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The Euler Equation from the Benveniste-Scheinkman equation.

The first order condition for maximum in the Bellman equation is

$$(*)\qquad \frac{\partial}{\partial u} r(x_t, h(x_t)) + \beta V'(x_{t+1}) \frac{\partial}{\partial u} g(x_t, u_t) \;=\; 0\,,$$

and the Benveniste-Scheinkman equation (holding of corners under regularity conditions) is:

$$V'(x_t) \;=\; \frac{\partial}{\partial x} r(x_t, h(x_t)) + \beta V'(x_{t+1}) \frac{\partial}{\partial x} g(x_t, u_t)\,.$$

In the case where $\frac{\partial}{\partial x} g(x_t, u_t) = 0$ (meaning that the researcher has formulated the model such that the transition equation is not a function of the state variable), the Benveniste-Scheinkman equation reduces to the envelope condition

$$V'(x_t) \;=\; \frac{\partial}{\partial x} r(x_t, h(x_t))\,.$$

If we substitute this into (*), we get the Euler equation:

$$\frac{\partial}{\partial u} r(x_t, h(x_t)) + \beta \frac{\partial}{\partial x} r(x_{t+1}, h(x_{t+1})) \frac{\partial}{\partial u} g(x_t, u_t) \;=\; 0\,.$$

(Note that the book lost β here, like I often do.)

Simple Example

I will show how the standard Euler equation from Macro II is a special case, looking at a simple investment problem. Consider an agent maximizing $\sum_{t=1}^{\infty} U(c_t)$ for given income y_t and interest r_t, subject to the law of motion:

$$A_{t+1} = y_t + A_t\,(1 + r_t) - C_t\,,$$

where A is assets invested in 1 period "bonds." It takes a little experimenting to choose the state variable $x_t = A_t$ (actually, that one is obvious) and $u_t = A_{t+1}$ (i.e., the function $g(A_{t+1})$ is just the identity function that returns A_{t+1}). Then the Euler equation becomes

$$\frac{\partial}{\partial A_{t+1}} U(\, y_t + A_t\,(1 + r_t) - A_{t+1}\,) + \beta \frac{\partial}{\partial A_{t+1}} U(\, y_{t+1} + A_{t+1}\,(1 + r_{t+1}) - A_{t+2}\,) * 1 \;=\; 0\,.$$

which, after substitution in for consumption, reduces to

$$-U'(C_t) + \beta\, U'(C_{t+1})(1 + r_{t+1}) = 0\,.$$