1. Consider the matrices
\[ A = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \]
and the corresponding ODEs \( \dot{x} = Ax \) and \( \dot{y} = By \) in \( \mathbb{R}^3 \). Let \( x(t) \) and \( y(t) \) be solutions of these ODEs with the same initial condition \( x(0) = y(0) = (z_1, z_2, z_3)^T \), where \( z_3 \neq 0 \).

(a) What is the behaviour of \( |x(t)| \) and \( |y(t)| \) as \( t \to \infty \)? (That is, do they decay to 0? Diverge to \( \infty \)? Remain bounded away from both?)

(b) Which of \( |x(t)| \) and \( |y(t)| \) dominates the other as \( t \to \infty \)? That is, does \( \lim_{t \to \infty} \frac{|x(t)|}{|y(t)|} \) equal 0 (in which case \( |y(t)| \) dominates), or does it equal \( \infty \) (in which case \( |x(t)| \) dominates)?

2. (a) Suppose \( A \) is an \( n \times n \) matrix and \( B(t) \) is a continuous map \( (B : \mathbb{R} \to \mathbb{R}^n) \). Prove that all solutions of
\[ \dot{x} = Ax + B(t) \]
are of the form
\[ x(t) = e^{At} \left[ \int_0^t e^{-As} B(s) \, ds + C \right] \]
where \( C \) is a constant vector in \( \mathbb{R}^n \).

(b) Using part (a) or otherwise show that if \( n = 2 \) then all solutions to
\[ \dot{x} = Ax + B(t), \quad x(0) = x_0 \]
tend to the origin if
\[ A = \begin{pmatrix} -1 & 1 \\ 1 & -4 \end{pmatrix} \]
and the continuous map \( B(t) \) is bounded.

3. (a) Find all equilibria for the system \( \dot{x} = f(x, \mu) \) where
\[ f(x, \mu) = \mu x - x^3 \]
and determine bifurcation points in the \( (\mu, x) \) plane. Plot a bifurcation diagram in the \( (\mu, x) \) plane indicating stable branches and unstable branches of equilibria.

(b) Consider a simple population growth model with harvesting,
\[ \dot{x} = x - x^2 - p \]
where \( x \) represents the size of the population and \( p \) is positive.

i. Find the equilibria of this model.

ii. Show that if the harvesting rate \( p \) satisfies \( p > \frac{1}{4} \) then the population dies out.

iii. Show that if \( 0 < p < \frac{1}{4} \) then the longterm fate of the population depends on the initial population size.

4. (a) Show that the equilibrium of the system
\[ \dot{x} = -2x + y^2 x \\
\dot{y} = -3y x^2 + y^2 x^3 \]
at the origin is stable by using a Lyapunov function (or otherwise).
(b) Find all equilibria for the system $\dot{x} = f(x, \mu)$ where

$$f(x, \mu) = \mu x - x^2$$

and determine bifurcation points in the $(\mu, x)$ plane. Plot a bifurcation diagram in the $(\mu, x)$ plane indicating stable branches and unstable branches of equilibria.

5. (a) Show that the Poincaré-Bendixson theorem for planar flows does not extend to smooth flows on $\mathbb{R}^4$ by giving an example of a system

$$\dot{p} = f(p),$$

where $f : \mathbb{R}^4 \to \mathbb{R}^4$ is smooth, which has a bounded flow invariant region without a fixed point yet no periodic solution exists in the region. **Hint:** it is possible to find an example of the form $f(p) = Ap$ where $A$ is a $4 \times 4$ matrix.

(b) Consider the planar system of differential equations

$$\dot{x} = \lambda + x + y - xy$$
$$\dot{y} = \lambda x(2 - x)$$

depending on the parameter $\lambda \neq 0$.

i. Find all equilibria for this system.

ii. Determine the type of each equilibrium (source, sink, saddle, etc.) as a function of $\lambda$.

6. (a) State the Picard-Lindelöf theorem on existence and uniqueness of solutions to the ODE $\dot{x} = f(x)$ for a vector field $f : U \to \mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is an open domain.

(b) State the Banach Fixed Point Theorem. Define Picard iteration and explain its role in the proof of the Picard-Lindelöf Theorem. **(You do not need to give a complete proof of the theorem, just describe how Picard iteration is used.)**

(c) Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the ODE $\dot{x} = f(x)$ has multiple solutions for some initial condition $x_0$. Write down at least two of these solutions.

(d) Give an example of a $C^1$ function $f : \mathbb{R} \to \mathbb{R}$ such that the ODE $\dot{x} = f(x)$ with initial condition $x(0) = x_0$ does not have a solution that is defined on all of $[0, \infty)$. Find the maximal value of $T$ such that your ODE has a solution on $[0, T)$, and explain what happens to $x(t)$ as $t \to T^-$.

7. (a) Consider ODEs of the form

$$x^{(n)} + a_{n-1}x^{(n-1)} + a_{n-2}x^{(n-2)} + \cdots + a_1 \dot{x} + a_0 x = 0, \quad (\star)$$

where $a_j \in \mathbb{R}$. What is the smallest value of $n$ for which the coefficients $a_j$ can be chosen so that each of $x(t) = t \sin(t)$ and $x(t) = t^2 e^t$ is a solution of $(\star)$?

(b) Let $A(t)$ be a time-dependent $n \times n$ matrix with real entries, and $\Phi(0)$ an $n \times n$ matrix. Is $\Phi(t) = e^{\int_0^t A(s) \, ds} \Phi(0)$ always a solution of $\dot{\Phi}(t) = A(t) \Phi(t)$? If so, prove it; if not, state why this may fail. **(A heuristic explanation is enough.)**

8. Determine the stability of the equilibrium at the origin for the system as a function of $\alpha$.

$$\dot{x} = x^2 - xy$$
$$\dot{y} = -y + \alpha x^2$$

**Hint:** construct a center manifold and analyze the flow on it.

9. (a) Show that if $A$ is a $2 \times 2$ matrix with eigenvalues $a + ib, a - ib$ with $a > 0$ and $b > 0$, then there is a basis for $\mathbb{R}^2$ such that $A$ has the form

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
(b) Give an example of a real $3 \times 3$ matrix $A$ such that $\dot{v} = Av$ has a nontrivial periodic solution of period $2\pi$ for the initial condition $v_0 = (0, 1, 1)^T \in \mathbb{R}^3$.

10. Show that the system
$$\ddot{x} + (x^2 + 2\dot{x}^2 - 1)\dot{x} + x = 0$$
has a non-constant periodic solution. *Hint: Find a flow-invariant region.*

11. Consider two conjugate flows $\varphi_t$ and $\psi_t$ on $\mathbb{R}^n$; that is, $\psi_t = h^{-1} \circ \varphi_t \circ h$ where $h : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism (both $h$ and $h^{-1}$ are continuous). Show that
$$h(\omega_\psi(x)) = \omega_\varphi(h(x))$$
where $\omega_\psi(x)$ is the omega limit set of the flow $\psi$ starting at $x$.

12. (a) Suppose $A$ is a real $3 \times 3$ matrix and $(A - 2I)^3 v = 0$ for all vectors $v \in \mathbb{R}^3$. Write down the possible Jordan canonical forms for $A$.

(b) What is the smallest degree $n > 0$ for which there is a differential equation of the form
$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_n x = 0$$
which has all the functions $\cos(2t)$, $t \cos(2t)$, $\sin(2t)$ as solutions? Find such a differential equation explicitly.

13. (a) Suppose that $H(x, y)$ is a $C^2$ function on the plane and
$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}.$$  
Show that $H$ is constant on the flow lines generated by the vector field, i.e. $H(x(t), y(t))$ is constant in $t > 0$.

(b) Using (a) or otherwise show that if $(x(0), y(0)) = (1, 0)$ is the initial condition for the flow generated by
$$\dot{x} = y, \quad \dot{y} = -x - x^3,$$
then the solution $(x(t), y(t))$ satisfies $y(t)^2 \leq \frac{3}{2}$ for all $t$.

14. (a) Suppose $A$ is a real $n \times n$ matrix and $B(t)$ is a continuous map $(B : \mathbb{R} \to \mathbb{R}^n)$. Prove that all solutions of
$$\dot{x} = Ax + B(t)$$
are of the form
$$x(t) = e^{At} \left[ \int_0^t e^{-As} B(s) \, ds + C \right],$$
where $C$ is a constant vector in $\mathbb{R}^n$.

(b) Suppose $A$ is a real $n \times n$ matrix and there exists a constant $C > 0$ such that $|e^{At} v| \leq C|v|$ for all $t > 0$ and $v \in \mathbb{R}^n$, where $|v|$ denotes the usual Euclidean length of a vector $v \in \mathbb{R}^n$. What does this imply about the Jordan canonical form of $A$? State your reasons carefully.
Suppose that $A$ is a real $n \times n$ matrix and there exists a constant $C > 0$ such that $|e^{At}v| \leq C|v|$ for all $t > 0$ and $v \in \mathbb{R}^n$, and furthermore that $B(t)$ is a continuous $\mathbb{R}^n$-valued function which satisfies $\lim_{t \to \infty} B(t) = 0$. Show that for all initial conditions $x_0$ with $|x_0| \leq 1$, there is a constant $K$ (uniform over all $|x_0| < 1$) such that the initial value problem

$$\dot{x} = Ax + B(t), \quad x(0) = x_0$$

satisfies $|x(t)| < K$ for all $t > 0$.

15. (a) Briefly explain (one paragraph or less) the technique of Picard iteration. Compute three Picard iteration terms $x_0(t), x_1(t), x_2(t)$ for the initial value problem:

$$\dot{x} = \sin(x); \quad x(0) = \pi/2.$$  

(b) Consider the system of differential equations

$$\begin{align*}
\dot{x} &= y - x \\
\dot{y} &= x - y - xz \\
\dot{z} &= xy - \alpha z
\end{align*}$$

where $\alpha \in \mathbb{R}$. Note that the origin is an equilibrium for the system. Determine the stability of the origin as a function of the parameter $\alpha$. Hint: it may be helpful to consider a Lyapunov function.

16. (a) Suppose $A$ is a $4 \times 4$ real-valued matrix and the solutions to $\det(A - \lambda I) = 0$, where $I$ is the $4 \times 4$ identity matrix are, counting multiplicity, $\lambda \in \{-1, -1, -i, i\}$. Write down all possible upper real Jordan canonical forms $B$ for $A$. Give the general solution to $\dot{x} = Bx$ for all such upper real Jordan canonical forms $B$.

(b) Show that if $A$ is a $2 \times 2$ real-valued matrix with eigenvalues $a + ib, a - ib$, with $a > 0, b > 0$, then there is a basis for $\mathbb{R}^2$ such that $A$ has the form

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$  

17. Fix a parameter $\beta \in \mathbb{R}$ and consider the ODE

$$\begin{align*}
\dot{x} &= \beta x + y - x^3, \\
\dot{y} &= -x - y^3.
\end{align*}$$  

(\dagger)

(a) Show that when $\beta \leq 0$, the fixed point at the origin is stable, and its basin of attraction is all of $\mathbb{R}^2$.

(b) Show that when $\beta > 0$, the fixed point at the origin is unstable. Determine for which values of $\beta$ it is

i. a saddle (both attracting and repelling directions);

ii. a proper unstable node (repelling, real eigenvalues, semisimple);

iii. an improper unstable node (repelling, real eigenvalues, a non-trivial Jordan block);

iv. an unstable focus (repelling, complex eigenvalues).

For each of the above cases that occur, fix a value of $\beta$ that realizes that case and draw the corresponding phase portrait for (\dagger) in a small neighborhood of the origin.

(c) It can be shown that for $\beta \leq 1$, the origin is the only equilibrium point of (\dagger). Use this fact to prove that for $0 < \beta \leq 1$, the system (\dagger) has a periodic orbit $\Gamma$.

(d) Extra credit: Find the smallest value of $\beta$ for which (\dagger) has an equilibrium point besides the origin.
18. Let \( f: \mathbb{R}^3 \to \mathbb{R}^3 \) be a \( C^1 \) vector field and \( \gamma: \mathbb{R} \to \mathbb{R}^3 \) a periodic solution of the ODE \( \dot{x} = f(x) \), with period \( T \). Let \( A(t) := Df(\gamma(t)) \) and let \( \Phi(t) \) be a fundamental matrix solution of \( \dot{\Phi} = A(t)\Phi \).

Let \( \Gamma = \{\gamma(t) \mid t \in [0, T]\} \subset \mathbb{R}^3 \) be the periodic orbit parameterized by \( \gamma \). Suppose that \( \Phi(T) \) has an eigenvalue inside the unit circle, and prove that there exists \( x \in \mathbb{R}^3 \setminus \Gamma \) such that \( \omega(x) = \Gamma \).

You may use any theorems proved during the lectures, but you must cite them by name and use the proper terminology.

19. Consider the two-dimensional system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + y(4 - 2x^2 - y^2).
\end{align*}
\]

Using the Poincaré-Bendixson theorem or otherwise, show that this system has a non-trivial periodic solution.

20. Using Liapunov’s method or otherwise, determine the stability of the origin \((0,0)\) for the following system. If you use Liapunov’s method, state carefully the assumptions and conclusion of the Liapunov stability criterion you use.

\[
\begin{align*}
\dot{x} &= -3x^3 + 2xy^2 \\
\dot{y} &= -y^3
\end{align*}
\]

21. (a) Let \( A \) be an \( n \times n \) matrix. Let \( \lambda \) be a real eigenvalue of \( A \) and \( E(\lambda, k) = \{v \in \mathbb{R}^n : (A - \lambda I)^k v = 0\} \).

Show that \( E(\lambda, k) \) is invariant under the flow generated by

\[
\begin{align*}
\dot{x} &= Ax, \\
&x(0) = x_0,
\end{align*}
\]

i.e. if \( x_0 \in E(\lambda, k) \), then \( x(t) \in E(\lambda, k) \).

(b) Give an example of a \( 3 \times 3 \) real matrix \( A \) such that the solution to

\[
\begin{align*}
\dot{x} &= A, \\
&x(0) = x_0
\end{align*}
\]

is

i. periodic with period \( 2\pi \) if \( x_0 \) lies in the plane spanned by the vectors

\[
\begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

ii. approaches the origin at an exponential rate if

\[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

(Editor’s note: There is possibly a typo in the first displayed line in part (b); perhaps it should read \( \dot{x} = Ax, \ x(0) = x_0. \))
22. (a) Determine whether or not the equilibrium of the system
\[\dot{x} = -2x + y^2x,\]
\[\dot{y} = -3yx^2 + y^3x^4,\]
at the origin is stable or asymptotically stable or neither by using a Lyapunov function or otherwise.

(b) Consider the two-dimensional system
\[\dot{x} = y,\]
\[\dot{y} = -x + y(4 - 2x^2 - y^2).\]
Show that this system has a non-trivial periodic solution.

Hint: Consider \(\frac{dr}{dt}\) where \(r = x^2 + y^2\) and use Poincaré-Bendixson.

23. Find an approximate center manifold \(W^c_x\) near the origin and the approximation to the flow on \(W^c_x\) for the system
\[\dot{x} = \alpha x^2 + y^2,\]
\[\dot{y} = -y + x^2.\]
Carry the expansion to high enough order to determine stability in terms of the real parameter \(\alpha\) (at least to third order).

24. Consider a simple population growth model with harvesting,
\[\dot{x} = \alpha x - \beta x^2 - \gamma,\]
where \(x\) represents the size of the population and \(\alpha, \beta,\) and \(\gamma\) are positive. We will consider \(\alpha\) and \(\beta\) as fixed and \(\gamma\) as a parameter to be varied.

(a) Show that if the harvesting rate \(\gamma\) satisfies \(\gamma > \frac{\alpha^2}{4\beta}\), then the population dies out.

(b) Show that if \(0 < \gamma < \frac{\alpha^2}{4\beta}\), then there are two equilibria, and describe how the longterm fate of the population depends on the initial population size.

25. Suppose \(g(t)\) is a real-valued continuous function and \(g(t) \geq 0\) for all \(t \geq 0\). Suppose in addition that for all \(t \geq 0\),
\[g(t) \leq C + K \int_0^t g(s) \, ds,\]
where \(C > 0\) and \(K > 0\) are positive constants. Show that
\[g(t) \leq Ce^{Kt}\]
(this is a version of Gronwall’s inequality). Hint: Define \(F(t) = C + K \int_0^t g(s) \, ds,\) bound \(\frac{F'(t)}{F(t)}\), and integrate.

26. (a) Consider the \(n^{th}\) order differential equation
\[\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_0 x = 0,\] (*)
where \(x \in \mathbb{R}\). Find the smallest \(n\) for which the following functions are all solutions to equation (*):
\[2t \sin(2t), 3e^{-3t}, 3e^{3t}.\]

(b) Give an explicit example of an equation (*) for which \(2t \sin(2t), 3e^{-3t}, 3e^{3t}\) are solutions, i.e. determine constants \(a_0, a_1, \ldots, a_{n-1}\) such that equation (*) has such solutions.
27. (a) Give an example of a one-dimensional ordinary differential equation with initial condition of the form
\[ \dot{x} = f(x), \quad x(0) = x_0, \]
where \( f : \mathbb{R} \to \mathbb{R} \) is infinitely differentiable, but the solution \( x(t) \) is not defined for all \( t > 0 \). (Make sure to prove your assertions.)
(b) Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is \( C^1 \) and \( f(x, y) = -f(x, -y) \). Is it possible for a solution to
\[ \dot{p} = f(p), \quad p(0) = (1, -1) \]
(where \( p \in \mathbb{R}^2 \)) to satisfy \( p(t) = (1, 1) \) for some \( t > 0 \)? Give reasons for your answer.
(c) Let \( f : \mathbb{R} \to \mathbb{R} \) be \( C^1 \) and assume \( f(0) = 0 \) and \( f(x) < 0 \) for \( x > 0 \), \( f(x) > 0 \) for \( x < 0 \).
Prove that the system of differential equations
\[ \dot{x} = y^2 + f(x) \]
\[ \dot{y} = -xy + f(y) \]
\[ \dot{z} = -z^3 \]
has an asymptotically stable equilibrium at the origin.

28. (a) Determine the stability of the equilibrium at the origin for the system
\[ \dot{x} = \alpha x - y^{3/2} + xy \]
\[ \dot{y} = -y + x^2 \]
as a function of \( \alpha \neq 0 \), stating briefly any theorem that you use.
(b) Determine the stability of the equilibrium at the origin for the system
\[ \dot{x} = -xy \]
\[ \dot{y} = -y + x^2. \]

*Hint: construct a center manifold and analyze the flow on it.*

29. (a) Consider ODEs of the form
\[ x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + a_{n-2}x^{(n-2)}(t) + \cdots + a_1 \dot{x} + a_0 x = 0, \tag{*} \]
where \( a_j \in \mathbb{R} \). What is the smallest value of \( n \) for which the coefficients \( a_j \) can be chosen so that each of \( x(t) = \cos(t), x(t) = te^{-t}, \) and \( x(t) = e^{2t} \) is a solution of (\( \star \))? 
(b) Fix \( A \in \mathbb{M}_{n \times n}(\mathbb{R}) \) and let \( B : \mathbb{R} \to \mathbb{R}^n \) be continuous. Solve the non-homogeneous ODE
\[ \dot{x} = Ax + B(t), \quad x : \mathbb{R} \to \mathbb{R}^n. \]

30. Consider the matrices
\[ A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \]
and the corresponding ODEs \( \dot{x} = Ax \) and \( \dot{y} = By \) in \( \mathbb{R}^3 \). Let \( x(t) \) and \( y(t) \) be solutions of these ODEs with the same initial conditions \( x(0) = y(0) = (z_1, z_2, z_3)^T \), where \( z_3 \neq 0 \). Which of \( |x(t)|, |y(t)| \) grows more quickly? That is, does \( \lim_{t \to \infty} \frac{|x(t)|}{|y(t)|} \) equal 0 (in which case \( |y(t)| \) grows more quickly), or does it equal \( \infty \) (in which case \( |x(t)| \) grows more quickly)?
31. Let \( \phi, \psi : \mathbb{R}^2 \to \mathbb{R}^2 \) be \( C^1 \) functions and consider the ODE
\[
\begin{align*}
\dot{x} &= \phi(x, y)x + \psi(x, y)y, \\
\dot{y} &= -\psi(x, y)x + \phi(x, y)y.
\end{align*}
\]

Give conditions on \( \phi, \psi \) which guarantee that the annulus \( A = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 2\} \) is forward invariant and contains a periodic orbit for (†). Ensure that your conditions are non-vacuous—that is, that there do exist \( \phi, \psi \) satisfying your conditions.

32. (a) Given a parameter \( \alpha \in \mathbb{R} \), consider the following system, which is (†) from Question 31 with \( \psi(x, y) = -1 \) and \( \phi(x, y) = \alpha - x^2 - y^2 \).
\[
\begin{align*}
\dot{x} &= -y + x(\alpha - x^2 - y^2), \\
\dot{y} &= x + y(\alpha - x^2 - y^2).
\end{align*}
\]

Find the stability of the equilibrium point at 0 for each value of \( \alpha \in \mathbb{R} \).

(b) Consider the system (‡) with \( 0 < \alpha < 1 \), and fix \( z_0 = (x_0, y_0) \) with \( x_0^2 + y_0^2 > \alpha \). Using the definition of \( \omega \)-limit set, show directly that \( \omega(z_0) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = \alpha\} \). Hint: It may help to use polar coordinates.