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Moment Generation Function (MFG) for the Normal distribution

The MGF is defined as $E\, e^{tX}$ which for the Normal $N(\mu, \sigma)$ becomes

$$M(t) \;=\; \int \frac{1}{\sigma\sqrt{2\pi}} e^{tx} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

or

$$M(t) \;=\; \int \frac{1}{\sigma\sqrt{2\pi}} \exp(\, tx \;+\; \frac{-(x-\mu)^2}{2\sigma^2}\,) dx$$

Now, recall that we don't know how to integrate difficult functions like this so we need to use a trick, namely to try and see if can rewrite the argument of the exponential function as "$-(x - \text{stuff})^2/(\text{more stuff})$" which has the same form as the argument of the normal distribution which we know what integrates to. (Note that for the purpose of integrating it is only the x-variable that "matters.") This is known as "completing the square."

So let us concentrate on the term $tx \,+\, \frac{-(x-\mu)^2}{2\sigma^2}$ and try to collect all terms involving x:

$$tx \,+\, \frac{-(x-\mu)^2}{2\sigma^2} \;=\; \frac{1}{2\sigma^2}(-(x^2 - 2\mu x + \mu^2) + 2tx\sigma^2) \;=\; \frac{-1}{2\sigma^2}(x^2 - 2\mu x + \mu^2 - 2tx\sigma^2) \;=$$

$$\frac{-1}{2\sigma^2}(x^2 - 2x(\mu + t\sigma^2) + \mu^2) \;=\; \frac{-1}{2\sigma^2}[(x - (\mu + t\sigma^2))^2 - (\mu + t\sigma^2)^2 + \mu^2] \;=$$

$$\frac{-1}{2\sigma^2}[(x - (\mu + t\sigma^2))^2 - 2\mu t\sigma^2 - t^2\sigma^4] \;=\; \frac{-(x - (\mu + t\sigma^2))^2}{2\sigma^2} + \frac{2\mu t\sigma^2 + t^2\sigma^4}{2\sigma^2} \;=$$

$$\frac{-(x - (\mu + t\sigma^2))^2}{2\sigma^2} + \mu t + \frac{1}{2} t^2 \sigma^2 \;=$$

So, finally, we get

$$M(t) \;=\; \frac{1}{\sigma\sqrt{2\pi}} \int e^{\frac{-(x-(\mu+t\sigma^2))^2}{2\sigma^2} + \mu t + \frac{1}{2}t^2\sigma^2} dx \;=\; \frac{1}{\sigma\sqrt{2\pi}} \int e^{\mu t + \frac{1}{2}t^2\sigma^2} e^{\frac{-(x-(\mu+t\sigma^2))^2}{2\sigma^2}} dx \;=$$

$$e^{\mu t + \frac{1}{2}t^2\sigma^2} \frac{1}{\sigma\sqrt{2\pi}} \int e^{\frac{-(x-(\mu+t\sigma^2))^2}{2\sigma^2}} dx$$

which, since the last term is a normal density, gives

$$M(t) \;=\; e^{\mu t + \frac{1}{2}t^2\sigma^2}$$

MGF for Poisson

$$E\, e^{tX} \;=\; \Sigma_{x=0}^{\infty}\, e^{tx}\; e^{-\lambda}\, \frac{\lambda^x}{x!} \;=\; \Sigma_{x=0}^{\infty}\, (e^t)^x\; e^{-\lambda}\, \frac{\lambda^x}{x!} \;=\; e^{-\lambda}\; \Sigma_{x=0}^{\infty}\, \frac{(e^t\lambda)^x}{x!} \;=\; e^{-\lambda}\; e^{e^t\lambda} \;=\; e^{\lambda\,(e^t-1)}$$

MGF for Binomial

$$E\,e^{tX} \;=\; \Sigma_{x=0}^{n}\, e^{tx}\, \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \;=\; \Sigma_{x=0}^{n}\, \frac{n!}{x!(n-x)!} (e^t p)^x (1-p)^{n-x}$$

$$=\; [\, e^t p + (1-p)\,]^n \;=\; [\, 1 + (e^t - 1)\, p\,]^n$$

MGF for Exponential

$$E\,e^{tX} \;=\; \frac{1}{\theta} \int_0^{\infty} e^{tx}\, e^{-x/\theta} \;=\; \frac{1}{\theta} \int_0^{\infty} e^{-x(1/\theta - t)} \;=\; \frac{1}{\theta} \int_0^{\infty} e^{-x(\frac{1-\theta\, t}{\theta})} \;=\; \frac{1}{\theta} \frac{\theta}{1-\theta\, t} \;=\; \frac{1}{1-\theta\, t}$$