The Euler Equation-general form

Consider a consumer who maximize the *expected* utility of present and future consumption subject to some constraints. (Typically, a life time budget constraint but there might be other constraints involved, for example, no access to credit in some periods.) The criterion function is

$$U(c_t) + \frac{1}{1+\rho}EU(c_{t+1}) + (\frac{1}{1+\rho})^2 EU(c_{t+2}) + ...$$

where the time horizon may be finite or infinite (if the time horizon is infinite, it is implicit that the functions are such that the criterion function remain finite). Assume that the consumer has access to a financial asset (or several, there will be an Euler equation for each!) with a random (safe as a special case) return r_{t+1} (the return, like the earning on a stock is generally not known at period t). The consumer, however, knows the distribution of the return and takes that into account (rational expectations). The consumer is free to adjust *in period t and in period t+1* the amount invested in the asset by an amount x and adjust consumption correspondingly. Now assume that the consumer has solved the optimization problem and solved for the optimal consumption c_t and the have planned optimal future consumption $c_{t+1}, c_{t+2}, ...$ (When the consumer gets to period $t + 1$ he or she will re-optimized based on what happens in between, for example, what the return on the consumers assets turned out to be.) At period t the consumer has the option of, for example, lowering consumption from its optimal level and invest it in the asset under consideration and add the gross returns to consumption in period $t + 1$. This is feasible whatever the initial contraints on the problem are, as long as they do not constrain period t and $t+1$ actions.

So we can consider the feasible value of the criterion function

$$W = U(c_t - x) + \frac{1}{1+\rho}EU(c_{t+1} + x * (1 + r_{t+1})) + (\frac{1}{1+\rho})^2 EU(c_{t+2}) + ...$$

as a function of x. Of course, if the original consumption plans were optimal, that means that the derivative $\frac{dW}{dx}$ of W with respect to x is zero for $x = 0$. This gives the result that

$$0 = -U'(c_t) + E\{U'(c_{t+1})\frac{1 + r_{t+1}}{1+\rho}\}$$

or

$$U'(c_t) = E\{U'(c_{t+1})\frac{1 + r_{t+1}}{1+\rho}\} \ .$$

This is the <u>Euler Equation</u>, which has a very large number of applications in economics. (NOTE: be sure to observe that the expectation on the right-hand side involves the product of the marginal utility and the random return; making mistakes here is very costly at exam time.) If you look back

at the way we solved the model with an infinite horizon and no uncertainty, the solution involved a) the Euler equation and b) the budget constraint. This is very typical.

Hall's PIH model

It is recommended that you read the original article in the JPE 1978; it is one of the most cited articles in economics (about 700 references last time I checked) and it was seen at the time as as an important contribution to the rational expectations revolution.

Hall assume that agents optimized utility under uncertainty as in the previous paragraph, but further assumed that agents have access to a risk free asset with return r and that $r = \rho$. Hall further assumed that agents have a quadratic utility function

$$U(c_t) = c_t - \frac{a}{2}c_t^2$$

such that marginal utility is linear in consumption

$$U'(c_t) = 1 - ac_t .$$

The Euler equation now becomes (after dividing over by a):

$$c_t = E\{c_{t+1}\} ,$$

which implies that

$$E\Delta c_{t+1} = 0 .$$

In words, the PIH model has the strong prediction that the change in consumption is zero! Econometricians have spent much time testing this prediction. In order to test the model, we need to assume that the relation holds each period, in other words, that at each period t,

$$E_t\Delta c_{t+1} = 0 ,$$

where "E_t'' means that we take the expectation taking everything known at period t and earlier as given. This is called a <u>conditional expectation</u> (and modern dynamic economics makes extensive use of conditional expectations).

In statistics a variables which at any period t has period $t+1$ expectation equal to the period t value is called a *martingale*, so Hall's model predicts that consumption is a martingale (although it is often, imprecisely, referred to as a random walk).

Hall tested the model on aggregate data (it has many times and often with more success been tested on micro-data, but we will restrict ourself to macro here) and found only weak evidence against it. Later papers have often found more evidence against the result and we will shortly cover some of this material.