Convolution Properties

DSP for Scientists

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Properties of Delta Function

- $\delta[n]$: Identity for Convolution

- $x[n] * \delta[n] = x[n]$
- $x[n] * k\delta[n] = kx[n]$
- $x[n] * \delta[n+s] = x[n+s]$

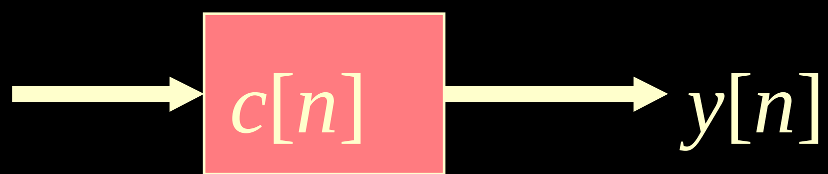
Mathematical Properties of Convolution (Linear System)

- Commutative: $a[n] * b[n] = b[n] * a[n]$
- $a[n] \longrightarrow \boxed{b[n]} \longrightarrow y[n]$
- *Then*
- $b[n] \longrightarrow \boxed{a[n]} \longrightarrow y[n]$

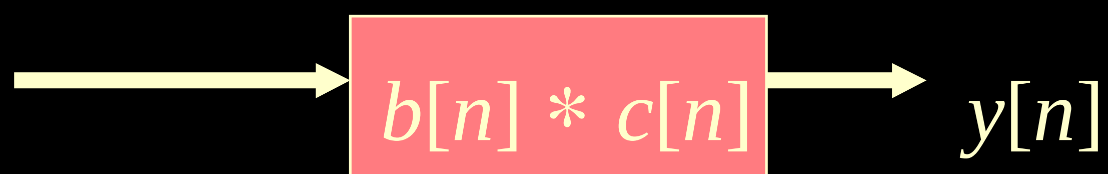
Properties of Convolution

- Associative:

  $\{a[n] * b[n]\} * c[n] = a[n] * \{b[n] * c[n]\}$
- *If*
- $a[n] * b[n]$



- *Then*
- $a[n]$



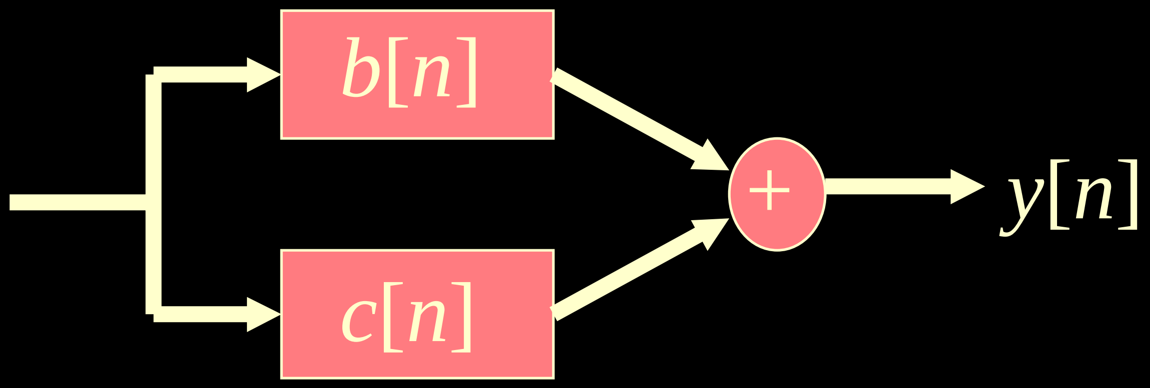
Properties of Convolution

- Distributive

$$a[n]*b[n] + a[n]*c[n] = a[n]*\{b[n] + c[n]\}$$

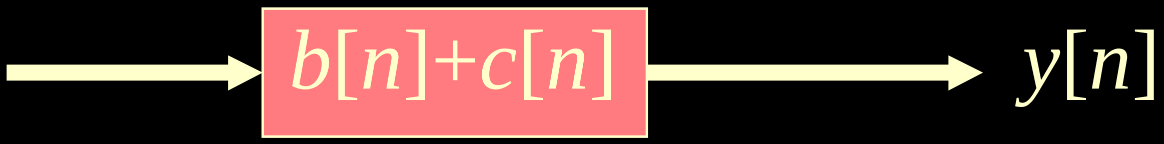
If

- $a[n]$



Then

- $a[n]$



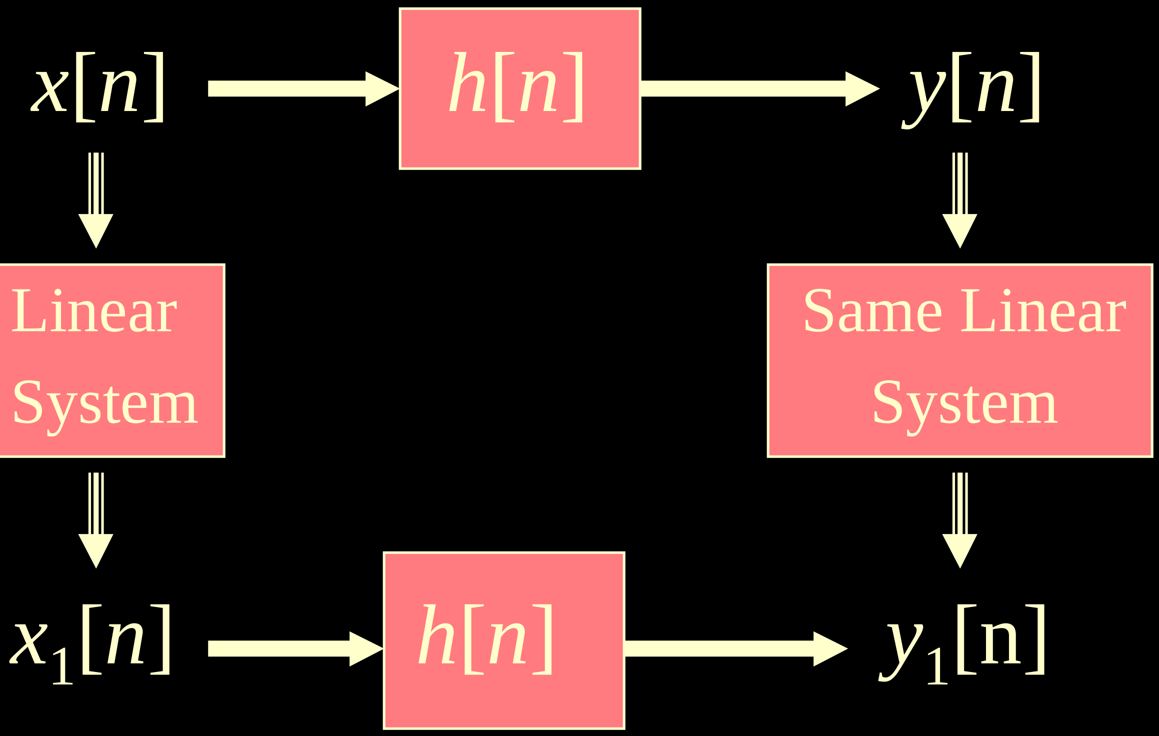
Properties of Convolution

- Transference: between Input & Output

- Suppose $x[n] * h[n] = y[n]$
- If L is a linear system,
- $x_1[n] = L\{x[n]\}$, $y_1[n] = L\{y[n]\}$
- Then
- $x_1[n] * h[n] = y_1[\mathrm{n}]$

Continue

- *If*



Then

Special Convolution Cases

- Auto-Regression (AR) Model
- $y[n] = \sum_{k = 0, M - 1.} h[k]x[n - k]$
- For Example: $y[n] = x[n] - x[n - 1]$
- (first difference)

Special Convolution Cases

- Moving Average (MA) Model
- $y[n] = b[0]x[n] + \sum_{k = 1, M - 1} b[k]\, y[n - k]$
- For Example: $y[n] = x[n] + y[n - 1]$
- (Running Sum)
- AR and MA are Inverse to Each Other

Example

- For One-order Difference Equation (MA Model)
- $y[n] = ay[n - 1] + x[n]$
- Find the Impulse Response, if the system is
- (a) Causal
- (b) Anti-causal

Causal System Solution

- Input: $\delta[n]$ Output: $h[n]$
- For Causal system, $h[n] = 0, n < 0$
- $h[0] = ah[-1] + \delta[0] = 1$
- $h[1] = ah[0] + \delta[1] = a$
- …
- $h[n] = a^n u[n]$

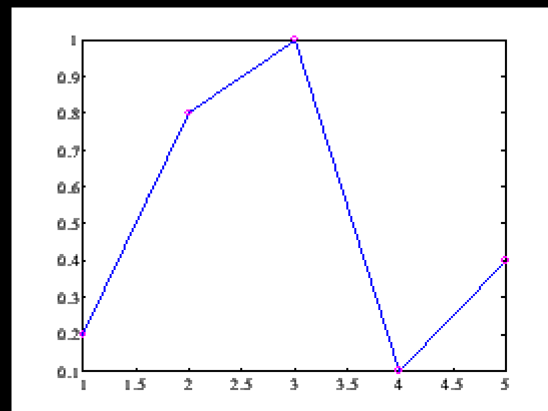
Anti-causal

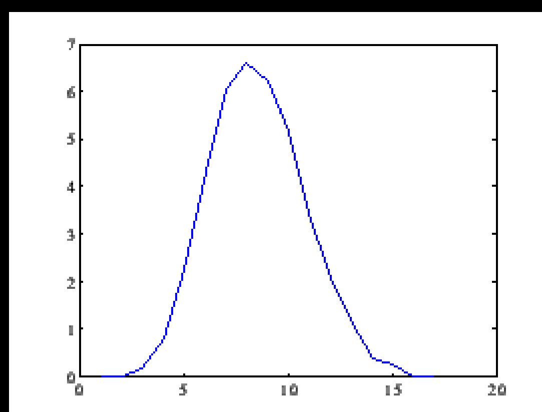
- Input: $x[n] = \delta[n]$ Output: $y[n] = h[n]$
- For Anti-Causal system, $h[n] = 0$, $n > 0$
- $y[n - 1] = (y[n] - x[n]) / a$
- $h[0] = (h[1] - \delta[1]) / a = 0$
- $h[-1] = (h[0] - \delta[0]) / a = - a^{-1}$
- …
- $h[- n] = - a^{-n} \Rightarrow h[n] = -a^{n}u[-n - 1]$

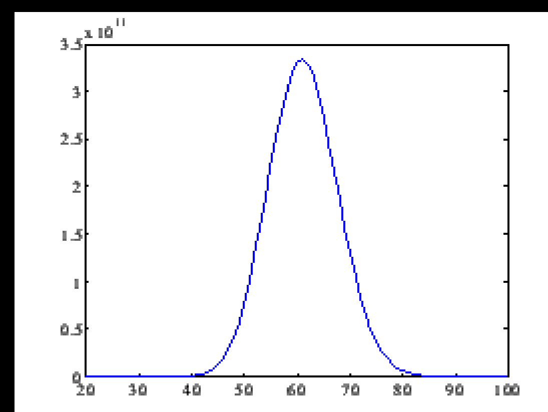
Central Limit Theorem

- If a pulse-like signal is convoluted with itself many times, a Gaussian will be produced.
- $a[n] \geq 0$
- $a[n] * a[n] * a[n] * \ldots * a[n] =$???

Central Limit Theorem





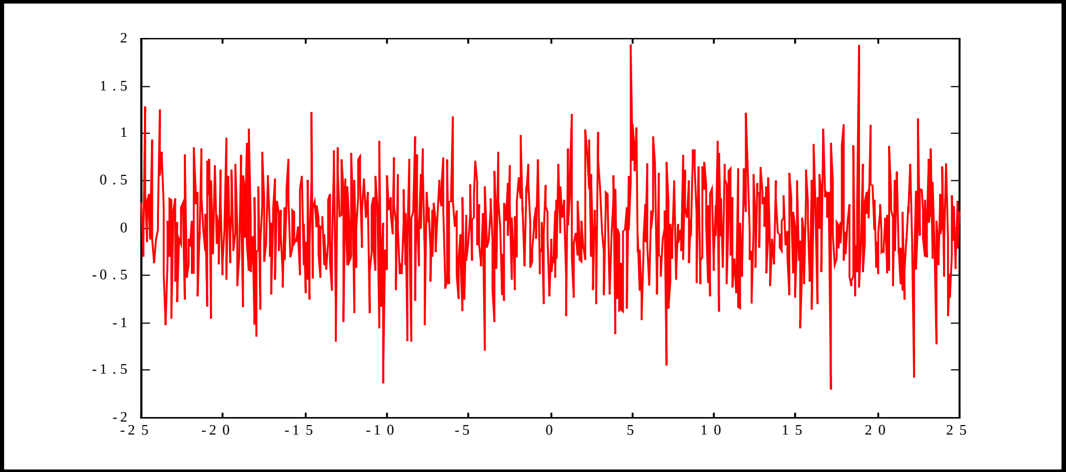


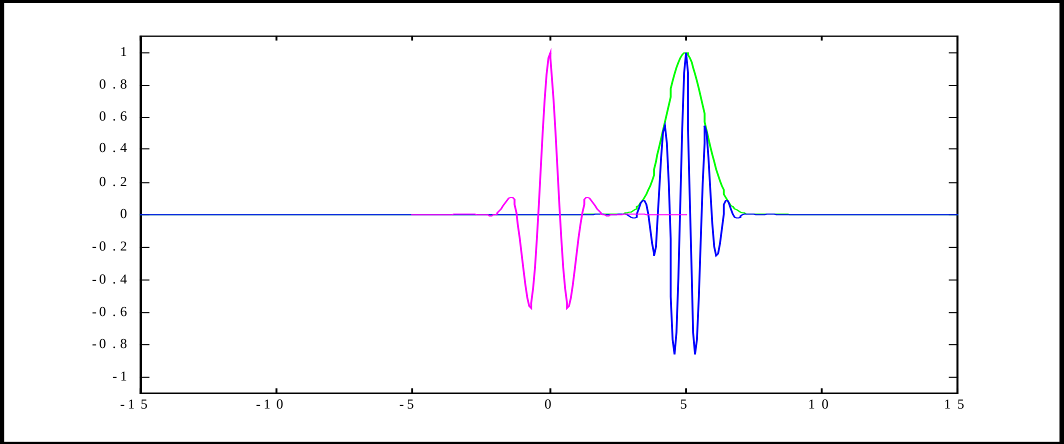
Correlation !!!

- Cross-Correlation
- $a[n] * b[-n] = c[n]$

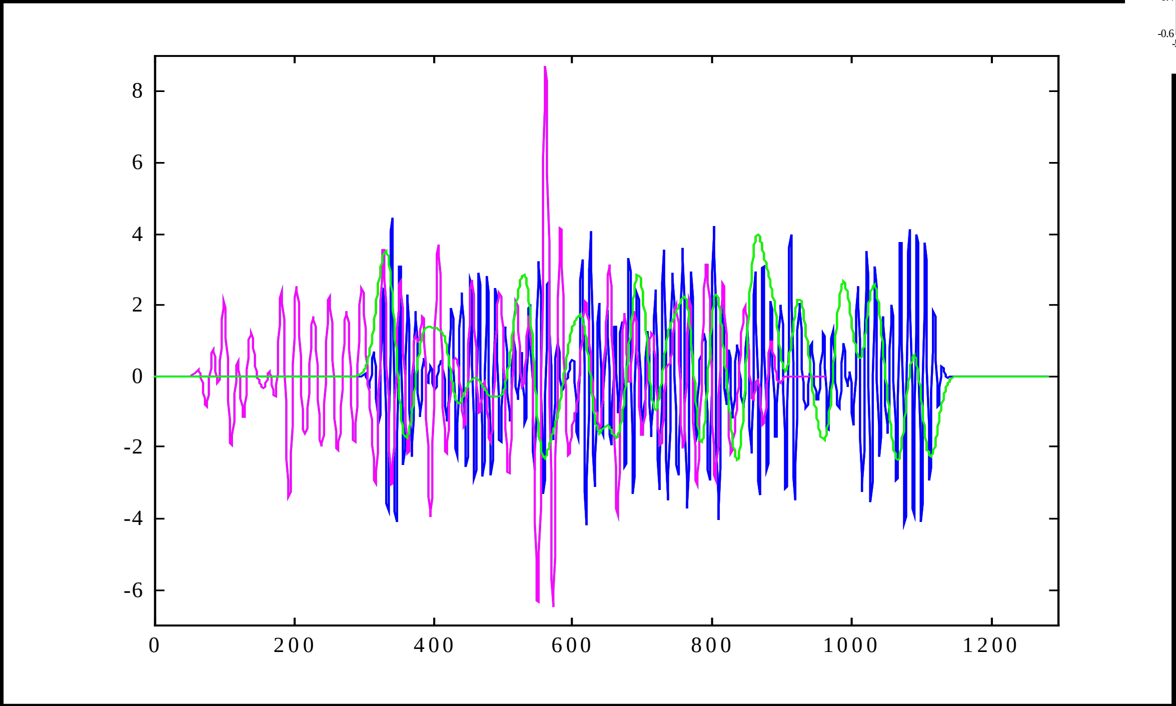
- Auto-Correlation:
- $a[n] * a[-n] = c[n]$
- Optimal Signal Detector (Not Restoration)

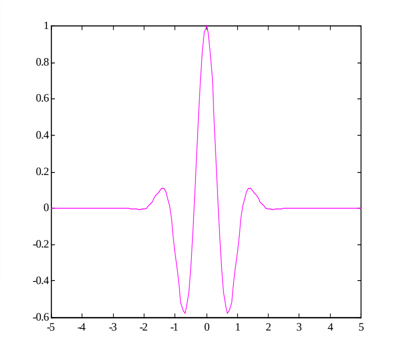
Correlation Detector





Correlation Results

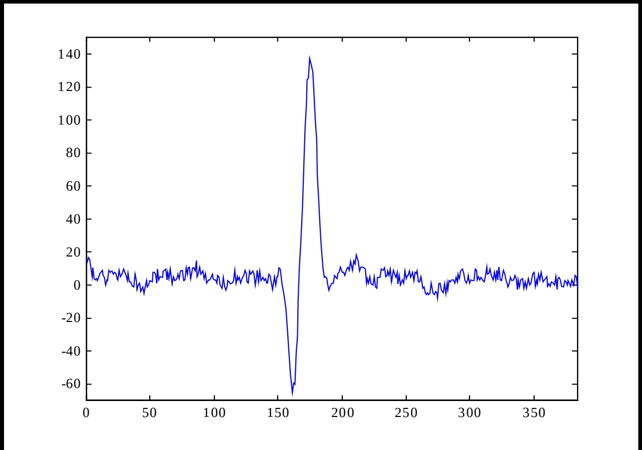


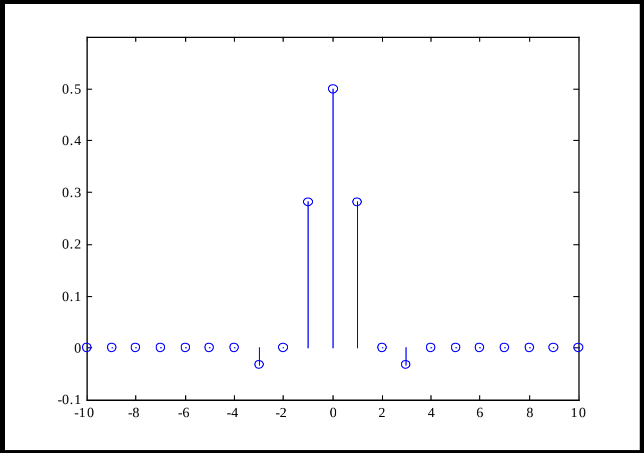


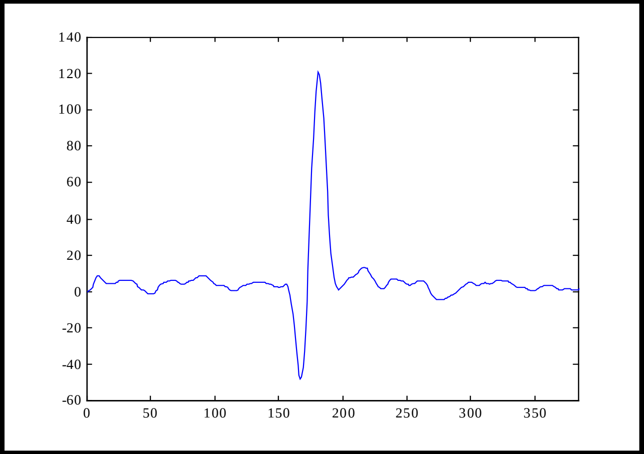
Low-Pass Filter

- Filter $h[n]$:
- Cutoff the high-frequency components (undulation, pitches), smooth the signal
- $\sum h[n] \neq 0, \qquad \sum (-1)^n h[n] = 0, \quad n = -\infty, \infty$

Example: Lowpass



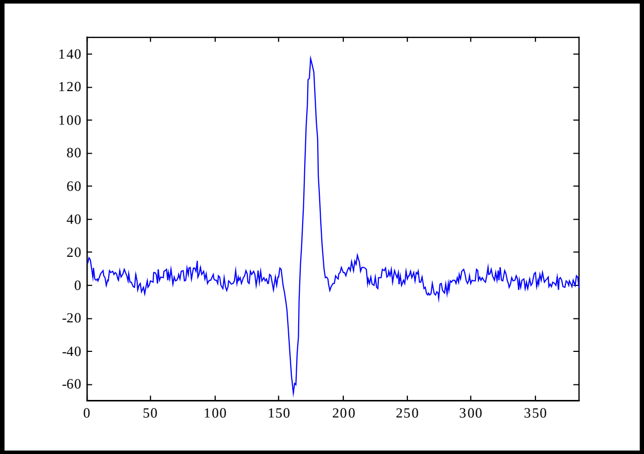


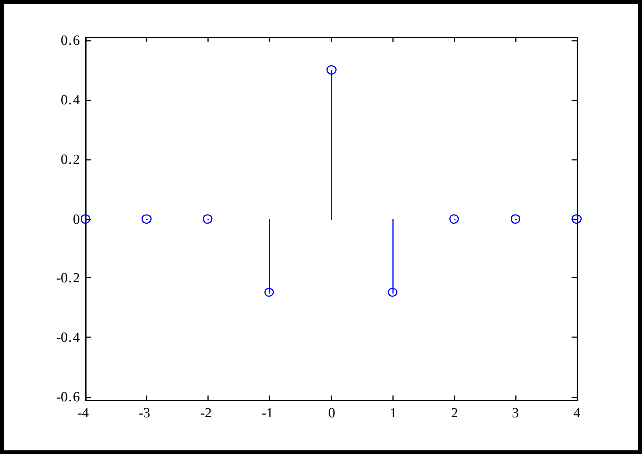


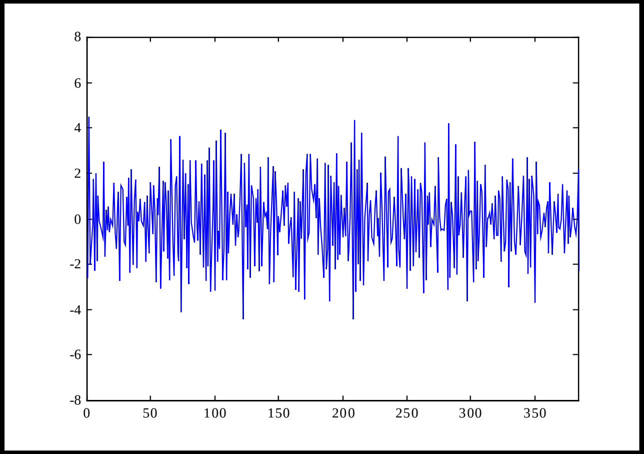
High-Pass Filter

- Filter $g[n]$:
- Remove the Average Value of Signal (Direct Current Components), Only Preserve the Quick Undulation Terms
- $\sum_n g[n] = 0$

Example: Highpass







Delta Function

- $x[n] * \delta[n] = x[n]$
- Do not Change Original Signal
- Delta function: All-Pass filter

- Further Change: Definition (Low-pass, High-pass, All-pass, Band-pass …)