 =Disjoint production, L_1will produce goods 1 to k , L_2 will produce goods (k+1) to N

I_2 =Common production of good k by both types, L_1will produce goods 1 to k , L_2 will produce goods k to N

I_3 =Disjoint production, L_1will produce goods 1 to (k-1) , L_2 will produce goods k to N

We may have different gains from trade depending on where the relative labor endowments of each country falls in this sequence.

6 Disjoint production in the same interval

Assume $\frac{L_2}{L_1} < \frac{L_2^*}{L_1^*}$ and both fall in I_1.

6.1 Autarky Solution

In equilibrium;

$w_2 L_2 \theta_k = w_1 L_1 (1-\theta_k)$

So;

$w_2 = w_1 \frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k}$

Equlibrium Prices:

$p_i = a_{1i} w_1$ for $i = 1$ to k

$p_j = a_{2j} w_2 = a_{2j} w_1 \frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k}$ for $j = k+1$ to n

Both types have the same Cobb-Douglas Utility Function.

$U = C_1^{b_1} C_2^{b_2} ... C_k^{b_k} ... C_n^{b_n}$

Where (for type 1 labor)

$C_i = \frac{w_1 b_i}{p_i} = \frac{w_1 b_i}{a_{1i} w_1} = \frac{b_i}{a_{1i}}$ for $i = 1$ to k,

$C_j = \frac{w_1 b_j}{p_j} = \frac{w_1 b_j}{w_1 \frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k} a_{2j}} = \frac{b_j}{\frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k} a_{2j}} = \frac{b_j}{a_{2j}}\frac{L_2}{L_1}\frac{\theta_k}{(1-\theta_k)}$ for $j = k+1$ to n

The utility of type 1 labor in home country under autarky will be;

$U_{L1} = \left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_2}{a_{12}}\right)^{b_2} ... \left(\frac{b_k}{a_{1k}}\right)^{b_k}\left(\frac{b_{(k+1)}}{a_{2(k+1)}}\frac{L_2}{L_1}\frac{\theta_k}{(1-\theta_k)}\right)^{b_{k+1}} ... \left(\frac{b_n}{a_{2n}}\frac{L_2}{L_1}\frac{\theta_k}{(1-\theta_k)}\right)^{b_n}$

After rearranging all the terms ,

$U_{L1} = \left\{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_2}{a_{12}}\right)^{b_2} ... \left(\frac{b_k}{a_{1k}}\right)^{b_k}\left(\frac{b_{(k+1)}}{a_{2(k+1)}}\right)^{b_{k+1}} ... \left(\frac{b_n}{a_{2n}}\right)^{b_n}\right\}\left(\frac{L_2}{L_1}\frac{\theta_k}{(1-\theta_k)}\right)^{1-\theta_k}$

$U_{L1} = \left\{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_2}{a_{12}}\right)^{b_2} ... \left(\frac{b_k}{a_{1k}}\right)^{b_k}\left(\frac{b_{(k+1)}}{a_{2(k+1)}}\right)^{b_{k+1}} ... \left(\frac{b_n}{a_{2n}}\right)^{b_n}\right\}\left(\frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k}\right)^{\theta_k}\left(\frac{L_2}{L_1}\frac{\theta_k}{(1-\theta_k)}\right)$

$U_{L2} = \left\{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_2}{a_{12}}\right)^{b_2} ... \left(\frac{b_k}{a_{1k}}\right)^{b_k}\left(\frac{b_{(k+1)}}{a_{2(k+1)}}\right)^{b_{k+1}} ... \left(\frac{b_n}{a_{2n}}\right)^{b_n}\right\}\left(\frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k}\right)^{\theta_k}$

For type 2 labor;

$C_i = \frac{w_2 b_i}{p_i} = \frac{w_2 b_i}{a_{1i} w_1} = \frac{w_1 \frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k} b_i}{w_1 a_{1i}} = \frac{b_i}{a_{1i}}\frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k}$ for $i = 1$ to k,

$C_j = \frac{w_2 b_j}{p_j} = \frac{w_2 b_j}{a_{2j} w_2} = \frac{b_j}{a_{2j}}$ for $j = k+1$ to n

The utility of type 2 labor in home country under autarky will be;

$U_{L2}=\left(\frac{b_1}{a_{11}}\frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k}\right)^{b_1}\left(\frac{b_2}{a_{12}}\frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k}\right)^{b_2}...\left(\frac{b_k}{a_{1k}}\frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k}\right)^{b_k}\left(\frac{b_{(k+1)}}{a_{2(k+1)}}\right)^{b_{(k+1)}}...\left(\frac{b_n}{a_{2n}}\right)^{b_n}$

After rearranging the terms;

$U_{L2}=\left(\frac{L_1}{L_2}\frac{(1-\theta_k)}{\theta_k}\right)^{\theta_k}\left\{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_2}{a_{12}}\right)^{b_2}...\left(\frac{b_k}{a_{1k}}\right)^{b_k}\left(\frac{b_{(k+1)}}{a_{2(k+1)}}\right)^{b_{(k+1)}}...\left(\frac{b_n}{a_{2n}}\right)^{b_n}\right\}$

The same calculations would apply for the foreign country and the utilities would be:

$U_{L_1^*}=\left\{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_2}{a_{12}}\right)^{b_2}...\left(\frac{b_k}{a_{1k}}\right)^{b_k}\left(\frac{b_{(k+1)}}{a_{2(k+1)}}\right)^{b_{k+1}}...\left(\frac{b_n}{a_{2n}}\right)^{b_n}\right\}\left(\frac{L_2^*}{L_1^*}\frac{\theta_k}{(1-\theta_k)}\right)^{1-\theta_k}$

$U_{L_2^*}=\left(\frac{L_1^*}{L_2^*}\frac{(1-\theta_k)}{\theta_k}\right)^{\theta_k}\left\{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_2}{a_{12}}\right)^{b_2}...\left(\frac{b_k}{a_{1k}}\right)^{b_k}\left(\frac{b_{(k+1)}}{a_{2(k+1)}}\right)^{b_{(k+1)}}...\left(\frac{b_n}{a_{2n}}\right)^{b_n}\right\}$

6.2 Equilibrium under Free Trade

Since both $\frac{L_2}{L_1}$ and $\frac{L_2^*}{L_1^*}$ are in the interval I_1, $\frac{L_2+L_2^*}{L_1+L_1^*}$ will also be in the same interval.

In equilibrium;

$w_2(L_2+L_2^*)\theta_k=w_1(L_1+L_1^*)(1-\theta_k)$

$w_2=w_1\frac{(L_1+L_1^*)}{(L_2+L_2^*)}\frac{(1-\theta_k)}{\theta_k}$

Due to factor price equilization, utility of the same type of labor in each country after trade will be the same.($U_{L_1}^{AT}=U_{L_1^*}^{AT}, U_{L_2}^{AT}=U_{L_2^*}^{AT}$)

Utility of type 1 & 2 labor will be;

$U_{L_1}^{AT}=\left\{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_2}{a_{12}}\right)^{b_2}...\left(\frac{b_k}{a_{1k}}\right)^{b_k}\left(\frac{b_{(k+1)}}{a_{2(k+1)}}\right)^{b_{k+1}}...\left(\frac{b_n}{a_{2n}}\right)^{b_n}\right\}\left(\frac{L_2+L_2^*}{L_1+L_1^*}\frac{\theta_k}{(1-\theta_k)}\right)^{1-\theta_k}$

$U_{L_2}^{AT}=\left(\frac{L_1^*+L_1}{L_2^*+L_2}\frac{(1-\theta_k)}{\theta_k}\right)^{\theta_k}\left\{\left(\frac{b_1}{a_{11}}\right)^{b_1}\left(\frac{b_2}{a_{12}}\right)^{b_2}...\left(\frac{b_k}{a_{1k}}\right)^{b_k}\left(\frac{b_{(k+1)}}{a_{2(k+1)}}\right)^{b_{(k+1)}}...\left(\frac{b_n}{a_{2n}}\right)^{b_n}\right\}$

6.3 Gains from Trade

The gain for the Home country;

$\frac{\sum U^{AT}-\sum U}{\sum U}=\theta_k\left(\frac{L_2+L_2^*}{L_1+L_1^*}\right)^{1-\theta_k}\left(\frac{L_1}{L_2}\right)^{1-\theta_k}+(1-\theta_k)\left(\frac{L_2}{L_1}\right)^{\theta_k}\left(\frac{L_1^*+L_1}{L_2^*+L_2}\right)^{\theta_k}-1$

The gain for type 1 labor will be;

$\left(\frac{\frac{L_2+L_2^*}{L_1+L_1^*}}{\frac{L_2}{L_1}}\right)^{1-\theta_k}-1$

Taking the natural log of both sides will give us;

$\ln(Gain-For-Type1)=(1-\theta_k)\left[\ln\left(\frac{L_2+L_2^*}{L_1+L_1^*}\right)-\ln\left(\frac{L_2}{L_1}\right)\right]$

$\ln(Gain-For-Type2)=\theta_k\left[\ln\left(\frac{L_1^*+L_1}{L_2^*+L_2}\right)-\ln\left(\frac{L_1}{L_2}\right)\right]$

Since $\frac{L_2}{L_1}<\frac{L_2^*}{L_1^*}$ home country is exporting Good 1 to k and importing good (k+1) to N. So the abundant factor in home country is L_1 , and L_1 will gain from trade. The gain for L_1 will be;

$\ln(Gain-For-Type1)=(1-\theta_k)\left[\ln\left(\frac{L_2+L_2^*}{L_1+L_1^*}\right)-\ln\left(\frac{L_2}{L_1}\right)\right]$

As it is seen, this expression is positive and as the demand for exported goods goes down(θ_k) , gain for the abundant factor goes up. This is because a decrease for the demand of exported goods means an increase for the demand of imports. Price of imported goods (which have higher demands than exported goods) will decrease and L_1 will enjoy the relative wage increase over imported goods.

Similarly, L_2 is the scarce factor. Home country imports good (k+1) to N and L_2 will be worse off.The gain for L_2 will be;

$\ln(Gain - For - Type2) = \theta_k \left[\ln\left(\frac{L_1^* + L_1}{L_2^* + L_2}\right) - \ln\left(\frac{L_1}{L_2}\right)\right]$

Since $\frac{L_2}{L_1} < \frac{L_2^*}{L_1^*}$, this expression is negative, in other words, this is the loss of the scarce factor. As the demand for exported goods goes down(θ_k) , the loss for the scarce factor will go down as well. This is because the relative wage of L_2 falls after trade , price of exported goods rises and, L_2 has to spend on exported goods(which have θ_k% of the budget) So as the demand for exports goes down, scarce factor will spend less of its income on exports and they will be less worse off.

Proposition 1 *If a country is exporting commodities for which the world demand is low, gain for the abundant factor will be higher and the loss of scarce factor will be less compared to their counterparts in a country that exports commodities for which the world demand is high.*

Interestingly, the gain from trade for the whole country is not much affected by the demand of exported or imported goods.

The gain for the home country=$\theta_k \left(\frac{L_2 + L_2^*}{L_1 + L_1^*}\right)^{1-\theta_k} \left(\frac{L_1}{L_2}\right)^{1-\theta_k} + (1-\theta_k)\left(\frac{L_2}{L_1}\right)^{\theta_k} \left(\frac{L_1^* + L_1}{L_2^* + L_2}\right)^{\theta_k} - 1$

The gain for the country is basicly a weighted average of the gains of the abundant and the scarce factors. Notice that the weight of the abundant factor is proportional to the demand for exports while the weight of the scarce factor is proportional to the demand for imports. As the world demand for the exports(θ_k) goes down, gain for the abundant factor rises while the importance (or the weight) of abundant factor will be less in the net gain for the whole country. Similarly, the loss of the scarce factor diminishes with the demand of exports, the importance (or the weight) of the scarce factor will be higher. That is why, even though the gains from trade are modest, we see severe distributional impacts as the world demand changes.

Notice that gains from trade depends on two things; Relative endowments of the countries and the world demand for exported and imported goods. The gains from trade diminishes as the two countries become similar in terms of their factor endowments.

7 The Role of Intermediate Goods

What is the role of intermediate goods in this analysis. Does it really make a difference to export intermediate goods or final goods. To see the role of intermediate goods, we incorporate intermediate goods to the model along with the final consumption goods. Assume, there are two goods one of which is an intermediate good (good 2) and not consumed at all. There are two factors of production, type 1 labor, L_1, and type 2 labor, L_2. Both spend their income onm good 2.

Let;a_{ij} = The amount of type i labor that will produce 1 unit of good j.

Assume;

$\frac{a_{21}}{a_{11}} > \frac{a_{22}}{a_{12}}$ (L_1 has a comparative advantage in Good 1).

Therefore, L_1 will produce the final good (good 1), and L_2 will produce the intermediate good (good2).

Each unit of good 1 requires s unit of good 2 s.t. $X_2 = sX_1$.

$P_2 = a_{22}w_2$

$P_1 = a_{11}w_1 + sP_1$, where $a_{11}w_1 = v_1$ is the value added.

$V_1 = P_1 - sP_2$

Assume that L_1 produces just the final good and L_2 produces just intermediate goods. In equilibrium;

$$w_1L_1 + w_2L_2 = TotalOutput \times P_1$$

$$\frac{V_1}{a_{11}}L_1 + \frac{P_2}{a_{22}}L_2 = \frac{L_1}{a_{11}}P_1$$

$$\frac{P_2}{a_{22}}L_2 = \frac{L_1}{a_{11}}(P_1 - V_1) = \frac{L_1}{a_{11}}sP_2$$

$$\frac{L_2}{a_{22}} = \frac{sL_1}{a_{11}}$$

This expression says, when perfect specialization by both labor types, good 2 has to be produced just as much as necessary to be soaked up in the production of final goods. Apperantly this is a razor's edge case.Rather than equality say, one type of labor can produce both goods such that;

If $\frac{L_2}{a_{22}} > \frac{sL_1}{a_{11}}$, then there are more intermediate goods than sufficient for the production of final goods and hence some L_2 produces final goods, (good 1) while all L_1 produces just good 1.

If$\frac{L_2}{a_{22}} < \frac{sL_1}{a_{11}}$, then there are more final goods than maximum feasible amount that can be produced using all intermediate goods and thus some of L_1 produces good 2 while all L_2 produces good 2.

Let's pick two symmetric countries and calculate the gains for the abundant factor in case it exports final goods and in case it exports intermediate goods and compare those two gains.

8 Autarky Solution

Assume in home country under autarky $\frac{L_2}{a_{22}} > \frac{sL_1}{a_{11}}$.

Let L_{21} be the type 2 labor that produces good 1, and L_{22} be the type 2 labor that produces good 2.

In equilibrium;

$\frac{L_{22}}{a_{22}} = s(\frac{L_1}{a_{11}} + \frac{L_{21}}{a_{21}})$

Prices will be;

$P_2 = a_{22}w_2$ (Good 2 will only be produced by L_2)

$P_1 = a_{11}w_1 + sP_2$ or $P_1 = a_{21}w_2 + sP_2$ (since good 1 can be produced either by L_1 or L_2)

Therefore an equilibrium condition will be; $a_{11}w_1 = a_{21}w_2$

$$\frac{w_2}{w_1} = \frac{a_{11}}{a_{21}}$$

Let $w_1 = 1$ be the numeraire;

$$\begin{aligned} P_1 &= a_{11} + s\frac{a_{22}a_{11}}{a_{21}} \\ P_2 &= \frac{a_{22}a_{11}}{a_{21}} \end{aligned}$$

$$C^A_{L_1} = \frac{w_1}{p_1} = \frac{1}{a_{11} + s\frac{a_{22}a_{11}}{a_{21}}} = \frac{a_{21}}{a_{11}(a_{21} + ba_{22})}$$

$$C^A_{L_2} = \frac{w_2}{p_1} = \frac{\frac{a_{11}}{a_{21}}}{a_{11} + b\frac{a_{22}a_{11}}{a_{21}}} = \frac{1}{(a_{21} + ba_{22})}$$

Further assume, in foreign country under autarky $\frac{L^*_2}{a_{22}} < \frac{sL^*_1}{a_{11}}$. (otherwise wages and prices would exactly be the same as home country and there would be no gain to any of the countries from trade)Let L^*_{12} be the type 1 labor that produces good 2, and L^*_{11} be the type 1 labor that produces good 1.In equilibrium;

$$\frac{L^*_2}{a_{22}} + \frac{L^*_{12}}{a_{12}} = s\frac{L^*_{11}}{a_{11}}$$

Prices will be;

$P^*_2 = a_{22}w^*_2$ or $P^*_2 = a_{12}w^*_1$ (Since good 2 can be produced either by L^*_1 or L^*_2)

$P^*_1 = a_{11}w^*_1 + sP^*_2$

Therefore an equilibrium condition will be; $P^*_2 = a_{22}w^*_2 = a_{12}w^*_1$

$$\frac{w^*_2}{w^*_1} = \frac{a_{12}}{a_{22}}$$

Similarly, let $w^*_1 = 1$ be the numeraire;

$$\begin{aligned} P^*_1 &= a_{11} + sa_{12} \\ P^*_2 &= a_{12} \end{aligned}$$

$C^A_{L^*_1} = \frac{w^*_1}{p^*_1} = \frac{1}{a_{11}+sa_{12}}$

$C^A_{L^*_2} = \frac{w^*_2}{p^*_1} = \frac{\frac{a_{12}}{a_{22}}}{a_{11}+sa_{12}} = \frac{a_{12}}{a_{22}(a_{11}+sa_{12})}$

9 Free Trade Solution

Since $\frac{L_2}{a_{22}} > \frac{sL_1}{a_{11}}$ and $\frac{L^*_2}{a_{22}} < \frac{sL^*_1}{a_{11}}$, when there is free trade the inequality can go for either direction s.t. $\frac{L_2+L^*_2}{a_{22}} \gtrless \frac{s(L_1+L^*_1)}{a_{11}}$. One way to compare the gains is to calculate gains for the abundant factor when home exports final goods and intermediate goods seperately using the inequality.

9.1 Final Good Produced in Common

Assume $\frac{L_2+L_2^*}{a_{22}} > \frac{s(L_1+L_1^*)}{a_{11}}$

This shows that good 1, the final good, is produced in common by both labor types. In other words total endowments are such that while all of L_1 produce good 1, some of L_2 produce good 1 too. Since under autarky in home country, some of L_2 was producing good 1, there would be no change for home country, prices and wages are the same under free trade as they were under autarky. When it comes to the foreign country, some of L_1^* was producing good 2 under autarky. After trade L_1^* produces just good 1 and foreign country exports good 1 in exchange of good 2. So L_1^* , as the abundant factor, gains from trade while L_2^* loses from trade.

Equilibrium prices and wages under free trade are as follows;

$$\frac{w_2}{w_1} = \frac{a_{11}}{a_{21}}$$

Let $w_1 = 1$ be the numeraire;

$$\begin{aligned} P_1 &= a_{11} + s\frac{a_{22}a_{11}}{a_{21}} \\ P_2 &= \frac{a_{22}a_{11}}{a_{21}} \end{aligned}$$

$C_{L_1}^T = \frac{w_1}{p_1} = \frac{1}{a_{11}+s\frac{a_{22}a_{11}}{a_{21}}} = \frac{a_{21}}{a_{11}(a_{21}+sa_{22})}$

$C_{L_2}^T = \frac{w_2}{p_1} = \frac{\frac{a_{11}}{a_{21}}}{a_{11}+s\frac{a_{22}a_{11}}{a_{21}}} = \frac{1}{(a_{21}+sa_{22})}$

Due to factor price equilization, these results hold for both of the countries.

Gain for the abundant factor in foreign country is;

$Gross\ Gain\ For\ L_1^* = \frac{C_{L_1}^T}{C_{L_1^*}^A} = \frac{\frac{a_{21}}{a_{11}(a_{21}+sa_{22})}}{\frac{1}{a_{11}+sa_{12}}} = \frac{a_{21}(a_{11}+sa_{12})}{a_{11}(a_{21}+sa_{22})}$

$Gross\ Gain\ For\ L_2^* = \frac{C_{L_2}^T}{C_{L_2^*}^A} = \frac{\frac{1}{(a_{21}+sa_{22})}}{\frac{a_{12}}{a_{22}(a_{11}+sa_{12})}} = \frac{a_{22}(a_{11}+sa_{12})}{a_{12}(a_{21}+sa_{22})}$

The gains from trade for labor types are;

$$\begin{aligned} GainForL_1 &= 0 \\ GainForL_2 &= 0 \\ GainForL_1^* &= \frac{a_{21}(a_{11}+sa_{12})}{a_{11}(a_{21}+sa_{22})} \\ GainForL_2^* &= \frac{a_{22}(a_{11}+sa_{12})}{a_{12}(a_{21}+sa_{22})} \end{aligned}$$

Notice that the gain for L_1^* is always positive and for L_2^* is always negative.

Since $\frac{a_{11}}{a_{21}} < \frac{a_{12}}{a_{22}}$, it follows that $\frac{a_{11}}{a_{21}} < \frac{(a_{11}+sa_{12})}{(a_{21}+sa_{22})}$ and by multiplying both sides by $\frac{a_{21}}{a_{11}}$, we can get $1 < \frac{a_{21}(a_{11}+sa_{12})}{a_{11}(a_{21}+sa_{22})}$.Similarly $\frac{a_{22}(a_{11}+sa_{12})}{a_{12}(a_{21}+sa_{22})} < 1$.

9.2 Intermediate Good Produced in Common

Now, assume $\frac{L_2+L_2^*}{a_{22}} < \frac{s(L_1+L_1^*)}{a_{11}}$

This shows that good 2, the intermediate good, is produced in common by both labor types. In other words total endowments are such that while all of L_2 produce good 2, some of L_1 produce good 2 too. Since under autarky in foreign country, some of L_1^* was producing good 2, there would be no change for foreign country, prices and wages are the same under free trade as they were under autarky. When it comes to the home country, some of L_2 was producing good 1 under autarky. After trade L_2 produces just good 2 and home country exports good 2 in exchange of good 1. So L_2 , as the abundant factor, gains from trade while L_1 loses from trade.

Therefore the equilibrium wages and prices are as follows;

$$\frac{w_2}{w_1} = \frac{a_{12}}{a_{22}}$$

Similarly, let $w_1^* = 1$ be the numeraire;

$$\begin{aligned} P_1 &= a_{11} + sa_{12} \\ P_2 &= a_{12} \end{aligned}$$

$C_{L_1}^T = \frac{w_1}{p_1} = \frac{1}{a_{11}+sa_{12}}$

$C_{L_2}^T = \frac{w_2}{p_1} = \frac{\frac{a_{12}}{a_{22}}}{a_{11}+sa_{12}} = \frac{a_{12}}{a_{22}(a_{11}+sa_{12})}$

Again due to FPE these results apply to both of the countries.

Gain for the abundant factor in home country is;

$Gross\ Gain\ For\ L_1 = \frac{C_{L_1}^T}{C_{L_1}^A} = \frac{\frac{1}{a_{11}+sa_{12}}}{\frac{a_{21}}{a_{11}(a_{21}+sa_{22})}} = \frac{a_{11}(a_{21}+sa_{22})}{a_{21}(a_{11}+sa_{12})}$

$Gross\ Gain\ For\ L_2 = \frac{C_{L_2}^T}{C_{L_2}^A} = \frac{\frac{a_{12}}{a_{22}(a_{11}+sa_{12})}}{\frac{1}{(a_{21}+sa_{22})}} = \frac{a_{12}(a_{21}+sa_{22})}{a_{22}(a_{11}+sa_{12})}$

The gains from trade for labor types are;

$$\begin{aligned} GainForL_1 &= \frac{a_{11}(a_{21}+sa_{22})}{a_{21}(a_{11}+sa_{12})} \\ GainForL_2 &= \frac{a_{12}(a_{21}+sa_{22})}{a_{22}(a_{11}+sa_{12})} \\ GainForL_1^* &= 0 \\ GainForL_2^* &= 0 \end{aligned}$$

Notice that gain for the abundant factor, L_2, is always positive.

Since $\frac{a_{22}}{a_{12}} < \frac{a_{21}}{a_{11}}$, it follows that $\frac{a_{22}}{a_{12}} < \frac{(a_{21}+sa_{22})}{(a_{11}+sa_{12})}$ and by multiplying both sides by $\frac{a_{12}}{a_{22}}$, we can get $1 < \frac{a_{12}(a_{21}+sa_{22})}{a_{22}(a_{11}+sa_{12})}$.Similarly $\frac{a_{11}(a_{21}+sa_{22})}{a_{21}(a_{11}+sa_{12})} < 1$.

Now, it is crucial to compare the gain for the abundant factor when it exports final goods to the gain when it exports intermediate goods. It will sufficient to compare the gain for L_1^* that exports final goods in the first case to the gain for L_2 that exports intermediate goods in the second case.

$$\begin{aligned} GainForL_1^* &= \frac{a_{21}(a_{11}+sa_{12})}{a_{11}(a_{21}+sa_{22})} \\ GainForL_2 &= \frac{a_{12}(a_{21}+sa_{22})}{a_{22}(a_{11}+sa_{12})} \end{aligned}$$

Proposition 2 *Abundant factor that exports final goods gains more than it would gain if it exports intermediate goods if and only if* $s > \frac{\left(\sqrt[2]{\frac{a_{12}a_{11}a_{21}}{a_{22}}}-a_{11}\right)}{\left(a_{12}-\sqrt[2]{\frac{a_{12}a_{11}a_{22}}{a_{21}}}\right)}$.

Proof. $GainForL_1^* = \frac{a_{21}(a_{11}+sa_{12})}{a_{11}(a_{21}+sa_{22})} > GainForL_2 = \frac{a_{12}(a_{21}+sa_{22})}{a_{22}(a_{11}+sa_{12})}; \frac{a_{21}(a_{11}+sa_{12})}{a_{11}(a_{21}+sa_{22})} > \frac{a_{12}(a_{21}+sa_{22})}{a_{22}(a_{11}+sa_{12})}$,
Rewrite this expression as;
$\frac{a_{11}+sa_{12}}{a_{21}+sa_{22}} > \sqrt[2]{\frac{a_{12}a_{11}}{a_{22}a_{21}}}$
$s\left(a_{12}-\sqrt[2]{\frac{a_{12}a_{11}}{a_{22}a_{21}}}a_{22}\right) > \left(\sqrt[2]{\frac{a_{12}a_{11}}{a_{22}a_{21}}}a_{21}-a_{11}\right)$
$s\left(a_{12}-\sqrt[2]{\frac{a_{12}a_{11}a_{22}}{a_{21}}}\right) > \left(\sqrt[2]{\frac{a_{12}a_{11}a_{21}}{a_{22}}}-a_{11}\right)$
Since ;
$a_{12} > \sqrt[2]{\frac{a_{12}a_{11}a_{22}}{a_{21}}}$ and $\sqrt[2]{\frac{a_{12}a_{11}a_{21}}{a_{22}}} > a_{11}$ both sides are positive, therefore a necessary and sufficient condition for the proposition to hold is
$s > \frac{\left(\sqrt[2]{\frac{a_{12}a_{11}a_{21}}{a_{22}}}-a_{11}\right)}{\left(a_{12}-\sqrt[2]{\frac{a_{12}a_{11}a_{22}}{a_{21}}}\right)}$. ■