ECONOMICS 7395, Spring 2005

Handout 17, April 8.

INTEREST RATE THEORY

We will now cover fixed income securities. The major categories of long-term fixed income securities are federal government bonds, corporate bonds, mortgages, and municipal bonds. Government bonds can be considered risk free whereas other bonds have a risk of default. Some bonds are callable or convertible. Fixed income securities are covered in chapters 13 and 14 in BKM and chapter 20 in EG. We will not consider default risk and the pricing of callable bonds in this class. So the development here is only directly applicable to government bonds. It seems to me that this is preferable. It is better to understand the simplest case before adding the complications that arise from callability, default risk, etc.

Most longer term bonds are coupon bonds. This means that they pay a fixed amount of interest at given points in time (the **coupon payment**) and a principal payment (the **nominal value** of the bond) at maturity. The nominal value is also called the "face value" or "par value". Normally the interest payments are equi-spaced, by which I mean that they pay interest every month, half year or year. Government bonds usually pay interest semiannually. We will typically use the convention that you can buy the bond at time $t = 0$ for a price P_0 and we denote the time to maturity by T. NOTE that T is the number of periods that the bond pays interest (so we set T=10, for a 5-year bond with semiannual payments). The cash flow to a bond will be denoted $C(1), ..., C(T)$. For a 4 year bond with a nominal value of 1000$ and annual interest payments of 50$ the cash-flows will be $C(1) = C(2) = C(3) = 50\$$ and $C(4) = 1050\$$. The coupon payment divided by the face value is called the **coupon rate**, or **current yield** and the coupon rate in the example will be .05 or 5%. The **yield to maturity** of a bond with a cash flow $C(t), t = 1, ..., T$ and a price of P_0 is the value y that solves the equation

$$P_0 \;=\; \sum_{t=1}^{T} \frac{C(t)}{(1+y)^t} \;.$$

For the 4 year bond considered above, assume that the price today is 900$. The yield to maturity will then be the y that solves

$$900 \;=\; \frac{50}{1+y} + \frac{50}{(1+y)^2} + \frac{50}{(1+y)^3} + \frac{1050}{(1+y)^4} \;.$$

You can solve this and find $y = 8.02\%$. Actually, since the equation is not linear, you may not be able to solve it unless you have a suitable calculator (or computer program). The yield correspond to a constant safe rate annual rate of interest. If the safe rate of interest was y then 1$ today could be saved and you would have $(1 + y)\$$ next year. Or to put it another way, 1$ next year would be worth $1/(1 + y)\$$ today. Similarly 50$ next year would be worth $50/(1 + y)\$$ today. And since 1$ saved for 2 years would be worth $(1 + y) * (1 + y) = (1 + y)^2$, we value 50$ in two years as $50/(1 + y)^2\$$ today. If the price of the bond today was 1000$, rather than 900$, you would find that the yield was 5%. In general, the yield to maturity is equal to the coupon rate if the price of the bond in the special case where the price of the bond is equal to its face value.

For a bond that pay semiannually the yield to maturity is calculated as

$$P_0 \;=\; \sum_{t=1}^{T} \frac{C(t)}{(1+y/2)^t} \; .$$

The yield to maturity is also called the **bond equivalent yield**. For government bonds the yield to maturity is actually not such a useful measure, since it does not take into account that the interest after a half year would be re-invested so to get a better measure of yield, you will often use the **effective annual yield** $(1+y/2)^2 - 1$. There are several other yield concepts, which are not all that useful (see EG or BKM).

We will make quite heavy use of yet another yield concept, namely the **spot rate**. Spot rates are the yields to bonds that only make one payment. Such bonds are called pure discount bonds or zero-coupon bonds. They play a central role in the theory for the term structure of interest rates, but they have also lately become important in the financial market. The government usually issues discount bonds at short maturities (treasury bills) and coupon bonds at longer maturities, but many brokerage firms now buy longer term coupon bonds and sell the right to the interest payments and the right to the principal separately – this way creating "artificial" long term government discount bonds. (These principals which are stripped of the interest rate payments are known as "strips").

The one year **spot rate** is defined as the yield on a pure discount bond of one year maturity. (Spot rates are normally quoted as 2 times the six-month rate but I will assume here that interest payments are annual in order to not clutter up the notation). So for a discount bond paying X$ in one year with a present price of P_0, you find the spot rate from solving

$$P_0 = \frac{X}{1+S_1} \Rightarrow S_1 = \frac{X}{P_0} - 1 \; .$$

For example: If $P_0 = 970.87$ and $X = 1000$, then $S_1 = 3\%$. You find the 2-year spot rate as the yield to a 2-period zero-coupon bond, i.e. for a zero-coupon bond paying off X after 2 years:

$$P_0 = \frac{X}{(1+S_2)^2} \;\Rightarrow\; S_2 = \sqrt{\frac{X}{P_0}} - 1 \; .$$

Example: If $P_0 = 950$ and $X = 1000$, then $S_2 = \sqrt{1000/950} - 1 = \sqrt{1.0526} - 1 = 2.6\%$
Similarly, for the 3-year spot rate with price P_0 and pay-off X:

$$P_0 = \frac{X}{(1+S_3)^3} \;\Rightarrow\; S_3 = (\frac{X}{P_0})^{(1/3)} - 1 \; .$$

Example: If $P_0 = 1700$ and $X = 2000$, then $S_3 = (2000/17000)^{(1/3)} - 1 = (1.1765)^{(1/3)} - 1 = 5.6\%$
In general, the T-year spot rate for a bond with price P_0 and pay-off X is

$$P_0 = \frac{X}{(1+S_T)^T} \;\Rightarrow\; S_T = (\frac{X}{P_0})^{(1/T)} - 1 \; .$$

The 1-year spot rate is usually different from the 2-year spot rate which again is different from the 3-year spot rate and so on. In other words, at any given date there is no single interest rate – even in the absence of default risk, callability, etc. The phrase **term structure of interest rates** refers to the patterns of interest rates for safe zero-coupon bonds at any given data.

The **yield curve** is a plot of the spot rate against time to maturity – S_T on the $Y-$ axis and T on the $X-$ axis (NOTE that "T" here is NOT the date, but the time to maturity at some given time). The yield curve can take various shapes, although an upward sloping yield curve is the most common.

Upward sloping yield curve (PUT THEM IN YOURSELF)

Downward sloping yield curve

Hump shaped yield curve

You can find pictures of today's yield curve in the Wall Street Journal. The "term structure of interest rates" is really just another word for the "yield curve". We will study some theories about the term structure of interest rates.

An important concept when you consider the term structure of interest rates is the **forward interest rate**. The forward interest rate f_{st} is the interest rate on a loan maturing t periods from now, but with the loan taken out s periods in the future ($s < t$ of course).
If you want to lend money (buy a bond) with a two year maturity you can do this in 2 ways:

a) Buy a 2-year discount bond with 2-year return $(1 + S_2)^2$, where S_2 is the spot-rate.
or
b) Buy a 1-year discount bond with 1-year return $1 + S_1$, and write a contract that states that you are going to spend $(1 + S_1)\$$ on a bond with return f_{12}, where f_{12} is the forward interest rate for a loan signed today, to be delivered at the end of year 1 and repaid at the end of year 2.
The contracts a) and b) should have equal value, since both contracts involved delivering 1$ (or $X\$$) today with a fixed return 2 years later. Therefore, the yield should be same, which means that:

$$(1 + S_2)^2 \;=\; (1 + S_1) * (1 + f_{12}) \quad ,$$

In general, any t year loan is equivalent to an s year loan and a $t - s$ period forward loan, taken out at time s, so we have

$$(1 + S_t)^t \;=\; (1 + S_s)^s * (1 + f_{st})^{(t-s)} \quad ,$$

which implies that

$$(1 + f_{st})^{(t-s)} \;=\; \frac{(1 + S_t)^t}{(1 + S_s)^s}$$

or

$$f_{st} \;=\; \left(\frac{(1 + S_t)^t}{(1 + S_s)^s} \right)^{1/(t-s)} - 1$$

For simplicity we will just denote the one period forward rate at time s by f_s, so that $f_s = f_{s,s+1}$ by definition. We will in particular use the one period forward rates. Using the above equation we find

$$f_1 \;=\; \frac{(1 + S_2)^2}{(1 + S_1)} \;-\; 1 \;,$$

$$f_2 \;=\; \frac{(1 + S_3)^3}{(1 + S_2)^2} \;-\; 1 \;,$$

and in general

$$f_t \;=\; \frac{(1 + S_{t+1})^{(t+1)}}{(1 + S_t)^t} \;-\; 1 \;.$$

EXAMPLE: Assume $S_2 = 6\%$ and $S_1 = 5\%$, so the yield curve (for maturities 1 and 2 periods) is upward sloping. What is the forward rate f_1? Using the equations above we find:

$$(1 + .06)^2 \;=\; (1 + 0.05)^1 * (1 + f_1)^1 \;.$$

or

$$1+f_1 \;=\; 1.06^2/1.05 = 1.0701 \;\Rightarrow\; f_1 \;=\; .0701 \;.$$

Note that the 2-period spot rate is an "average" of the 1-period spot rate and the forward rate. The spot rate is close to, but not exactly equal to the simple arithmetic average; here (.0701+.05)/2 = .06005. Therefore, when the 2-period spot rate is 1 point above the 1-period spot rate, it implies that the forward rate is approximately 2 percentage points higher than the 1-period spot rate.
EXAMPLE: Assume $S_4 = 8\%$ and $S_2 = 7\%$. We can then find the forward rate f_{24} as

$$f_{24} \;=\; \sqrt{\frac{1.08^4}{1.07^2}} - 1 \;=\; .09 \;.$$

If you observe an upward sloping yield curve, this means that the forward rates are all higher than the spot rates, in the sense that $f_t > S_{t+1}$ for all t (remember that f_t is the interest rate on a loan delivered at t maturing in $t+1$ just as S_{t+1} is the current spot rate on a loan maturing in $t+1$). This follows since

$$1+f_t \;=\; \frac{(1+S_{t+1})^{(t+1)}}{(1+S_t)^t} \;=\; \frac{(1+S_{t+1})^t}{(1+S_t)^t} * (1+S_{t+1}) \;>\; 1+S_{t+1} \;,$$

since

$$S_{t+1} > S_t \Rightarrow (1+S_{t+1}) > (1+S_t) \Rightarrow (1+S_{t+1})/(1+S_t) > 1 \Rightarrow \left(\frac{1+S_{t+1}}{1+S_t}\right)^t > 1 \;.$$

The **pure expectations hypothesis** states that forward interest rates are unbiased predictors of corresponding future interest rates. For example, if the two-year spot rate is 6% and the one-year spot rate is 5%, then the forward rate for a 1-year loan taken out next year is (approximately) 7%. The pure expectations hypothesis states that the expected 1-year spot rate in a year is 7%. Define the one year spot rate in year t as $S_{t,1}$. The pure expectations hypothesis then can be stated as,

$$f_1 \;=\; ES_{1,1} \;,$$

or more generally as

$$f_t \;=\; ES_{t,1} \;.$$

How to test the pure expectations hypothesis? You run the regression

$$f_t \;=\; b_0 + b_1 S_{t,1} + u_t \;,$$

and perform an F-test for $b_0 = 0,\; b_1 = 1$. The pure expectations hypothesis is an example of a rational expectations model. It implies that investors on average can forecast the future interest rate, i.e. $f_t = S_{t,1} + u_t$, where u_t is a mean zero error term. In the literature you will often see that authors performing the "reverse regression" $S_{t,1}$ on f_t (e.g. Fama), but that does not seem to make sense. The (future) interest rates are the exogenous variables and investors forecast those subject to some errors, so if you do the reverse regression you have a "measurement error" in the regressor and as I have shown earlier, this leads to a downward bias in the estimated coefficient. Similarly, you can test if the forward rate at period 0 for a $(t-s)$-period discount bond sold in period s is the unbiased predictor for the $(t-s)$-period spot rate in period s, although testing is usually done on 1-year spot and forward rates.

In practice, you create a data set where the Y-variable is, say, the 1-year forward rates in 1960, 1961,..,1989, i.e. $\{f_{1960,1}, f_{1961,1}, .., f_{1989,1}\}$, and your X variable is the 1-year spot rates in 1961, 1962,..,1990, i.e. $\{S_{1961,1}, S_{1962,1}, .., S_{1990,1}\}$. In other words, you regress the forward rate on the spot rate that existed the following year.

The pure expectations hypothesis will be rejected if investors demand a risk premium for buying long term bonds. For example, if short term investors are risk averse, then they may require a positive risk premium (often called a **liquidity premium** in interest rate models) in order to buy long-term bonds, which they may have to sell at a random price one year later. In the example where the one-year spot rate is 5% and the two-year spot rate is 6%, it may actually be the case that investors expect that on average the one-year spot rate will be 5% the following year. So how come the forward rate is 7%? The difference, 2%, will be the risk premium. In this case we will say that the ("impure") expectations hypothesis hold if $f_t = k + ES_{t,1}$ where k is a constant – the risk premium. (Usually we simply refer to this model as the expectations hypothesis, and the **pure expectations hypothesis** will denote the situation without any risk premium). We can still test the (impure) expectations hypothesis. In the example, the risk premium for buying 2-year bonds rather than 1-year bonds is 2%. If in another year the expected future one-year spot rate is 8% then the one-year forward rate should be 8+2=10%. We test this by running the regression

$$f_t \;=\; b_0 + b_1 S_{t,1} + u_t \;,$$

(the same regression as before) but now we only test if $b_1 = 1$—the coefficient b_0 will measure the risk premium. The risk premium will not necessarily be positive, in the case where long-period investors dominate the market, it will be negative, since such investors prefer long bonds and only invest in short bonds if they receive a suitable risk premium. Typically, though, we expect to find a positive risk premium. In this case the yield curve will be upward sloping, even if investors expect future interest rates to equal present interest rates on average.

So far we have been talking about "interest rates" and "spot rates". If I_t is the annual rate of inflation, then for example (1+the spot rate) divided by (1+the inflation rate) minus 1 will be called the **real rate of return** and we denote it by, e.g. S_1^r—using a superscript r for real returns. We have

$$(1 + S_t^r) = (1 + S_t)/(1 + I_t) \;.$$

Note that, more precisely, I_t is the average rate of inflation over the life (t periods) of the underlying discount bond. We want to test if forward rates reflect future inflation. The relation involving the inflation rate is non-linear and it is somewhat intractable to deal with. We therefore use the **continuously compounded** interest and inflation rates.

Define the continuously compounded spot rate, s_t as

$$s_t = \ln(1 + S_t) \;.$$

This of course implies $S_t = \exp(s_t) - 1$. Denote the continuously compounded real spot rate in period t by s_t^r and the continuously compounded rate of inflation by i_t. Then $s_t^r = \ln(1 + S_t^r)$ and $i_t = \ln(1 + I_t)$, and we then have

$$s_t^r = s_t - i_t \ .$$

The continuously compounded rates give us a linear relation between the real spot rate, the nominal spot rate and the rate of inflation. One often think of the real rate of interest as the nominal rate of interest minus the inflation rate, but this is only strictly correct if formulated in terms of continuously compounded rates. Economists often assume that the real rate is given by the general amount of growth in the economy and that nominal rates are given by the real rate times the inflation rate. Furthermore, it is often assumed that the real rate of interest is constant. Let us assume that the real (continuously compounded) spot rate is constant and equal to k. We can now test if the forward rate is a good predictor of inflation or, more precisely, if the forward rate is equal to the (constant for now) real interest rate plus the expected rate of inflation, i.e. if $f_t = k + Ei_t$, where i_t is the inflation rate in period t or, equivalently, if $f_t = k + i_t + u_t$, where u_t has mean 0. We test this by running the regression

$$f_t = d_0 + d_1 i_t + u_t \ .$$

You then test if the forward rate is a perfect predictor of future inflation by testing if $d_1 = 1$ (if this is the case then $\hat{d}_0$ will be the estimate of the constant real interest rate). This is, however, a somewhat unreasonable hypothesis since the real spot rate is not necessarily constant. If you were willing to assume that expected inflation is constant, you could also test the "opposite" hypothesis if the forward rate is a predictor of the future real spot rate by regressing

$$f_t = a_0 + a_1 s_{t,1}^r + u_t \ .$$

Here you could test if the forward rate is a perfect predictor of the future real interest rate by testing if $a_1 = 1$. In this regression you assume that the future inflation rate is constant (and equal to a_0), which is not a realistic assumption if we look far into the future, but it might be reasonable to assume that over a short time interval, like 3 months, the inflation rate is constant. Finally, you could try and sort out if the forward rate primarily reflects rational expectations about future inflation rates or about future real interest rates by running the regression

$$f_t = c_0 + c_1 s_{t,1}^r + c_2 i_t + u_t \ .$$

In this regression the interpretation of c_1 is the extent to which the forward rates reflect future spot rates, c_2 is the extent to which forward rates reflect future inflation, and c_0 is a risk premium. In this regression you can test the expectations hypothesis by testing if $c_1 = c_2 = 1$. In my view the later regression is the most sensible to test, since real interest rates and (certainly) inflation rates vary quite a lot over time.

To round off this topic, let me mention that one can formulate interesting models of the governments expectations versus the market expectations. If the government observes that the market

expects high future inflation, but the government is determined to keep future inflation low, then the government will not issue long bonds for which it has to pay a high "inflation premium". If on the other hand the government expects high future inflation but the market does not, the government can issue long bonds and subsequently erode the value of those by allowing for high inflation. (This typically happens during wars, deliberately or not). If we realize that market participants use the typical maturity of government bond sales as an indicator for the governments resolve to keep inflation low the model becomes a complicated game theoretic model. This you may encounter in advanced monetary macro courses—you need not worry about this issue for the purpose of this class (i.e. not for the exam). I am not sure how realistic such "games" between the government and investors are—I just wanted to point out that many interesting models of monetary policy and inflation include models of the term structure of interest rates.
Consider then the log-price of a discount bond paying off 1$ after t years:

$$\ln P_0 = \ln \frac{1}{(1+S_t)^t} = -t \ \ln(1+S_t) = -t \ s_t \ .$$

It is well known that if you calculate (say) $\ln P_1 - \ln P_0$ then you approximately get the percentage by which P_1 is larger than P_0. This follows from the formula $\ln(1+x) \approx x$, when x is small. Here

$$\ln P_1 - \ln P_0 \ = \ \ln\left(\frac{P_1}{P_0}\right) \ = \ \ln\left(\frac{P_0 + P_1 - P_0}{P_0}\right) \ = \ \ln\left(1 + \frac{P_1 - P_0}{P_0}\right) \ \approx \ \frac{P_1 - P_0}{P_0} \ .$$

Now consider an investor who uses the equation above to determine the price he is willing to pay for a discount bond. He knows what kind of return he needs to invest and sets the price correspondingly. Assume that the rate of return he needs goes up by 1 percent. He then changes the price he is willing to pay to (say) P_0'. The approximate percentage price change will be

$$\ln P_0' - \ln P_0 \approx -t\ (s_t + .01) - (-ts_t) \ = \ -t * .01 \ .$$

EXAMPLE: Consider a 30 year discount bond. Assume that the required yield goes up from 5% to 6%. This is not an unusual change. But the implication is that the 30 year bond immediately looses about $(-30 * .01 =)$ 30% of its value. We can of course calculate the exact change in price: Assume the nominal value is 1000$. Originally you would be willing to pay $1000/(1.05)^{30} = 231.38$\$. When the required return changes to 6% the price becomes $1000/(1.06)^{30} = 174.11$\$. You can find that $(231.38 - 174.11)/231.38 = 25\%$, which is approximately the 30% loss in the value of the discount bond that you would expect. Note that the 25% is somewhat far from 30%, since the approximate formula only is precise for small changes, so in practice you would use the precise formula for changes of this large magnitude.

One of most common headlines in the Wall Street Journal is (approximately) "bonds markets shaken by renewed inflation fears". Why do the bond markets shake so much? The yields and spot rates that we have considered are nominal. Usually we expect the real rates of returns to be roughly stable – in other words we expect that when the rate of inflation goes up by 1% a year the one period interest rates will also go up by about 1%. Therefore the required yields of investors will also go up by about 1% and therefore the price of a 30 year discount bond will approximately drop 30% in price.

It is therefore clear that if you intend to make a short term investment, then the investment strategy of buying 30 year discount bonds and selling them in the market a short period later, is a highly risky strategy. If you have known expenses in 30 years (relevant for pension funds, insurance companies) a 30 year government discount bond is fully safe, but for short term investors this is absolutely not a riskless strategy, and we therefore consider bond as risky assets - even when they (like federal government bonds) are without default risk.

If you buy a discount bond and sell it one year later you will obtain a return, which we denote the 1-year **holding period return**. The one-period holding period return, H_t, on a dollar invested in a t-period zero-coupon bond will be

$$H_t = \frac{P_1 - P_0}{P_0} ,$$

the percent change in price from period 0 to period 1. Here P_0 is the price of the bond at period 0 and P_1 is the price of the same bond at period 1 (the following year). We can of course also calculate holding bond for longer or shorter periods – I leave our the formulas since it is pretty obvious. For a 1-year discount bond, the price in the next period is 1, and we observe that $H_1 = (1 - P_t)/P_t = 1/P_t - 1 = S_t$. For a one year bond, which you hold for a year, the holding period return is equal to the spot rate as it should be. For longer maturity bond this is, however, not the case, since such bonds are subject to price risk, if they are sold before maturity. It is instructive to consider the continuously compounded holding period return, $h_t = \ln(1 + H_t)$. On handout 18 we defined the continuously compounded spot rate s_t, but here we will use the notation $s_{0,t}$ for the continuously compounded spot rate at period 0, and $s_{k,t}$ for the continuously compounded spot rate at period k. With this notation the formula we showed last time is $\ln(P_0) = -t\, s_{0,t}$. It follows that

$$\ln(P_k) = -(t - k)s_{k,t-k} .$$

The log-price in period k is minus $t - k$ times the spot rate of bond with maturity $t - k$. A t period maturity discount bond has transmogrified into a $t - k$ period discount bond after k years. We have

$$h_t = \ln(P_1) - \ln(P_0) = -(t-1)s_{1,t-1} + ts_{0t} ,$$

which implies

$$h_t = s_{0t} - (t-1)(s_{1,t-1} - s_{0t}) . \tag{1}$$

The equation (1) has a simple interpretation. Your holding period return will be equal to the (continuously compounded) spot rate, s_{0t}, if the $(t - 1)$-period spot rate next year is equal to the present t-period spot rate, but if e.g. the spot rate over the year has increased by 1%, then the bond has lost value and the return will go down by $t - 1$ percent. Note that we are comparing t-period spot rates in period 0 with $t-1$ period spot rates in period 1. It is therefore not surprising that 1 year holding period returns are very volatile.

EXAMPLE: Assume that you buy a 21 year bond with a continuously compounded spot rate of 6%, which is the return you would get if you held the bond for 21 years. Which return you get if you sell the bond after a year, depends on the 20 year spot rate next year. Suppose it turns out to be 8%. Then your continuously compounded holding period return would be

$$h_t \;=\; 6\% - 20 * (8\% - 6\%) \;=\; -34\% \;,$$

a pretty bad investment. When the interest rate goes up, the prices of long discount bonds go down, and the holding period return therefore goes down.

The variability of holding period returns on long bonds is usually denoted **interest rate risk**. Elton and Gruber (p. 495) show the 1-year holding period returns to various assets. For long government bonds the numbers are

t	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992
h_t	42.08	2.23	14.80	31.54	24.08	-2.67	9.24	19.03	6.29	18.68	8.09

As you can observe, 1-year holding period returns are indeed very volatile.

In reality, investors differ a lot among themselves. For example, pension funds and life insurance companies may not care about 1-year holding period returns, but only about yield to maturity if they are able to match the payment streams from maturing bonds to future pay-outs. This is sometimes called **perfect matching**. It may be very hard (and not necessarily optimal) to attempt to obtain perfect matching, but nevertheless portfolio managers want to limit interest rate risk. Methods for limiting interest rate risk are known as **immunization**.

Various methods for immunization exists, but rather than using some mechanical rule, you should try and quantify your utility function and then find out which investment maximizes expected utility. Recall, that the higher the curvature of the utility function the more averse you are to risk.

EXAMPLE 1: Assume that you only care about your balance sheet 2 periods from now. Also assume that you can invest in a 2-period discount bond with a spot rate $S_{0,2}$ of 4.5%, or you can invest in a 1-period bond with a spot rate $S_{0,1}$ of 5%. If you invest in a 1-period bond, then you of course have to reinvest the return at an uncertain spot rate $S_{1,1}$ at the end of period 1. We assume that the future spot rate is normally distributed with expected value $ES_{1,1} = 0.05$, and standard deviation 0.2. If we assume you invest 1$ (to be more realistic if might be 100,000$) and evaluate your utility using a log utility function, then the utility of buying a 2-period bond would be

$$\ln((1 + .045)^2) \;=\; 0.088 \;,$$

while the utility of buying a period 1 bond plus reinvesting would be

$$E \ln((1 + S_{0,1})(1 + S_{1,1})) \;=\; \ln(1.05) + E \ln(1 + S_{1,1})) \;=\; 0.078 \;.$$

So for such an investor it is clearly better to buy the long bond with a spot rate of 4.5% than to buy the 1-year bond with a spot rate of 5%, even if the expected spot rate next period is also 5%.

(Note, that the standard error of the return may be unrealistically high in this example, but in the case of long maturity bonds, it would not be. Also note, that it is not so simple to find the expected value of the logarithm of a normally distributed variable - I will not expect you to be able to do this.)

EXAMPLE 2: You might also be in a situation where you only care about your balance sheet 1 period from now. Again you may have the opportunity to invest in either a 1-period bond, which now gives you a safe return, or in a 2-year bond, which you buy at the price P_0 and then sell at the end of period 1, at a random price P_1. Now assume that the 1-period spot rate, $S_{0,1}$, is 5% and the price of the 2-year bond after the end of the first period is such that your return, $P_1/P_0 - 1$ is normally distributed with a mean of 0.05 and a standard deviation of 0.10. You then find that the (expected) utility of investing in a 1-year bond is

$$\ln(1 + .05) \;=\; 0.049 \;,$$

while the expected utility of investing in a 2-year bond and selling at the random price P_1 at the end of period 1 is

$$E \ln(P_1/P_0) \;=\; 0.045 \;.$$

In this situation, it is preferable to buy the 1-period bond since the expected utility of the investment is lower due to the interest rate risk of the 1-year bond.
The important point to get here is that investors should choose between various bond maturities based on maximizing expected utility. (If I ask questions on this in homeworks or exams, I will either use a simple utility function like the quadratic or directly give outcomes and probabilities as, e.g., in homework 4.)

Segmented markets, preferred habitat, liquidity premiums.
Historically the term structure has varied a lot over time. Part of this is explained by varying expectations about the inflation rate (we return to this), but it seems that part of the explanation is the relative demand and supply of bonds of different maturities. As mentioned, insurance companies have fixed obligations far into the future and may not wish to buy short term bonds (the 2-period investor in the examples can be thought of as a simple model of insurance companies in this regard). Similarly, agents with short term obligations may not want to buy long term bonds (the 1-period investors in the example).
Segmented markets refer to the situation where some investors only buy long bonds and some investors only buy short bonds. Since the markets are not connected in such a situation, the term structure will be determined by the demand and supply of long bonds without relation to the price of short bonds and vice versa. This is an extreme model that is not taken very serious by present day economists. **Preferred habitat** refers to the fact that some agents prefer long bonds and some prefer short bonds, but an agent who would prefer to buy long bonds, if the spot rate was the same for long and short bonds, will switch to short bonds if the short spot rate is high enough relative to the long yield. This is exactly the types of agents described in the examples. If the market is dominated by investors in long bonds, while the supply of long bonds is no larger than the supply

of short bonds, then you would expect the short spot rate to be above the long spot rate. Typically we do, however, expect that there are more short term investors in the market. In this case the long spot rate will be higher than the short spot rate. The difference is called the **liquidity premium** or just the **risk premium**. It reflects that the short term investor needs a "premium" in order to buy long bonds which carry interest rate risk for him. The yield curve is typically upward sloping, reflecting liquidity premiums at longer maturities. Also note that the size of the risk premium will depend on two things: a) the relative number of short term investors to long term investors (for given supply) – if a higher number of people want to invest short we would expect the liquidity premium to increase and b) how risk averse the typical investor is – when investors are very risk averse they will only invest long term if the receive a large risk premium, so the yield curve become steeper.

If there is a big change in the demand for long versus short bonds (for example because investors whose preferred habitat is short bonds pull out of the market), should we expect a large change in the term structure? Probably not, due to the structure of the supply side. The supply side is dominated by an agent whose is not likely to have strong preferences for any habitat and who is not likely to have much risk aversion at all – namely Uncle Sam. What are the objectives of the U.S. government when borrowing in the market? The primary objective is of course to regulate the money supply. Why? This is topic of monetary economics. But given that the government wants to sell a certain amount of bonds, the most important objective will be to sell them at as good a price as possible in order to keep interest expenses down. So if investors (the lenders) change their preference towards long bonds driving down the long yields, it is highly likely that the government (the borrower) will take advantage of this, and now issue more of the long bonds which now carries less interest. So due to reactions like this you might expect the yield curve to be a lot more stable over time then it would otherwise be.