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Complete Markets

Consider assets $i = 1, ..., N$ with returns r_i. Assume there are S states of the world (to be revealed after trading) and that $N = S$. Consider dummies D_s that take the value 1 if state $s = 1, ..., N$ occurs. Asset i has return

$$r_i = a_{i1} D_1 + ... + a_{iS} D_N \ .$$

We can define the vector $R = \{r_1, ..., r_N\}'$ and the vector $D = \{D_1, ... D_N\}'$ we have

$$R = A\,D$$

where a_{ij} is the typical element of the $N \times N$ matrix A.

We say that markets are complete (or we need to stress the Arrow-Debreu framework: that a full set of Arrow securities exist) if trading in the assets with returns r_i is equivalent to trading in a full set of Arrow securities. This holds if the matrix A can be inverted, so that

$$D = A^{-1} R \ .$$

In this case, if you need to find how much of each assets if sold or bought, you might solve for Arrow securities and then find the amount of the assets by multiplying by A. Note:

- We need at least as many assets as states-of-the-world.
- The returns need to be linearly independent (i.e., none of the rows in A can be written as a linear combination of the others
- If $N > S$, you still have complete markets if you can choose a subset of assets so that the matrix A can be inverted.
- If PO is the vector of payouts, then the gross returns are (for case of $N = 3$, with subscripts denoting assets and superscripts denoting state-of-the-world) but this is easily generalized):

$$\begin{pmatrix} R_1^1 & R_1^2 & R_1^3 \\ R_2^1 & R_2^2 & R_2^3 \\ R_3^1 & R_3^2 & R_3^3 \end{pmatrix} = \begin{pmatrix} P_1^{-1} & 0 & 0 \\ 0 & P_2^{-1} & 0 \\ 0 & 0 & P_3^{-1} \end{pmatrix} \begin{pmatrix} PO_1^1 & PO_1^2 & PO_1^3 \\ PO_2^1 & PO_2^2 & PO_2^3 \\ PO_3^1 & PO_3^2 & PO_3^3 \end{pmatrix}$$

  So if the payouts are linearly independent, so are the returns and *vice versa*.

Example: Two states of the world, A and B. You have an Arrow security for state A and a safe asset with return r^f. We have

$$
\begin{aligned}
r^A &= 1\,D_A + 0 && (1)\\
r^f &= 1\,D_A + 1\,D_B && (2)\\
& && (3)
\end{aligned}
$$

We can find the Arrow securities as

$$
\begin{pmatrix} D_A \\ D_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} r^A \\ r^f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} r^A \\ r^f \end{pmatrix} = \begin{pmatrix} r^A \\ r^f - r^A \end{pmatrix}
$$