Pseudoconformal Curvature and the Embeddability of a CR Hypersurface

A Dissertation Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Brandon M. Lee

May 2014

Pseudoconformal Curvature and the Embeddability of a CR Hypersurface

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Abstract

Under certain conditions on the codimension and curvature, the image of a Cauchy-Riemann (CR) hypersurface of revolution under a CR embedding is proved to be totally geodesic. We also prove a similar statement for the image of a Kähler manifold under a holomorphic conformal embedding.

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1 Introduction

The invariant properties under biholomorphic mappings of a real hypersurface into complex spaces are one of the primary objectives in Cauchy-Riemann (CR) geometry. These manifolds possess an integrable, non-degenerate subbundle of the tangent bundle, known simply as its CR structure bundle.

The study of CR geometry originated from a paper by H. Poincaré (cf. [14]), who showed that certain non-constant holomorphic maps $\partial\mathbb{B}^2 \to \partial\mathbb{B}^2$ must be automorphisms. N. Tanaka (cf. [16]) later extended this result to higher dimensional cases. By the works of E. Cartan (cf. [1]) and S. S. Chern-J. Moser (cf. [2]), complete sets of invariants were constructed for local Levi non-degenerate real hypersurfaces. S. M. Webster, [17], later gave formulas for the fourth-order curvature tensor of Chern-Moser by use of a real, non-vanishing one-form which annihilates the CR bundle on the hypersurface.

More recently, the study of CR geometry has concentrated towards the embeddability of CR manifolds, which continues to be an outstanding challenge. S.-Y. Kim and J.-W. Oh, (cf. [15]) gave necessary and sufficient conditions for local embeddability into a sphere of a strictly pseudoconvex pseudohermitian CR manifold in terms of its pseudoconformal curvature tensors. The studies of P. Ebenfelt, X. Huang, and D. Zaitsev (see [5]) found rigidity results of CR embeddings of CR manifolds, with CR codimension of less than $n/2$, into spheres, which generalized the result of S. M. Webster's of codimension one (cf. [18]). S. Ji and Y. Yuan, [13], recently showed that if the CR second fundamental form is zero, then a CR hypersurface is the image of a sphere by a linear map. In addition, around the same time as [5], F. Forstneric's argument (cf. [8]) has shown that most analytic CR manifolds are not holomorphically embeddable into algebraic ones of the same CR codimension.

We will concentrate on a particular kind of CR manifold - those hypersurfaces admitting unitary symmetry, or real hypersurfaces of revolution. Formally, we will focus on hypersurfaces of the form

$$M = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : r(z, \overline{z}, w, \overline{w}) = 0\}, \tag{1.1}$$

where the defining function r satisfies

$$r = p(z, \overline{z}) + q(w, \overline{w}),$$
$$p = \overline{z}^t H z, \overline{q} = q,$$

with the $n \times n$ matrix H being (constant) hermitian positive definite. Examples of these types of hypersurfaces include spheres and ellipsoids.

Associated to each hypersurface M is the domain D_0, defined by

$$D_0 = \{w \in \mathbb{C} : q(w, \overline{w}) < 0\}. \tag{1.2}$$

Here, we will need to assume that $q_{\overline{w}} \neq 0$ and $dq \neq 0$ whenever $q = 0$. As it turns out, the function

$$h = \frac{q_w q_{\overline{w}} - q q_{w\overline{w}}}{q^2}$$

defines a hermitian metric on D_0.

In [19], S. M. Webster studied the case when the Gaussian curvature K of h is equal to -2, proving the following:

Theorem 1.1 (Webster; [19], Theorem 1)**.** *Let $w \in D_0$ and $(z, w) \in M$. Then, at points where $dq \neq 0$, $S(z, w) = 0$ if and only if $K(w) = -2$.*

Here, S is the pseudoconformal curvature of M. In addition, if we assume this case, then the result of Chern-Moser (see [2]) shows that M is spherical; that is, it is locally embeddable into the sphere $S^{\hat{n}}$.

By making use of the pseudoconformal Gauss equation, we can extend this result:

Theorem 1.2. *Let M be a real hypersurface of $\mathbb{C}^{n+1}$ defined by (1.1). Let (D_0, h) be the associated domain of M. If $1 < n \leq \hat{n} < 2n-1$ and the Gaussian curvature of h satisfies $K > -2$, then there does not exists a smooth CR embedding of M into the sphere $S^{\hat{n}}$.*

By combining the two theorems, we obtain the following corollary:

Corollary 1.3. *Let M be a real hypersurface of $\mathbb{C}^{n+1}$ defined by (1.1) and (D_0, h) its associated domain. Let $1 < n \leq \hat{n} < 2n-1$ and $f : M \to S^{\hat{n}}$ be a smooth CR embedding. If the Gaussian curvature K of h satisfies $K \geq -2$, then f is totally geodesic and $K \equiv -2$.*

This joint work of Huang-Ji-Lee has been accepted and will appear soon in [11].

The $K > -2$ condition in Theorem 1.2 is needed because of the following example. Let $\epsilon > 0$ and

$$q(w, \overline{w}) = |w|^2 + \epsilon |w|^4 - 1.$$

By (5.33) and direct calculation, the Gaussian curvature K is given by

$$K = -2 - 4\epsilon + o(1) < -2.$$

The mapping given by

$$F : M \to \mathbb{S}^{\hat{n}}, (z, w) \mapsto (z, w, \sqrt{\epsilon} w^2)$$

is a CR embedding that is not totally geodesic by the pseudoconformal Gauss equation (4.7).

In addition to the above, by using similar techniques from the CR case, we will also prove the Kähler version of Theorem 1.2:

Theorem 1.4. *Let $f : (X, \omega_X) \to (Y, \omega_Y)$ be a holomorphic conformal embedding between Kähler manifolds. Let $\dim_{\mathbb{C}} X = n$ and $\dim_{\mathbb{C}} Y = \hat{n}$, and suppose that the curvature tensors of X and Y are pseudoconformally flat. If $1 < n \leq \hat{n} < 2n - 1$, then $f(X)$ is a totally geodesic submanifold of Y.*

2 Preliminaries

2.1 CR Manifolds

If M is a real submanifold of $\mathbb{C}^n$, the tangent space of M may then inherit some of the complex structure from the larger space $\mathbb{C}^n$. The idea of a CR structure is based on the real hypersurface case, which we will review briefly.

2.1.1 Tangent Bundles

Let us denote by $T\mathbb{C}^n$ to be the (real) tangent bundle of $\mathbb{C}^n$ and by

$$\mathbb{C}T\mathbb{C}^n = T\mathbb{C}^n \otimes \mathbb{C}$$

to denote the complexification of $T\mathbb{C}^n$.

A smooth section X of $\mathbb{C}T\mathbb{C}^n$ is called a *complex vector field* on $\mathbb{C}^n$. Locally, a complex vector field can be written as a linear combination of the basis operators $\partial/\partial z^j$ and $\partial/\partial \bar{z}^k$:

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial z^j} + \sum_{k=1}^{n} b_k \frac{\partial}{\partial \bar{z}^k}, \tag{2.1}$$

where the coefficient functions are assumed to be smooth and complex-valued. Here, we write $z = (z^1, \ldots, z^n)$ for the local coordinates of $\mathbb{C}^n$. (2.1) shows that we may write $\mathbb{C}T\mathbb{C}^n$ as a direct sum:

$$\mathbb{C}T\mathbb{C}^n = T^{1,0}\mathbb{C}^n + T^{0,1}\mathbb{C}^n, \tag{2.2}$$

where $T^{1,0}\mathbb{C}^n$ denotes the subbundle whose sections are linear combinations of the $\partial/\partial z^j$ and $T^{0,1}\mathbb{C}^n$ denotes its complex conjugate bundle.

2.1.2 Real Hypersurfaces and CR Manifolds

A primary interest in CR geometry is in the boundaries of domains in $\mathbb{C}^n$. If such a boundary is a smooth manifold, then it is a real hypersurface; that is, it can be considered as a real submanifold with real codimension one.

Let us first suppose that M is a smooth real hypersurface in $\mathbb{C}^n$. For any point $p \in M$, let us define $T^{1,0}_p M$, which we will call the *bundle of* $(1,0)$*-vectors of* M *over* p, to be the intersection

$$T^{1,0}_p M = \mathbb{C}T_p M \cap T^{1,0}_p \mathbb{C}^n. \tag{2.3}$$

We define its complex conjugate bundle by $T^{0,1}_p M := \overline{T^{1,0}_p M}$.

For any point $p \in M$, $T^{1,0}_p M$ is a complex vector space with complex dimension

$$\dim_{\mathbb{C}} T^{1,0}_p M = n - 1.$$

In addition, the subbundle $T^{1,0}M$ satisfies the following properties:

- $T^{1,0}M$ is *integrable*; that is, it is closed under the Lie bracket operation,

$$[T^{1,0}M, T^{1,0}M] \subseteq T^{1,0}M;$$

- $T^{1,0}M \cap T^{0,1}M = \{0\}$;

- $HM = T^{1,0}M + T^{0,1}M$ is a subbundle of codimension one in $\mathbb{C}TM$; that is, at each point $p \in M$, the complex vector space H_pM has codimension one in $\mathbb{C}T_pM$.

A CR manifold is a real differentiable manifold together with a geometric structure modeled on that of a real hypersurface in $\mathbb{C}^n$. More precisely, we make the following definition:

Definition 2.1. *A* CR manifold *is a real differentiable manifold M whose complexified tangent bundle $\mathbb{C}TM$ contains a subbundle $T^{1,0}M$ that satisfies:*

(i) $T^{1,0}M$ is integrable; that is, it is closed under the Lie bracket operation;

(ii) $T^{1,0}M \cap T^{0,1}M = \{0\}$.

Here, we define the conjugate subbundle by $T^{0,1}M = \overline{T^{1,0}M}$. We call such a subbundle $T^{1,0}M$ the CR structure (bundle) *of M.*

The complex dimension $\dim_{\mathbb{C}} T^{1,0}M$*, which is independent of p, is called the* CR dimension *of M. The* CR codimension *of the CR structure is the codimension of $HM = T^{1,0}M + T^{0,1}M$ in $\mathbb{C}TM$. In the case that the CR codimension is one, we say that the CR manifold is of* hypersurface type*, or that it is a* CR hypersurface*.*

A smooth section of $T^{1,0}M$ is called a CR vector field *over M. A C^1-smooth function f is called a* CR function *if it is locally annihilated by any CR vector field. A* CR mapping *is a smooth mapping F between CR manifolds M and N whose differential satisfies $dF(T^{1,0}M) \subseteq T^{1,0}N$.*

2.1.3 Contact Forms

A real, non-vanishing smooth one-form θ over a CR manifold M is called a *contact form* if

$$\theta \wedge (d\theta)^n \neq 0.$$

Equivalently, a contact form is a real, non-vanishing smooth section of

$$T^0 M := HM^{\perp}.$$

Associated with any contact form θ over M, one has the uniquely defined *characteristic*, or *Reeb*, *vector field* T. T is a real vector field defined by

$$T \lrcorner d\theta = 0, \langle \theta, T \rangle = 1, \tag{2.4}$$

where $\lrcorner$ denotes contraction (or interior multiplication). Since $d\theta$ is a degenerate two-form on TM, but non-degenerate on the hyperplane defined by $\theta = 0$ in TM, we can always find such a T in the kernel of $d\theta$.

2.1.4 Levi Form

For a CR manifold M, and for any point $p \in M$, the *Levi form at* p is a mapping

$$h_p : T_p^{1,0} M \to (\mathbb{C}T_p M)/(H_p M), v_p \mapsto \frac{1}{2i}\pi_p([v, \overline{v}]), \tag{2.5}$$

where v is any vector field in $T^{1,0}M$ that equals v_p at p and

$$\pi_p : \mathbb{C}T_p M \to (\mathbb{C}T_p M)/(H_p M)$$

is the natural projection. This definition of h_p is independent of the choice of v.

2.1.5 Levi and Contact Forms

The Levi form of a CR manifold can be defined in terms of a given contact form. By fixing a contact form θ on a CR manifold M, we define the Levi form of

(M, θ) by

$$h_\theta(v, w) := -d\theta(v, \overline{w}) = \theta([v, \overline{w}]), \tag{2.6}$$

for all $v, w \in HM$. Here, we used Cartan's formula

$$\langle d\theta, v \wedge \overline{w} \rangle = v\langle \theta, \overline{w} \rangle - \overline{w}\langle \theta, v \rangle - \langle \theta, [v, \overline{w}] \rangle \tag{2.7}$$

and the fact that $\langle \theta, V \rangle = 0$ for all $V \in HM$, which implies

$$\langle \theta, v \rangle = \langle \theta, \overline{w} \rangle = 0.$$

Observe that the Levi form of M can be regarded as a hermitian metric on the subbundle $T^{1,0}M$. This metric can be defined by

$$h_\theta : T^{1,0}M \otimes T^{1,0}M \to \mathbb{C}, (v, \overline{v}) \mapsto h_\theta(v, \overline{v}) = \theta([v, \overline{v}]) = \langle \theta, [v, \overline{v}] \rangle.$$

Definition 2.2. *We say that* (M, θ) *is* Levi non-degenerate at p *if*

$$h_\theta(v_p, w_p) = 0$$

for all w_p *implies* $v_p = 0$. (M, θ) *is* Levi non-degenerate *if* M *is Levi non-degenerate at every point* $p \in M$.

(M, θ) *is called* pseudoconvex *if* h_θ *is (positive) semidefinite. It is called* strongly pseudoconvex *if* h_θ *is (positive) definite.*

2.1.6 Levi Form of Hypersurfaces

Let M be a smooth real hypersurface of $\mathbb{C}^n$. Then, around any point $p \in M$, there exists a neighborhood U_p of p in $\mathbb{C}^n$ and a smooth real-valued function ρ

defined in U_p such that

$$M \cap U_p = \{(z,w) \in U \cap (\mathbb{C}^{n-1} \times \mathbb{C}) : \rho(z,w,\overline{z},\overline{w}) = 0\}$$

with $d\rho \neq 0$ in U_p. The function ρ is called a *defining function of M at p*.[1] In this case, the one-form $\theta = -i\partial\rho$ is a contact form of M. From (2.6), we obtain

$$h_\theta(v,\overline{w}) = -\langle d\theta, v \wedge \overline{w}\rangle;$$

that is, the Levi form with respect to θ satisfies

$$h_\theta = -d\theta = -i\overline{\partial}\partial\rho = i\partial\overline{\partial}\rho.$$

2.2 Pseudohermitian Structures

Throughout this section, we follow the summation convention laid out in [5]. In particular, lowercase Greek indices will run over the set $\{1,\ldots,n\}$. A general tensor will be written as $T_{\alpha}{}^{\beta}{}_{\mu\overline{\nu}}$, where the indices without conjugation will indicate $\mathbb{C}$-linear dependence in the corresponding argument and indices with conjugation will indicate $\mathbb{C}$-antilinear dependence. Recall that such a tensor $T_{\alpha}{}^{\beta}{}_{\mu\overline{\nu}}$ can be considered as an $\mathbb{R}$-multilinear complex-valued function on $V \times V^* \times V \times V$.

We will not assume that a tensor is symmetric in its indices. Hence, the ordering of the indices may carry important information. Simultaneous conjugation of all the indices of a given tensor will correspond to conjugation of that tensor. For example,

$$T_{\overline{\alpha}}{}^{\overline{\beta}}{}_{\overline{\mu}\nu} = \overline{T_{\alpha}{}^{\beta}{}_{\mu\overline{\nu}}}.$$

[1] Note that the defining function ρ is not unique. For any non-zero smooth function h, $h\rho$ is also a defining function for the manifold.

The Levi form matrix $(g_{\alpha\overline{\beta}})$ of M (with respect to a given contact form θ), and its inverse $(g^{\overline{\beta}\alpha})$, will be used to raise and lower indices (without changing their ordering):

$$\theta_\alpha = g_{\alpha\overline{\beta}}\theta^{\overline{\beta}}, A^{\overline{\alpha}}{}_\beta = g^{\overline{\alpha}\gamma}A_{\gamma\beta}.$$

2.2.1 Admissible Coframes

Let M be a CR manifold and θ a given contact form on M. Let us suppose that $\{L_\alpha\}_{\alpha=1,\dots,n}$ is a basis of $(1,0)$-vector fields on $T^{1,0}M$ such that $(T, L_\alpha, L_{\overline{\alpha}})$ is a frame on $\mathbb{C}TM$. Here, T will be the characteristic vector field associated to θ. Then, the first equation in (2.4) is equivalent to

$$d\theta = ig_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}, \tag{2.8}$$

where $(g_{\alpha\overline{\beta}})$ is the Levi form matrix and $(\theta, \theta^\alpha, \theta^{\overline{\alpha}})$ is the coframe dual to $(T, L_\alpha, L_{\overline{\alpha}})$ (for convenience of notation, we will usually say that (θ, θ^α) is the *coframe dual to the frame* (T, L_α)). Note that θ and T are real, whereas θ^α and L_α always have non-trivial real and imaginary parts.

In general, (2.8) will not always be the case. Hence, we define:

Definition 2.3. *If θ is a contact form on M, we call a coframe (θ, θ^α) (and its dual frame (T, L_α))* admissible *if*

$$d\theta = ig_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}},$$

holds; or equivalently, if T is the characteristic vector field for θ with respect to (2.4).

2.2.2 Pseudohermitian Structures

A choice of contact form θ on a CR manifold M is referred to as a *pseudohermitian structure*. This defines a hermitian metric on $T^{1,0}M$ via the (positive definite) Levi form.

For every pseudohermitian structure θ, [5] (and [17]) defines a *pseudohermitian connection* ∇ on $T^{1,0}M$ (and also on $\mathbb{C}TM$) which is expressed relative to an admissible coframe (θ, θ^α) by

$$\nabla L_\alpha := {\omega_\alpha}^\beta \otimes L_\beta, \tag{2.9}$$

where the one-forms ${\omega_\alpha}^\beta$ on M are uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge {\omega_\alpha}^\beta \, (mod\, \theta \wedge \theta^{\overline{\alpha}}), \\ dg_{\alpha\overline{\beta}} &= \omega_{\alpha\overline{\beta}} + \omega_{\overline{\beta}\alpha}. \end{aligned} \tag{2.10}$$

The first equation of (2.10) can be rewritten as

$$d\theta^\beta = \theta^\alpha \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\beta, \tau^\beta = {A^\beta}_{\overline{\nu}} \theta^{\overline{\nu}}, A^{\alpha\beta} = A^{\beta\alpha}, \tag{2.11}$$

for a suitable, uniquely determined, torsion matrix $({A^\beta}_{\overline{\alpha}})$ (cf. [17], [5]).

2.2.3 Pseudohermitian Curvature

The *curvature of the pseudohermitian connection* ∇ is given by

$$\begin{aligned} d{\omega_\alpha}^\beta - {\omega_\alpha}^\gamma \wedge {\omega_\gamma}^\beta = {R_\alpha}^\beta{}_{\mu\bar{\nu}} \theta^\mu \wedge \theta^{\bar{\nu}} + {W_\alpha}^\beta{}_\mu \theta^\mu \wedge \theta \\ - {W^\beta}_{\alpha\bar{\nu}} \theta^{\bar{\nu}} \wedge \theta + i\theta_\alpha \wedge \tau^\beta - i\tau_\alpha \wedge \theta^\beta, \end{aligned} \tag{2.12}$$

where the functions $R_\alpha{}^\beta{}_{\mu\bar\nu}$ and $W_\alpha{}^\beta{}_\nu$ represent the pseudohermitian curvature of (M, θ).

As can be seen in [5], the components $W_\alpha{}^\beta{}_\mu$ can be obtained as covariant derivatives of the torsion matrix $A^\beta{}_{\bar\alpha}$ (see (2.11)). Following [5], we denote the covariant differentiation operator with respect to the pseudohermitian connection ∇ also by ∇, and its components by indices preceeded by a semicolon. An index of 0 will be used to denote the covariant derivative with respect to T. Thus, for example,

$$\begin{aligned} \nabla A^\beta{}_{\bar\alpha} &= dA^\beta{}_{\bar\alpha} + A^\mu{}_{\bar\alpha}\omega_\mu{}^\beta - A^\beta{}_{\bar\nu}\omega_{\bar\alpha}{}^{\bar\nu} \\ &= A^\beta{}_{\bar\alpha;0}\theta + A^\beta{}_{\bar\alpha;\nu}\theta^\nu + A^\beta{}_{\bar\alpha;\bar\nu}\theta^{\bar\nu}. \end{aligned} \tag{2.13}$$

In this notation, the above mentioned relation reads (see [5]):

$$W_\alpha{}^\beta{}_\mu = A_{\alpha\mu;}{}^\beta, W^\beta{}_{\alpha\bar\nu} = A^\beta{}_{\bar\nu;\alpha}. \tag{2.14}$$

2.2.4 Pseudoconformal Connection

Recall (see [2], [17], and [5]) that the *Chern-Moser coframe bundle* Y over M is defined as the bundle of the coframes $(\omega, \omega^\alpha, \omega^{\bar\alpha}, \varphi)$ on the real line bundle $\pi_E : E \to M$ of all contact forms that satisfies

$$d\omega = i g_{\alpha\bar\beta}\omega^\alpha \wedge \omega^{\bar\beta} + \omega \wedge \varphi,$$

where ω^α is in $\pi_E^*(T'M)^2$ and ω is the canonical form on E given by $\omega(\theta)(X) := \theta((\pi_E)_*X)$, for $\theta \in E$, $X \in T_\theta E$. The canonical forms ω, ω^α, $\omega^{\bar\alpha}$, φ are similarly defined on Y (following [5], the same notation is also used for this coframe).

It was shown in [2] and [5] that these forms can be completed to a natural

[2] Note that $T'M := (T^{0,1}M)^\perp$ will be used later in the second fundamental form.

parallelism on Y given by the coframe of 1-forms

$$(\omega, \omega^{\alpha}, \omega^{\bar{\alpha}}, \varphi, {\varphi_{\beta}}^{\alpha}, \varphi^{\alpha}, \varphi^{\bar{\alpha}}, \psi) \tag{2.15}$$

defining the *pseudoconformal connection* on Y and satisfying the following equations:

$$\begin{aligned}
g_{\alpha\bar{\beta}}\varphi &= \varphi_{\alpha\bar{\beta}} + \varphi_{\bar{\beta}\alpha}, \\
d\omega &= i\omega^{\mu} + \omega_{\mu} + \omega \wedge \varphi, \\
d\omega^{\alpha} &= \omega^{\mu} \wedge {\varphi_{\mu}}^{\alpha} + \omega \wedge \varphi^{\alpha}, \\
d\varphi &= i\omega_{\bar{\nu}} \wedge \varphi^{\bar{\nu}} + i\varphi_{\bar{\nu}} \wedge \omega^{\bar{\nu}} + \omega \wedge \psi, \\
d{\varphi_{\beta}}^{\alpha} &= {\varphi_{\beta}}^{\mu} \wedge {\varphi_{\mu}}^{\alpha} + i\omega_{\beta} \wedge \varphi^{\alpha} - i\varphi_{\beta} \wedge \omega^{\alpha} \\
&\quad - i{\delta_{\beta}}^{\alpha}\varphi_{\mu} \wedge \omega^{\mu} - \frac{{\delta_{\beta}}^{\alpha}}{2}\psi \wedge \omega + {\Phi_{\beta}}^{\alpha}, \\
d\varphi^{\alpha} &= \varphi \wedge \varphi^{\alpha} + \varphi^{\mu} \wedge {\varphi_{\mu}}^{\alpha} - \frac{1}{2}\psi \wedge \omega^{\alpha} + \Phi^{\alpha}, \\
d\psi &= \varphi \wedge \psi + 2i\varphi^{\mu} \wedge \varphi_{\mu} + \Psi,
\end{aligned} \tag{2.16}$$

where the curvature two-forms ${\Phi_{\beta}}^{\alpha}$, Φ^{α}, Ψ can be decomposed as

$$\begin{aligned}
{\Phi_{\beta}}^{\alpha} &= {S_{\beta}}^{\alpha}{}_{\mu\bar{\nu}}\omega^{\mu} \wedge \omega^{\bar{\nu}} + {V_{\beta}}^{\alpha}{}_{\mu}\omega^{\mu} \wedge \omega + {V^{\alpha}}_{\beta\bar{\nu}}\omega \wedge \omega^{\bar{\nu}}, \\
\Phi^{\alpha} &= {V^{\alpha}}_{\mu\bar{\nu}}\omega^{\mu} \wedge \omega^{\bar{\nu}} + {P_{\mu}}^{\alpha}\omega^{\mu} \wedge \omega + {Q_{\bar{\nu}}}^{\alpha}\omega^{\bar{\nu}} \wedge \omega, \\
\Psi &= -2iP_{\mu\bar{\nu}}\omega^{\mu} \wedge \omega^{\bar{\nu}} + R_{\mu}\omega^{\mu} \wedge \omega + R_{\bar{\nu}}\omega^{\bar{\nu}} \wedge \omega.
\end{aligned} \tag{2.17}$$

The functions ${S_{\beta}}^{\alpha}{}_{\mu\bar{\nu}}$, ${V_{\beta}}^{\alpha}{}_{\mu}$, ${P_{\mu}}^{\alpha}$, ${Q_{\bar{\nu}}}^{\alpha}$, R_{μ} together represent the *pseudoconformal curvature* of M.

As in [2] and [5], we will restrict our attention to coframes $(\theta, \theta^{\alpha})$ for which the Levi form $(g_{\alpha\bar{\beta}})$ is constant. The one-forms φ^{α}, $\varphi^{\bar{\alpha}}$, ${\varphi_{\beta}}^{\alpha}$, ψ are uniquely determined by requiring the coefficients in (2.17) to satisfy certain symmetry and trace condition; for example,

$$S_{\alpha\bar{\beta}\mu\bar{\nu}} = S_{\mu\bar{\beta}\alpha\bar{\nu}} = S_{\mu\bar{\nu}\alpha\bar{\beta}} = S_{\bar{\nu}\mu\bar{\beta}\alpha},$$
$$S_{\mu}{}^{\mu}{}_{\alpha\bar{\beta}} = V_{\alpha}{}^{\mu}{}_{\mu} = P_{\mu}{}^{\mu} = 0. \tag{2.18}$$

2.2.5 Pseudoconformal Formula

Let us fix any contact form θ on M. Then, any admissible coframe (θ, θ^α) defines a unique section $M \to Y$ for which the pull-backs of (ω, ω^α) coincide with (θ, θ^α), and the pull-back of φ vanishes. As in [17] and [5], this section is used to pull back the forms in (2.15) to M. Following [5], the same notation is used for the pulled-back forms on M (however, these forms will now depend on the choice of the admissible coframe (θ, θ^α)). With this convention, we have

$$\theta = \omega, \theta^\alpha = \omega^\alpha, \varphi = 0, \tag{2.19}$$

on M.

In view of [17] (cf. [5]), the (pulled-back tangential) *pseudoconformal curvature tensor* $S_\alpha{}^\beta{}_{\mu\bar{\nu}}$ can then be obtained from the tangential pseudohermitian curvature tensor $R_\alpha{}^\beta{}_{\mu\bar{\nu}}$ in (2.12) by

$$\begin{aligned} S_{\alpha\bar{\beta}\mu\bar{\nu}} = R_{\alpha\bar{\beta}\mu\bar{\nu}} &- \frac{R_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + R_{\mu\bar{\beta}}g_{\alpha\bar{\nu}} + R_{\alpha\bar{\nu}}g_{\mu\bar{\beta}} + R_{\mu\bar{\nu}}g_{\alpha\bar{\beta}}}{n+2} \\ &+ \frac{R(g_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + g_{\alpha\bar{\nu}}g_{\mu\bar{\beta}})}{(n+1)(n+2)}, \end{aligned} \tag{2.20}$$

where

$$R_{\alpha\bar{\beta}} := R_{\mu}{}^{\mu}{}_{\alpha\bar{\beta}}, R := R_{\mu}{}^{\mu} \tag{2.21}$$

are, respectively, the *pseudohermitian Ricci* and *scalar curvature of* (M, θ).

2.2.6 Traceless Components

Equation (2.20) expresses the pseudoconformal curvature tensor $S_{\alpha\bar{\beta}\mu\bar{\nu}}$ as the *traceless component* of the pseudohermitian curvature tensor $R_{\alpha\bar{\beta}\mu\bar{\nu}}$, with respect to the decomposition of the space of all tensors $T_{\alpha\bar{\beta}\mu\bar{\nu}}$ with the symmetry condition in (2.18) into the direct sum of the subspace of such tensors of trace zero (i.e., those tensors such that $T_{\mu}{}^{\mu}{}_{\alpha\bar{\beta}} = 0$) and the subspace of tensors of the form

$$T_{\alpha\bar{\beta}\mu\bar{\nu}} = H_{\alpha\bar{\beta}} g_{\mu\bar{\nu}} + \hat{H}_{\mu\bar{\beta}} g_{\alpha\bar{\nu}} + \tilde{H}_{\alpha\bar{\nu}} g_{\mu\bar{\beta}} + \check{H}_{\mu\bar{\nu}} g_{\alpha\bar{\beta}}, \tag{2.22}$$

where $(H_{\alpha\bar{\beta}})$, $(\hat{H}_{\mu\bar{\beta}})$, $(\tilde{H}_{\alpha\bar{\nu}})$, and $(\check{H}_{\mu\bar{\nu}})$ are hermitian matrices.

We will call two tensors, as above, *conformally equivalent* if their difference is of the form (2.22). In this terminology, the right-hand side of (2.20) (together with (2.21)) gives, for any tensor $T_{\alpha\bar{\beta}\mu\bar{\nu}}$ with the symmetry relation (2.18), its *traceless component*, which is the unique tensor of trace zero that is conformally equivalent to $T_{\alpha\bar{\beta}\mu\bar{\nu}}$.

Proposition 2.4. *(Webster, [17]; [5], Proposition 3.1.) Let M be a strictly pseudoconvex CR manifold of hypersurface type of CR dimension n. Let $\omega_{\beta}{}^{\alpha}, \tau^{\alpha}$ be defined by* (2.10)-(2.11) *with respect to an admissible coframe $(\theta, \theta^{\alpha})$, and, let $\varphi_{\beta}{}^{\alpha}, \varphi^{\alpha}, \psi$ be the forms in* (2.15)*, pulled back to M using $(\theta, \theta^{\alpha})$ as above. Then we have the following relations:*

$$\begin{aligned} \varphi_{\beta}{}^{\alpha} &= \omega_{\beta}{}^{\alpha} + D_{\beta}{}^{\alpha}\theta, \\ \varphi^{\alpha} &= \tau^{\alpha} + D_{\mu}{}^{\alpha}\theta^{\mu} + E^{\alpha}\theta, \\ \psi &= iE_{\mu}\theta^{\mu} - iE_{\bar{\nu}}\theta^{\bar{\nu}} + B\theta, \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} D_{\alpha\bar\beta} &:= \frac{iR_{\alpha\bar\beta}}{n+2} - \frac{iRg_{\alpha\bar\beta}}{2(n+1)(n+2)}, \\ E^\alpha &:= \frac{2i}{2n+1}(A^{\alpha\mu}{}_{;\mu} - D^{\bar\nu\alpha}{}_{;\bar\nu}), \\ B &:= \frac{1}{n}(E^\mu{}_{;\mu} + E^{\bar\nu}{}_{;\bar\nu} - 2A^{\beta\mu}A_{\beta\mu} + 2D^{\bar\nu\alpha}D_{\bar\nu\alpha}). \end{aligned} \tag{2.24}$$

Proof. The formulas for ${\varphi_\beta}^\alpha$ and $D_{\alpha\overline{\beta}}$ were proved in [17]. The formula for φ^α follows from the third equation in (2.16) and (2.11). Indeed, these two equations yield

$$\theta^\alpha \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\beta = \theta^\alpha \wedge {\varphi_\alpha}^\beta + \theta \wedge \varphi^\beta.$$

Substituting the formula for ${\varphi_\beta}^\alpha$ in (2.23), we obtain

$$\theta \wedge \tau^\beta = {D_\alpha}^\beta \theta^\alpha \wedge \theta + \theta \wedge \varphi^\beta, \tag{2.25}$$

which implies the formula for φ^α in (2.23), with some E^α. Similarly, the formula for ψ in (2.23), with some B, follows from equating the coefficients of θ in the pulled-back fourth equation of (2.16) and using (2.19) (whence $d\varphi = 0$ on M).

To obtain the formula for E^α in (2.24), we substitute the formulas (2.23) for ${\varphi_\beta}^\alpha$, φ^α, and ψ in the pulled-back sixth equation of (2.16) and use (2.8), (2.11), the covariant derivative (2.13), the analogues for ${D_\beta}^\alpha$, and the formula for Φ^α in (2.17):

$$\nabla A^\alpha{}_{\overline{\nu}} \wedge \theta^{\overline{\nu}} + \nabla {D_\beta}^\alpha \wedge \theta^\beta + i g_{\mu\overline{\nu}} E^\alpha \theta^\mu \wedge \theta^{\overline{\nu}} = -\frac{1}{2}\psi \wedge \theta^\alpha + V^\alpha{}_{\mu\overline{\nu}} \theta^\mu \wedge \theta^{\overline{\nu}} \bmod \theta. \tag{2.26}$$

By identifying the coefficient in front of $\theta^\mu \wedge \theta^{\overline{\nu}}$ in (2.26), and using the formula for ψ in (2.23), we obtain

$$A^{\alpha}{}_{\overline{\nu};\mu} - D_{\mu}{}^{\alpha}{}_{;\overline{\nu}} + ig_{\mu\overline{\nu}}E^{\alpha} = -\frac{1}{2}iE_{\overline{\nu}}\delta_{\mu}{}^{\alpha} + V^{\alpha}{}_{\mu\overline{\nu}}.$$

The formula for E^{α} in (2.24) is now obtained by summing over μ and $\overline{\nu}$ and using the trace condition $V^{\alpha}{}_{\mu}{}^{\mu} = 0$. Similarly, the formula for B follows by substituting the formula for ψ in the pulled-back last equation of (2.16) (mod θ) and using the trace condition $P_{\nu}{}^{\nu} = 0$. □

3 CR Second Fundamental Form

In this section, we will let M be a strictly pseudoconvex CR manifold of (real) dimension $2n+1$ and $f : M \to \hat{M}$ a CR embedding of M into another strictly pseudoconvex CR manifold $\hat{M}$ of (real) dimension $2\hat{n}+1$ with rank $\hat{n}$ CR bundle. We will also use the $\hat{}$ symbol to denote objects associated to $\hat{M}$.

We continue to use the summation convention from the previous section. In addition, capital Latin indices $A, B, \ldots$, will run over the set $\{1, \ldots, \hat{n}\}$ whereas lowercase Greek indices $\alpha, \beta, \ldots$, will run over $\{1, 2, \ldots, n\}$. Lowercase Latin indices $a, b, \ldots$, will run over the complementary set $\{n+1, \ldots, \hat{n}\}$.

3.1 Admissible for the Pair

Since $\hat{M}$ is strictly pseudoconvex and f an embedding, according to [5], for every contact form $\hat{\theta}$ on $\hat{M}$, the pull-back $f^*\hat{\theta}$ is non-vanishing, and hence, a contact form on M. In general, $f^*\hat{\theta}$ may vanish (an example is when $f(M)$ is contained in a complex-analytic subvariety of $\hat{M}$). Hence, we follow [5] by always choosing the coframe $(\hat{\theta}, \hat{\theta}^A)$ on $\hat{M}$ such that the pull-back of $(\hat{\theta}, \hat{\theta}^{\alpha})$ is a coframe for M. Because of this, the $\hat{}$ will sometimes be dropped over the frames and coframes.

We also follow [5] by identifying M with the submanifold $f(M)$ of $\hat{M}$, and write $M \subset \hat{M}$. Hence, $T^{1,0}M$ becomes a rank n subbundle of $T^{1,0}\hat{M}$ along M. It then follows that the (real) codimension of M in $\hat{M}$ is $2(\hat{n}-n)$ and that there is a rank $\hat{n}-n$ subbundle $N'M$ of $T'\hat{M}$ along M consisting of one-forms on $\hat{M}$ whose pull-backs to M (under f) vanish. $N'M$ is called the *holomorphic conormal bundle of M in $\hat{M}$.*

Definition 3.1. *We say that the pseudohermitian structure $(\hat{M}, \hat{\theta})$ (or simply that of $\hat{\theta}$) is* admissible for the pair $(M, \hat{M})$ *if the characteristic vector field $\hat{T}$ of $\hat{\theta}$ is tangent to M, and hence, coincides with the characteristic vector field of the pull-back of $\hat{\theta}$.*

3.2 Adapted Coframes

It can easily be seen that not all contact forms $\hat{\theta}$ on $\hat{M}$ are admissible for the pair $(M, \hat{M})$. However, we do have the following:

Lemma 3.2 ([5], Lemma 4.1)**.** *Let $M \subset \hat{M}$ be as above. Then any contact form on M can be extended to a contact form θ in a neighborhood M in $\hat{M}$ such that θ is admissible for $(M, \hat{M})$.*

Proof. Let θ be any fixed extension of the given contact form on M to a neighborhood of M in $\hat{M}$. Any other extension is clearly of the form $\tilde{\theta} = u\theta$, where u is a smooth function on $\hat{M}$ near M with $u|_M \equiv 1$. Let T be the characteristic vector field of the restriction of θ to M. Then $\tilde{\theta}$ is admissible for $(M, \hat{M})$ if and only if $T \lrcorner d\tilde{\theta} = 0$. That is, if $T \lrcorner d\theta - du = 0$ along M. By the assumptions, the latter identity holds when pulled back to M. Now it is clear that there exists a unique choice of du along M for which it holds also in the normal direction. The required function u can now be constructed in local coordinate charts and glued together via a partition of unity, completing the proof. □

By taking admissible coframes and using the Gram-Schmidt algorithm, we can obtain the following corollary:

Corollary 3.3 ([5], Corollary 4.2)**.** *Let M and $\hat{M}$ be strictly pseudoconvex CR manifolds of dimension $2n+1$ and $2\hat{n}+1$, respectively, and suppose that $f : M \to \hat{M}$ is a CR embedding. If (θ, θ^α) is any admissible coframe on M, then in a neighborhood of any point $\hat{p} \in f(M)$ in $\hat{M}$, there exists an admissible coframe $(\hat{\theta}, \hat{\theta}^A)$ on $\hat{M}$ with $f^*(\hat{\theta}, \hat{\theta}^\alpha, \hat{\theta}^a) = (\theta, \theta^\alpha, 0)$. In particular, $\hat{\theta}$ is admissible for the pair $(f(M), \hat{M})$; that is, the characteristic vector field $\hat{T}$ is tangent to $f(M)$. If the Levi form of M with respect to (θ, θ^α) is $(\delta_{\alpha\bar{\beta}})$, then $(\hat{\theta}, \hat{\theta}^A)$ can be chosen such that the Levi form of $\hat{M}$ relative to it is also $(\delta_{A\bar{B}})$. With this additional property, the coframe $(\hat{\theta}, \hat{\theta}^A)$ is uniquely determined along M up to unitary transformations in $U(n) \times U(\hat{n}-n)$.*

Definition 3.4. *If we fix an admissible coframe (θ, θ^α) on M, and let $(\hat{\theta}, \hat{\theta}^A)$ be an admissible coframe on $\hat{M}$ near $f(M)$, we say that $(\hat{\theta}, \hat{\theta}^A)$ is adapted to (θ, θ^α) on M (or simply to M if the coframe on M is understood) if it satisfies the conclusion of Corollary 3.3, with the requirement for the Levi form.*

3.3 CR Second Fundamental Form - Intrinsic Version

The fact that (θ, θ^A) (here, we omit the $\hat{\cdot}$) is adapted to M implies, in view of (2.10), that if the pseudohermitian connection matrix of $(\hat{M}, \theta)$ is $\hat{\omega}_B^A$, then that of (M, θ) is the pull-back of $\hat{\omega}_\beta^\alpha$. A similar statement holds for the pulled-back torsion matrix $\hat{\tau}^\alpha$. Hence, we follow [5] by omitting the $\hat{\cdot}$ over these pull-backs.

Theorem 3.5 (Webster, [17])**.** *Let (M^{2n+1}, θ) be a strictly pseudoconvex pseudohermitian manifold and let (θ, θ^α) be an admissible coframe. Then there exists a unique way to write*

$$d\theta^\alpha = \sum_{\gamma=1}^{n} \theta^\gamma \wedge \omega_\gamma^\alpha + \theta \wedge \tau^\alpha, \tag{3.1}$$

where τ^α *are* $(0,1)$*-forms over* M *that are linear combinations of* $\theta^{\bar\alpha} = \overline{\theta^\alpha}$*, and* ω_α^β *are one-forms over* M *such that*

$$0 = dg_{\alpha\bar\beta} - g_{\gamma\bar\beta}\omega_\alpha^\gamma - g_{\alpha\bar\gamma}\omega_{\bar\beta}^{\bar\alpha}. \tag{3.2}$$

We may write $\omega_{\alpha\bar\beta} = g_{\gamma\bar\beta}\omega_\alpha^\gamma$ and $\overline{\omega_{\beta\bar\alpha}} = g_{\alpha\bar\gamma}\omega_{\bar\beta}^{\bar\gamma}$ by lowering the indices via the Levi matrix. In particular, by the normalization of the Levi form (that is, $g_{\alpha\beta} = \delta_{\alpha\beta}$) the second equation in (2.10) reduces to

$$\omega_{B\bar A} + \omega_{\bar A B} = 0, \tag{3.3}$$

where $\omega_{\bar A B} = \overline{\omega_{A\bar B}}$.

Now, if $(\hat\theta, \hat\theta^A)$ is adapted to (θ, θ^α), by (3.3), we have $\theta = f^*\hat\theta$, $\theta^\alpha = f^*\hat\theta^\alpha$,

$$d\theta^\alpha = \sum_{\gamma=1}^{n} \theta^\gamma \wedge \omega_\gamma^\alpha + \theta \wedge \tau^\alpha, 0 = \omega_\alpha^\beta + \omega_{\bar\beta}^{\bar\alpha}, 1 \le \alpha, \beta \le n,$$

and

$$d\hat\theta^A = \sum_{C=1}^{\hat n} \hat\theta^C \wedge \hat\omega_C^A + \hat\theta \wedge \hat\tau^A, 0 = \hat\omega_A^B + \hat\omega_{\bar B}^{\bar A}, 1 \le A, B \le N.$$

For simplicity, we will denote $f^*\hat\omega_B^A$ by ω_B^A. We also denote $f^*\hat\omega_{A\bar B}$ by $\omega_{A\bar B}$, where $\omega_{A\bar B} = \omega_A^B$.

Let us write ${\omega_\alpha}^a = {\omega_\alpha}^a{}_\beta \theta^\beta$. The matrix $({\omega_\alpha}^a{}_\beta)$, $1 \le \alpha, \beta \le n$, $n+1 \le a \le \hat n$, defines the *(intrinsic) CR second fundamental form of* M, or, more precisely, of the embedding f. It was used in [18] and [7].

Note that, since θ^b is 0 on M, we deduce by using (2.11) that, on M,

$${\omega_\alpha}^b \wedge \theta^\alpha + \tau^b \wedge \theta = 0, \tag{3.4}$$

which implies that

$$\omega_\alpha{}^b = \omega_\alpha{}^b{}_\beta \theta^\beta, \omega_\alpha{}^b{}_\beta = \omega_\beta{}^b{}_\alpha, \tau^b = 0. \tag{3.5}$$

3.4 CR Second Fundamental Form - Extrinsic Version

This version of the CR second fundamental form was used in [5]. Let M be a CR hypersurface of dimension $2n+1$, and we denote by $\mathcal{V} = T^{0,1}M \subset \mathbb{C}TM$ its (0,1)-vector bundle and $T'M = \mathcal{V}^\perp \subset \mathbb{C}T^*M$. Recall that a mapping

$$f = (f_1, \ldots, f_k) : M \to \mathbb{C}^k$$

is called a CR mapping if $f_*(T_p^{0,1}M) \subset T_{f(p)}^{0,1}\mathbb{C}^k$, for all $p \in M$. This is equivalent to saying that $Lf_j = 0$, for all $j = 1, \ldots, k$, and every $(0,1)$-vector field L.

Let $\hat{M} \subset \mathbb{C}^{\hat{n}+1}$ be another real hypersurface (and hence, a CR manifold) and $f : M \to \mathbb{C}^{\hat{n}+1}$ a CR mapping sending M into $\hat{M}$. We let $d = \hat{n} - n$ be the codimension of f. Thus, $\hat{M}$ is a real hypersurface in $\mathbb{C}^{n+d+1}$.

Let $p \in M$ and $\hat{\rho}$ a local defining function for $\hat{M}$ near $\hat{p} := f(p) \in \hat{M}$. Define an increasing sequence of subspaces $E_k(p)$ of the space $T_p'\mathbb{C}^{n+d+1}$ of $(1,0)$-covectors as follows. Let $L_{\bar{1}}, \ldots, L_{\bar{n}}$ be a basis of $(0,1)$-vector fields on M near p and define

$$\begin{aligned} E_k(p) &:= span_{\mathbb{C}}\{L^{\bar{J}}(\hat{\rho}_{Z'} \circ f)(p) : J \in (\mathbb{Z}_+)^n, 0 \le |J| \le k\} \\ &\subset T_p'\mathbb{C}^{n+d+1}, \end{aligned} \tag{3.6}$$

where $\hat{\rho}_{Z'} = \partial\hat{\rho}$ is represented by vectors in $\mathbb{C}^{n+d+1}$ in some local coordinate system Z' near $\hat{\rho}$. Here, we used multi-index notation $L^{\bar{J}} := L_{\bar{1}}^{\bar{J}_1} \ldots L_{\bar{n}}^{\bar{J}_n}$ and $|J| := J_1 + \ldots + J_n$. It was shown in [5] that $E_k(p)$ is independent of the choice of local defining function $\hat{\rho}$ and coordinates Z', as well as the choice of basis

$L_{\bar{1}}, \ldots, L_{\bar{n}}$.

The *(extrinsic) CR second fundamental form* for a CR mapping $f : M \to \hat{M}$ between real hypersurfaces $M \subset \mathbb{C}^{n+1}$ and $\hat{M} \subset \mathbb{C}^{n+d+1}$ can be defined (up to a scalar factor) by

$$\Pi(X_p, Y_p) := \overline{\pi(XY(\hat{\rho}_{Z'} \circ f)(p))} \in \overline{T'_{\hat{p}}\hat{M}/E_1(p)}, \tag{3.7}$$

where $\pi : T'_{\hat{p}}\hat{M} \to T'_{\hat{p}}\hat{M}/E_1(p)$ is the projection and X, Y are any $(1,0)$-vector fields on M extending given vectors $X_p, Y_p \in T^{1,0}_p M$.

In the case when $\hat{M}$ (and hence also M) is strictly pseudoconvex, the Levi form of $\hat{M}$ (at $\hat{p}$) with respect to $\hat{\rho}$ defines an isomorphism $\overline{T'_{\hat{p}}\hat{M}/E_1(p)} \cong T^{1,0}_{\hat{p}}\hat{M}/f_* T^{1,0}_p M$, and hence, the second fundamental form can be viewed as a $\mathbb{C}$-bilinear symmetric form

$$\Pi_p : T^{1,0}_p M \times T^{1,0}_p M \to T^{1,0}_{\hat{p}}\hat{M}/f_* T^{1,0}_p M \tag{3.8}$$

that does not depend on the choice of $\hat{\rho}$. We say that the second fundamental form of f is *nondegenerate at p* if its values span the target space.

3.5 Relationship Between Extrinsic and Intrinsic

We now want to relate the (intrinsic) second fundamental form $(\omega_\alpha{}^b{}_\beta)$ with the (extrinsic) second fundamental form Π_M, in the case that $\hat{M}$ is embedded as a real hypersurface in $\mathbb{C}^{\hat{n}+1}$. This calculation is due to [5], which we review here.

Given any admissible contact form θ for the pair $(M, \hat{M})$, and a point $p \in M$, let us choose a defining function $\hat{\rho}$ of $\hat{M}$ near a point $\hat{p} = f(p) \in M$ such that $\theta = i\overline{\partial}\hat{\rho}$ on $\hat{M}$. That is, in local coordinates Z' in $\mathbb{C}^{\hat{n}+1}$ vanishing at $\hat{p}$, we have

$$\theta = i \sum_{k=1}^{\hat{n}+1} \frac{\partial \hat{\rho}}{\partial \overline{Z}'_k} d\overline{Z}'_k, \tag{3.9}$$

where we pull back the forms $d\overline{Z}'_1, \ldots, d\overline{Z}'_{\hat{n}+1}$ to $\hat{M}$. Given further a coframe (θ, θ^A) on $\hat{M}$ near $\hat{p}$ adapted to M and its dual frame (T, L_A), we have

$$L_\beta(\hat{\rho}_{\overline{Z}'} \circ f) = -iL_\beta \lrcorner d\theta = g_{\beta\overline{C}}\theta^{\overline{C}} = g_{\beta\bar{\gamma}}\theta^{\bar{\gamma}}. \tag{3.10}$$

Recall that we are assuming that the Levi form matrix has been normalized, i.e., $(g_{A\bar{B}}) = (\delta_{A\bar{B}})$. Following [5], we will continue to use the notation $g_{A\bar{B}}$. After conjugating (3.10), we see that the subspace $E_1(p) \subset T'_p\mathbb{C}^{\hat{n}+1}$ in (3.6) is spanned by (θ, θ^α), where we use the standard identification $T'_p\hat{M} \cong T'_p\mathbb{C}^{\hat{n}+1}$. Applying L_α to both sides of (3.10), and using the analogue of (2.10) for $\hat{M}$ and (3.5), we conclude that

$$L_\alpha L_\beta(\hat{\rho}_{\overline{Z}'} \circ f) = g_{\beta\bar{\gamma}} L_\alpha \lrcorner d\theta^{\bar{\gamma}} = -\omega_{\bar{a}\beta\alpha}\theta^{\bar{a}} = \omega_{\alpha\bar{a}\beta}\theta^{\bar{a}}, mod \theta, \theta^{\bar{\alpha}}, \tag{3.11}$$

where we have used (3.3) for the last identity. Comparing with the extrinsic definition of the second fundamental form (3.7), and identifying the spaces in (3.7) and (3.8) via the Levi form of $\hat{M}$, we conclude that

$$\Pi(L_\alpha, L_\beta) = \omega_\alpha{}^a{}_\beta L_a, \tag{3.12}$$

where we have identified L_a with its equivalence class in $T^{1,0}_{\hat{p}}\hat{M}/T^{1,0}_p M$. By conjugating (3.11) and comparing with (3.6), we see that the space $E_2 = E_2(p)$ is spanned (via the identification above) by the forms

$$\theta, \theta^\alpha, \omega_{\bar{\alpha}a\bar{\beta}}\theta^a. \tag{3.13}$$

By this relation, the second fundamental form can now be viewed as a bilinear mapping

$$\Pi_p : T^{1,0}_p M \times T^{1,0}_p M \to T^{1,0}_p \hat{M}/T^{1,0}_p M,$$

defined by $(\omega_\alpha{}^a{}_\beta)$. It is independent of the choice of adapted coframe when $\hat{M}$ is locally CR embeddable in $\mathbb{C}^{\hat{n}+1}$.

4 Gauss Equations and Applications

The Gauss equation of Riemannian geometry relates the Riemannian curvature tensors of a manifold and its submanifold with the second fundamental form of a function composed with the Riemannian metric. In this section, we will review the process laid out in [5] in order to establish pseudohermitian and pseudoconformal analogues of the Gauss equation.

4.1 Pseudohermitian Gauss Equation

Let $M \subset \hat{M}$ be as in the previous section. Let us fix a coframe (θ, θ^A) on $\hat{M}$ and we suppose that this coframe is adapted to M. We first compare the pseudohermitian curvature tensors $R_\alpha{}^\beta{}_{\mu\bar{\nu}}$ and $\hat{R}_A{}^B{}_{C\bar{D}}$ of (M, θ) and $(\hat{M}, \theta)$, respectively.

By comparing (2.12), and the corresponding equation for $\hat{R}_A{}^B{}_{C\bar{D}}$ pulled back to M, and using $\hat{\omega}_\alpha^\beta = \omega_\alpha{}^\beta$, $\hat{\tau}^\alpha = \tau^\alpha$, and $\hat{W}_\alpha{}^\beta{}_\mu = W_\alpha{}^\beta{}_\mu$, as a consequence of (2.14), we conclude that, on M,

$$\hat{R}_\alpha{}^\beta{}_{\mu\bar{\nu}}\theta^\mu \wedge \theta^{\bar{\nu}} + \omega_\alpha{}^a \wedge \omega_a{}^\beta = R_\alpha{}^\beta{}_{\mu\bar{\nu}}\theta^\mu \wedge \theta^{\bar{\nu}}. \tag{4.1}$$

By using the symmetry (3.3), we conclude that, on M, we have

$$\hat{R}_{\alpha\bar{\beta}\mu\bar{\nu}}\theta^\mu \wedge \theta^{\bar{\nu}} - \omega_\alpha{}^a \wedge \omega_{\bar{\beta}\alpha} = R_{\alpha\bar{\beta}\mu\bar{\nu}}\theta^\mu \wedge \theta^{\bar{\nu}}. \tag{4.2}$$

This can also be written, in view of (3.5), after equating the coefficients of

$\theta^\mu \wedge \theta^{\bar{\nu}}$ as

$$\hat{R}_{\alpha\bar{\beta}\mu\bar{\nu}} = R_{\alpha\bar{\beta}\mu\bar{\nu}} + g_{a\bar{b}}\omega_\alpha{}^a{}_\mu \omega_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}}. \tag{4.3}$$

The identity (4.3) relates the pseudohermitian curvature tensors of M and $\hat{M}$ with the second fundamental form of the embedding f of M into $\hat{M}$, and hence, this equation can be considered as a pseudohermitian analogue of the Gauss equation. As in [5], we state it in an invariant form using the previously established relation (3.12) between the extrinsic and intrinsic second fundamental forms Π and $(\omega_\alpha{}^a{}_\beta)$, given respectively by (3.7)-(3.8) and (3.5). For this, we view the pseudohermitian curvature tensors as $\mathbb{R}$-multilinear functions

$$R, \hat{R} : T^{1,0}M \times T^{1,0}M \times T^{1,0}M \times T^{1,0}M \to \mathbb{C}.$$

We further identify the quotient space $T_p^{1,0}\hat{M}/T_p^{1,0}M$ for $p \in M$ with the orthogonal complement of $T_p^{1,0}M$ in $T_p^{1,0}\hat{M}$ with respect to the Levi form of $\hat{M}$ relative to θ, and then use this Levi form to define a canonical hermitian scalar product $\langle \cdot, \cdot \rangle$ on $T_p^{1,0}\hat{M}/T_p^{1,0}M$. The identity (4.3) now yields the following:

Proposition 4.1 (Pseudohermitian Gauss Equation; [5], Proposition 5.1)**.** *Let $M \subset \hat{M}$ be as above and θ be a contact form on $\hat{M}$ that is admissible for the pair $(M, \hat{M})$. Then, for all $p \in M$, the following holds:*

$$\hat{R}(X, Y, Z, V) = R(X, Y, Z, V) + \langle \Pi(X, Z), \Pi(Y, V) \rangle, \tag{4.4}$$

for $X, Y, Z, V \in T_p^{1,0}M$.

4.2 Pseudoconformal Gauss Equation

The pseudoconformal analogue of the Gauss equation follows immediately from (4.3) and (4.4) by taking traceless components (see [5]). Let us denote by $[T_{\alpha\bar{\beta}\mu\bar{\nu}}]$

to be the traceless component of a tensor $T_{\alpha\bar\beta\mu\bar\nu}$, that can be computed by the analogue of the equations (2.20)-(2.21). Hence, (2.20) can be rewritten as

$$S_{\alpha\bar\beta\mu\bar\nu} = [R_{\alpha\bar\beta\mu\bar\nu}]. \tag{4.5}$$

By taking traceless components of both sides in (4.3), and using (4.5) we now obtain

$$[\hat{R}_{\alpha\bar\beta\mu\bar\nu}] = S_{\alpha\bar\beta\mu\bar\nu} + [g_{a\bar b}\omega_\alpha{}^a{}_\mu \omega_{\bar\beta}{}^{\bar b}{}_{\bar\nu}]. \tag{4.6}$$

As noted in [5], the left-hand side of (4.6) may not be, in general, equal to $\hat{S}_{\alpha\bar\beta\mu\bar\nu}$, which is the (restriction to M of the) traceless component of $\hat{R}_{A\bar BC\bar D}$ However, [5] showed that

$$[\hat{R}_{\alpha\bar\beta\mu\bar\nu}] = [\hat{S}_{\alpha\bar\beta\mu\bar\nu}].$$

Indeed, by the decomposition into trace zero components and multiples of the Levi form matrix $(g_{A\bar B})$ on $\hat{M}$, the tensors $\hat{R}_{A\bar BC\bar D}$ and $\hat{S}_{A\bar BC\bar D}$ are conformally equivalent with respect to the Levi form $(g_{A\bar B})$; that is, their difference is of the form similar to (2.22), with lowercase Greek indices replaced by capital Latin indices. Since the Levi form of $\hat{M}$ restricts to that of M, the restrictions $\hat{R}_{\alpha\bar\beta\mu\bar\nu}$ and $\hat{S}_{\alpha\bar\beta\mu\bar\nu}$ are conformally equivalent with respect to $(g_{\alpha\bar\beta})$. Hence, the claim holds. Now (4.6) immediately yields the desired relation between the pseudoconformal curvature tensors of M and $\hat{M}$ and the second fundamental form:

$$[\hat{S}_{\alpha\bar\beta\mu\bar\nu}] = S_{\alpha\bar\beta\mu\bar\nu} + [g_{a\bar b}\omega_\alpha{}^a{}_\mu \omega_{\bar\beta}{}^{\bar b}{}_{\bar\nu}], \tag{4.7}$$

or using formulas (2.20)-(2.21) for the traceless part,

$$\begin{aligned} S_{\alpha\bar\beta\mu\bar\nu} = \hat{S}_{\alpha\bar\beta\mu\bar\nu} &- \frac{\hat{S}_\gamma{}^\gamma{}_{\alpha\bar\beta} g_{\mu\bar\nu} + \hat{S}_\gamma{}^\gamma{}_{\mu\bar\beta} g_{\alpha\bar\nu} + \hat{S}_\gamma{}^\gamma{}_{\alpha\bar\nu} g_{\mu\bar\beta} + \hat{S}_\gamma{}^\gamma{}_{\mu\bar\nu} g_{\alpha\bar\beta}}{n+2} \\ &+ \frac{\hat{S}_\gamma{}^\gamma{}_\delta{}^\delta}{(n+1)(n+2)} - g_{a\bar b}\omega_\alpha{}^a{}_\mu \omega_{\bar\beta}{}^{\bar b}{}_{\bar\nu} \end{aligned}$$

$$+\frac{\omega_\gamma{}^a{}_\alpha\omega^\gamma{}_{a\bar\beta}g_{\mu\bar\nu}+\omega_\gamma{}^a{}_\mu\omega^\gamma{}_{a\bar\beta}g_{\alpha\bar\nu}+\omega_\gamma{}^a{}_\alpha\omega^\gamma{}_{a\bar\nu}g_{\mu\bar\beta}+\omega_\gamma{}^a{}_\mu\omega^\gamma{}_{a\bar\nu}g_{\alpha\bar\beta}}{n+2}$$
$$-\frac{\omega_\gamma{}^a{}_\delta\omega^\gamma{}_a{}^\delta(g_{\alpha\bar\beta}g_{\mu\bar\nu}+g_{\alpha\bar\nu}g_{\mu\bar\beta})}{(n+1)(n+2)}. \tag{4.8}$$

As for the pseudohermitian curvature, we follow [5] by viewing the pseudoconformal curvature tensors, as well as their trace zero components, as $\mathbb{R}$-multilinear functions

$$S,\hat S: T^{1,0}M\times T^{1,0}M\times T^{1,0}M\times T^{1,0}M\to\mathbb{C},$$

but now they are independent of θ. Then, with the above notation, (4.7) yields the following:

Proposition 4.2 (Pseudoconformal Gauss Equation; [5], Proposition 5.2)**.** *For $M\subset\hat M$ as above and every $p\in M$, the following holds:*

$$[\hat S(X,Y,Z,V)]=S(X,Y,Z,V)+[\langle\Pi(X,Z),\Pi(Y,V)\rangle], \tag{4.9}$$

for $X,Y,Z,V\in T_p^{1,0}M$.

4.3 Rigidity Lemmas

We now state the rigidity lemmas needed for the proof of Theorem 1.2:

Lemma 4.3 ([10], Lemma 3.2)**.** *Let $\{\psi_j\}_{j=1}^k$ and $\{\chi_j\}_{j=1}^k$ be holomorphic functions in $z\in\mathbb{C}^n$ near the origin. Assume that $\psi_j(0)=\chi_j(0)=0$. Let $H(z,\overline{z})$ be a real analytic function for $z\approx 0$ such that*

$$\sum_{j=1}^k\psi_j(z)\overline{\chi_j(z)}=|z|^2H(z,\overline{z}). \tag{4.10}$$

Then when $k\le n-1$, $H(z,\overline{z})\equiv 0$.

Proof. By complexifying (4.10), we have

$$\sum_{j=1}^{k} \psi_j(z)\overline{\chi_j(\xi)} = \langle z, \overline{\xi} \rangle H(z, \overline{\xi}),$$

where z, ξ are treated as independent variables. Without loss of generality, we may assume that $\psi_j \not\equiv 0$ for every j. Hence, we can find a point z_0 sufficiently close to the origin such that $\psi_j(z_0) = \epsilon_j \neq 0$, for each j. By the assumption on k, we see that

$$V_{z_0} = \{z : \psi_j(z) = \psi_j(z_0), j = 1, \ldots, k\}$$

defines a complex analytic variety of dimension at least one near z_0. By the choice of z_0, and by $\psi_j(0) = 0$, V_{z_0} cannot contain a complex line passing through the origin. Hence, there exists a point $z' \in V_{z_0}$ such that V_{z_0} contains a complex curve C near z' parametrized by an equation of the form

$$z(t) = z' + vt + o(t), \tag{4.11}$$

where $\{z', v\}$ are independent vectors, and $|t| < 1$. Note that for each $z \in C$ and a ξ close to 0 with $\langle z, \overline{\xi} \rangle = 0$, we have

$$\sum \overline{\epsilon_j} \chi_j(\xi) = 0.$$

Also, we note that a direction computation based on (4.11) shows that all such ξ fill in an open subset of $\mathbb{C}^n$. Hence, we conclude that $\sum \overline{\epsilon_j} \chi_j(\xi) \equiv 0$. Therefore, (4.10) can be reduced to

$$\sum_{j=1}^{k-1} (\psi_j(z) - \frac{\epsilon_j}{\epsilon_k} \psi_k(z)) \overline{(\chi_j(z))} = \langle z, \overline{z} \rangle H(z, \overline{z}).$$

Applying an induction argument, it follows that $\sum \psi_j \chi_j \equiv 0$ and $H \equiv 0$. $\square$

Now, we show the tensor version that will be used. For the sake of readability, we define

$$\mathcal{H}^l_{\alpha\overline{\beta}\mu\overline{\nu}} := H^l_{\alpha\overline{\beta}} g_{\mu\overline{\nu}} + \hat{H}^l_{\mu\overline{\beta}} g_{\alpha\overline{\nu}} + \tilde{H}^l_{\alpha\overline{\nu}} g_{\mu\overline{\beta}} + \check{H}^l_{\mu\overline{\nu}} g_{\alpha\overline{\beta}}.$$

Corollary 4.4 ([11], Lemma 2.1)**.** *Let* $A_\alpha{}^a{}_\beta$ *and* $B_\alpha{}^a{}_\beta$ *be complex numbers, where* $1 \leq \alpha, \beta \leq n$*,* $n+1 \leq a \leq \hat{n}$*, and* $n \leq \hat{n}$*. Let* $(g_{\alpha\overline{\beta}})$ *and* $(G_{a\overline{b}})$ *be hermitian matrices with* $(g_{\alpha\overline{\beta}})$ *positive definite. Let* $(H^l_{\alpha\overline{\beta}})$*,* $(\hat{H}^l_{\alpha\overline{\beta}})$*,* $(\tilde{H}^l_{\alpha\overline{\beta}})$*, and* $(\check{H}^l_{\alpha\overline{\beta}})$ *be hermitian matrices where* $1 \leq l \leq k$*. Suppose that* $\hat{n} - n \leq n - 1$ *and that*

$$\sum_{a,b=n+1}^{\hat{n}} G_{a\overline{b}} A_\alpha{}^a{}_\beta X^\alpha X^\beta \overline{B_\mu{}^b{}_\nu} \overline{X^\mu X^\nu} = \sum_{l=1}^{k} \mathcal{H}^l_{\alpha\overline{\beta}\mu\overline{\nu}} X^\alpha \overline{X^\beta} X^\mu \overline{X^\nu} \tag{4.12}$$

holds for any $X = (X^\alpha) = (X^\beta) = (X^\mu) = (X^\nu) \in \mathbb{C}^n$*. Then*

$$\sum_{a,b=n+1}^{\hat{n}} G_{a\overline{b}} A_\alpha{}^a{}_\beta X^\alpha X^\beta \overline{B_\mu{}^b{}_\nu} \overline{X^\mu X^\nu} \equiv 0$$

for all $X \in \mathbb{C}^n$*.*

Proof. Note that the right-hand side of (4.12) is equal to

$$\begin{aligned}
\sum_{l=1}^{k} \mathcal{H}^l_{\alpha\overline{\beta}\mu\overline{\nu}} X^\alpha \overline{X^\beta} X^\mu \overline{X^\nu} &= \sum_{l=1}^{k} \left(H^l_{\alpha\overline{\beta}} X^\alpha \overline{X^\beta} |X|^2 + \hat{H}^l_{\mu\overline{\beta}} X^\mu \overline{X^\beta} |X|^2 \right) \\
&\quad + \sum_{l=1}^{k} \left(\tilde{H}^l_{\alpha\overline{\nu}} X^\alpha \overline{X^\nu} |X|^2 + \check{H}^l_{\mu\overline{\nu}} X^\mu \overline{X^\nu} |X|^2 \right) \\
&= |X|^2 \sum_{l=1}^{k} \left(H^l_{\alpha\overline{\beta}} X^\alpha \overline{X^\beta} + \hat{H}^l_{\mu\overline{\beta}} X^\mu \overline{X^\beta} \right) \\
&\quad + |X|^2 \sum_{l=1}^{k} \left(\tilde{H}^l_{\alpha\overline{\nu}} X^\alpha \overline{X^\nu} + \check{H}^l_{\mu\overline{\nu}} X^\mu \overline{X^\nu} \right) \\
&= |X|^2 A(X, \overline{X}),
\end{aligned}$$

where $A(X,\overline{X})$ is some real analytic function of X and $|X|^2 = g_{\alpha\overline{\beta}} X^\alpha \overline{X^\beta}$. The left-hand side is equal to

$$\sum_{a,b=n+1}^{\hat{n}} G_{a\bar{b}} A_\alpha{}^a{}_\beta X^\alpha X^\beta \overline{B_\mu{}^b{}_\nu} \overline{X^\mu X^\nu} = \sum_{a=n+1}^{\hat{n}} g_a(X) \overline{h_a(X)},$$

where

$$g_a(X) = \sum_{a,b} A_\alpha{}^a{}_\beta X^\alpha X^\beta,$$
$$h_a(X) = \sum_{b=n+1}^{\hat{n}} \sum_{a,b} \overline{G_{ab}} B_\alpha{}^b{}_\beta X^\alpha X^\beta$$

are holomorphic functions. Hence, (4.12) becomes

$$\sum_{a=n+1}^{\hat{n}} g_a(X)\overline{h_a(X)} = |X|^2 A(X,\overline{X})$$

for all $X \in \mathbb{C}^n$. By the assumption $\hat{n} - n < n$ and Lemma 4.3, we have that $A(X,\overline{X}) \equiv 0$. □

5 Proof of Theorem 1.2

At this time, we would like to set up some of the notation that we will use throughout the rest of this section. Let us denote the coordinates of $\mathbb{C}^n \times \mathbb{C}$ by (z,w), where we denote by $z = (z^1,\ldots,z^n)$ to be the coordinates of $\mathbb{C}^n$. In addition, we will continue to use the same summation convention from the previous sections. We also follow the partial derivative conventions of [19], setting

$$f_\alpha = \tfrac{\partial f}{\partial z^\alpha},\ f_w = \tfrac{\partial f}{\partial w}, \text{ etc.}$$

We will denote by D to be the domain defined by

$$D = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : r(z, \bar{z}, w, \bar{w}) < 0\},$$

where the defining function r is given by

$$r(z, \bar{z}, w, \bar{w}) = p(z, \bar{z}) + q(w, \bar{w}),$$
$$p(z, \bar{z}) = \bar{z}^t H z$$
$$\overline{q(w, \bar{w})} = q(w, \bar{w}),$$

such that $H = (h_{\alpha\bar{\beta}})$ is a (constant) hermitian positive definite matrix. The boundary M of D is the real hypersurface

$$M = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : r(z, \bar{z}, w, \bar{w}) = 0\},$$

For our calculations below, we will assume that $q_{\overline{w}} \neq 0$.

The associated domain D_0, and its boundary M_0, are given by

$$D_0 = \{w \in \mathbb{C} : q(w, \overline{w}) < 0\},$$
$$M_0 = \{w \in \mathbb{C} : q(w, \overline{w}) = 0\}.$$

In this case, we assume that $dq \neq 0$ whenever $q = 0$.

5.1 Admissible Coframe on M

For our calculations, we choose to use the one-form θ defined by

$$\theta = -i\partial r = -i(p_\alpha dz^\alpha + q_w dw), \tag{5.1}$$

which is a contact form on M. Note that, since θ is a real one-form on M, we have that $\theta = \overline{\theta}$. This implies that

$$-i\partial r = \theta = \overline{\theta} = i\overline{\partial} r;$$

that is,

$$-i(p_\alpha dz^\alpha + q_w dw) = i(p_{\overline{\beta}} dz^{\overline{\beta}} + q_{\overline{w}} d\overline{w}). \tag{5.2}$$

Rearranging (5.2), we obtain, on M,

$$p_\alpha dz^\alpha + p_{\overline{\beta}} dz^{\overline{\beta}} = -(q_w dw + q_{\overline{w}} d\overline{w}). \tag{5.3}$$

By definition of the exterior derivative, we have that

$$\begin{aligned} d\theta &= -i\overline{\partial}\partial r \\ &= -i\overline{\partial}(p_\alpha dz^\alpha + q_w dw) \\ &= -i\Big(p_{\alpha\overline{\beta}} dz^{\overline{\beta}} \wedge dz^\alpha + p_{\alpha\overline{w}} d\overline{w} \wedge dz^\alpha \\ &\qquad + q_{w\overline{\beta}} dz^{\overline{\beta}} \wedge dw + q_{w\overline{w}} d\overline{w} \wedge dw\Big). \end{aligned} \tag{5.4}$$

Since p is only a function of z and q a function of only w, we have that

$$p_{\alpha\overline{w}} = q_{w\overline{\beta}} = 0.$$

Hence, (5.4) becomes

$$\begin{aligned} d\theta &= -i(p_{\alpha\overline{\beta}} dz^{\overline{\beta}} \wedge dz^\alpha + q_{w\overline{w}} d\overline{w} \wedge dw) \\ &= i(p_{\alpha\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}} + q_{w\overline{w}} dw \wedge d\overline{w}). \end{aligned} \tag{5.5}$$

By using the equality in (5.3), (5.5) becomes the following:

$$\begin{aligned}
d\theta &= ip_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} + iq_{w\overline{w}}dw \wedge d\overline{w} \\
&= ip_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} + i\frac{q_{w\overline{w}}q_w}{q_{\overline{w}}}dw \wedge dw + iq_{w\overline{w}}dw \wedge d\overline{w} \\
&= ip_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} + iq_{w\overline{w}}dw \wedge \left(\frac{q_w}{q_{\overline{w}}}dw + d\overline{w}\right) \\
&= ip_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} + i\frac{q_{w\overline{w}}}{q_{\overline{w}}}dw \wedge (q_w dw + q_{\overline{w}}d\overline{w}) \\
&= ip_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} - i\frac{q_{w\overline{w}}}{q_{\overline{w}}}dw \wedge \left(p_\alpha dz^\alpha + p_{\overline{\beta}}dz^{\overline{\beta}}\right). \qquad (5.6)
\end{aligned}$$

Let us define

$$Q = \frac{q_{w\overline{w}}}{q_w q_{\overline{w}}}. \qquad (5.7)$$

(5.6) can now be written as

$$\begin{aligned}
d\theta &= ip_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} + iQq_w p_\alpha dz^\alpha \wedge dw + iQq_w p_{\overline{\beta}}dz^{\overline{\beta}} \wedge dw \\
&= ip_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} + iQq_w p_\alpha dz^\alpha \wedge dw + iQq_w p_{\overline{\beta}}dz^{\overline{\beta}} \wedge dw \\
&\quad + iQp_\alpha p_{\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} - iQp_\alpha p_{\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} + iQp_\alpha p_\alpha dz^\alpha \wedge dz^\alpha \\
&= ig_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} + iQp_\alpha dz^\alpha \wedge (p_\alpha dz^\alpha + q_w dw) \\
&\quad + iQp_{\overline{\beta}}dz^{\overline{\beta}} \wedge (p_\alpha dz^\alpha + q_w dw) \\
&= ig_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} - Qp_\alpha dz^\alpha \wedge \theta - Qp_{\overline{\beta}}dz^{\overline{\beta}} \wedge \theta, \qquad (5.8)
\end{aligned}$$

where we set the Levi form matrix to be

$$g_{\alpha\overline{\beta}} = p_{\alpha\overline{\beta}} + Qp_\alpha p_{\overline{\beta}}. \qquad (5.9)$$

If we set $\eta_\alpha = -Qp_\alpha$ and $\eta_{\overline{\beta}} = \overline{\eta_\beta}$, (5.8) becomes

$$d\theta = ig_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} + \eta_\alpha dz^\alpha \wedge \theta + \eta_{\overline{\beta}}dz^{\overline{\beta}} \wedge \theta$$

$$
\begin{aligned}
&= ig_{\alpha\overline{\beta}} \left(dz^{\alpha} \wedge dz^{\overline{\beta}} - i\eta^{\overline{\beta}} dz^{\alpha} \wedge \theta - i\eta^{\alpha} dz^{\overline{\beta}} \wedge \theta \right) \\
&= ig_{\alpha\overline{\beta}} \left(dz^{\alpha} \wedge dz^{\overline{\beta}} - dz^{\alpha} \wedge (i\eta^{\overline{\beta}}\theta) + (i\eta^{\alpha}\theta) \wedge dz^{\overline{\beta}} - (i\eta^{\alpha}\theta) \wedge (i\eta^{\overline{\beta}}\theta) \right) \\
&= ig_{\alpha\overline{\beta}} \left((dz^{\alpha} + i\eta^{\alpha}\theta) \wedge (dz^{\overline{\beta}} - i\eta^{\overline{\beta}}\theta) \right). \qquad (5.10)
\end{aligned}
$$

Now we define

$$
\begin{aligned}
\theta^{\alpha} &= dz^{\alpha} + i\eta^{\alpha}\theta, \\
\theta^{\overline{\beta}} &= \overline{\theta^{\beta}}. \qquad (5.11)
\end{aligned}
$$

Thus, by (5.10), $d\theta = ig_{\alpha\overline{\beta}}\theta^{\alpha} \wedge \theta^{\overline{\beta}}$. By definition, $(\theta, \theta^{\alpha}, \theta^{\overline{\beta}})$ is an admissible coframe on M.

5.2 A Metric on D_0

Let us define

$$
h = \frac{q_w q_{\overline{w}} - q q_{w\overline{w}}}{q^2}.
$$

Then we have the following:

Lemma 5.1 (Webster, [19])**.** *M is strictly pseudoconvex if and only if, on* D_0*,*

$$
\frac{q_w q_{\overline{w}} - q q_{w\overline{w}}}{q^2} > 0.
$$

Proof. Let $w \in D_0$ and $(z, w) \in M$. By definition, $w \in D_0$ implies that $q(w, \overline{w}) < 0$. Hence, by definition of the defining function r, $p(z, \overline{z}) > 0$ at (z, w). By the definiteness of H, we have that $z \neq 0$. In addition, we recall that these calculations are only valid where $q_{\overline{w}} \neq 0$.

($\Rightarrow$) Let us suppose that M is strictly pseudoconvex. By definition, the Levi matrix $(g_{\alpha\overline{\beta}})$ is positive definite. Thus, for every non-zero $X = X^{\alpha}\frac{\partial}{\partial z^{\alpha}} \in T^{1,0}_{(z,w)}M$, we have that $g_{\alpha\overline{\beta}}X^{\alpha}X^{\overline{\beta}} > 0$. By definition of the Levi matrix $(g_{\alpha\overline{\beta}})$, and the partial derivatives of p, we have that

$$\begin{aligned}0 &< (p_{\alpha\overline{\beta}} + Qp_\alpha p_{\overline{\beta}})X^\alpha X^{\overline{\beta}} \\ &= h_{\alpha\overline{\beta}}X^\alpha X^{\overline{\beta}} + Q|h_{\alpha\overline{\varepsilon}}X^\alpha z^{\overline{\varepsilon}}|^2 \\ &= p(x,\overline{x}) + Q|p(x,\overline{z})|^2,\end{aligned}$$

where we write $x = (X^1, \ldots, X^n)^t$. Taking $X^\alpha = z^\alpha$ for all α, we obtain

$$0 < p(z,\overline{z}) + Q|p(z,\overline{z})|^2 = -q(w,\overline{w}) + Q(q(w,\overline{w}))^2.$$

By definition of Q, this implies $qq_w q_{\overline{w}} < q^2 q_{w\overline{w}}$. Since $q < 0$ on D_0, we have

$$q_w q_{\overline{w}} - qq_{w\overline{w}} > 0.$$

Hence,

$$h = \frac{q_w q_{\overline{w}} - qq_{w\overline{w}}}{q^2} > 0$$

on D_0.

($\Leftarrow$) Conversely, suppose that

$$h = \frac{q_w q_{\overline{w}} - qq_{w\overline{w}}}{q^2} > 0$$

on D_0. Then,

$$q_w q_{\overline{w}} - qq_{w\overline{w}} > 0.$$

Hence, by definition of Q, we have that $Qq < 1$. Let $X = X^\alpha \frac{\partial}{\partial z^\alpha}$ be a non-zero vector in $T^{1,0}_{(z,w)}M$. By the previous argument,

$$g_{\alpha\overline{\beta}}X^\alpha X^{\overline{\beta}} = p(x,\overline{x}) + Q|p(x,\overline{z})|^2,$$

where we again write $x = (X^1, \ldots, X^n)^t$.

When $Q \geq 0$, then

$$g_{\alpha\overline{\beta}} X^\alpha X^{\overline{\beta}} = p(x, \overline{x}) + Q|p(x, \overline{z})|^2 > 0$$

by the definiteness of H.

When $Q < 0$, then by the Cauchy-Schwarz inequality[3],

$$|p(x, \overline{z})|^2 \leq p(x, \overline{x}) p(z, \overline{z}).$$

This implies that

$$0 > Q|p(x, \overline{z})|^2 \geq Qp(x, \overline{x})p(z, \overline{z}) = -Qqp(x, \overline{x}).$$

Thus,

$$\begin{aligned} g_{\alpha\overline{\beta}} X^\alpha X^{\overline{\beta}} &= p(x, \overline{x}) + Q|p(x, \overline{z})|^2 \\ &\geq p(x, \overline{x}) - Qqp(x, \overline{x}) \\ &> 0, \end{aligned}$$

because $Qq < 1$. Hence, the Levi matrix $(g_{\alpha\overline{\beta}})$ is positive definite, and by definition, M is strictly pseudoconvex. □

Remarks. From now on, we will assume that M is strictly pseudoconvex; that is, we assume that the Levi matrix $(g_{\alpha\overline{\beta}})$ is positive definite. In this case,

$$h = \frac{q_w q_{\overline{w}} - q q_{w\overline{w}}}{q^2} > 0$$

[3]**Cauchy-Schwarz Inequality**: For all vectors x and y in an inner product space $(X, \langle \cdot, \cdot \rangle)$, we have that

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle.$$

and h defines a hermitian metric on D_0.

5.3 Curvature Formulas for M

Let us first define the following:

$$\Sigma_l^m = \sum_{\gamma=l}^{m} p_\gamma p_{\overline{\gamma}},$$
$$\Sigma_{l,k}^m = \sum_{\gamma=l,\gamma\neq k}^{m} p_\gamma p_{\overline{\gamma}},$$

where $p_\alpha = \frac{\partial}{\partial z^\alpha} p$ as before. If I_n is the $n \times n$ identity matrix, we will denote the augmented matrix obtained by deleting the first α rows and β columns by[4]

$$I_n^{\alpha,\beta} := I_n(\{1,\ldots,\alpha\}', \{\overline{1},\ldots,\overline{\beta}\}'),$$

with $I_n^\alpha := I_n^{\alpha,\alpha}$. These will ease the readability of the following identities concerning the inverse of the Levi matrix that will appear throughout the rest of this paper.

Lemma 5.2. *For $n > 1$, let g be the $n \times n$ matrix defined by*

$$g_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} + Qp_\alpha p_{\overline{\beta}},$$

The $(\alpha, \overline{\beta})$-minors, $A_{\alpha\overline{\beta}}$, of g satisfy

$$A_{\alpha\overline{\beta}} = \begin{cases} 1 + Q\Sigma_{1,\alpha}^n, & \alpha = \beta, \\ -Qp_\beta p_{\overline{\alpha}}, & \alpha \neq \beta, \alpha+\beta \text{ even}, \\ +Qp_\beta p_{\overline{\alpha}}, & \alpha \neq \beta, \alpha+\beta \text{ odd}. \end{cases}$$

[4] By deleting the first α rows and the first β columns, the first row of $I_n^{\alpha,\beta}$ will have an index of $\alpha + 1$ and the first column of $I_n^{\alpha,\beta}$ will have index $\beta + 1$

In particular, $\det(g) = 1 + Q\sum_{\gamma=1}^{n} p_\gamma p_{\overline{\gamma}}$ *and the entries of the adjugate matrix* $adj(g)$ *of* g *satisfy*

$$adj(g)_{\alpha\overline{\beta}} = \begin{cases} 1 + Q\Sigma^n_{1,\alpha}, & \alpha = \beta, \\ -Qp_\alpha p_{\overline{\beta}}, & \alpha \neq \beta. \end{cases}$$

Proof. We prove this by induction. For $n = 2$,

$$g = \begin{pmatrix} 1 + Qp_1p_{\overline{1}} & Qp_1p_{\overline{2}} \\ Qp_2p_{\overline{1}} & 1 + Qp_2p_{\overline{2}} \end{pmatrix}.$$

The minors of g are

$$A_{1\overline{1}} = 1 + Qp_2p_{\overline{2}}, A_{1\overline{2}} = Qp_2p_{\overline{1}},$$
$$A_{2\overline{1}} = Qp_1p_{\overline{2}}, A_{2\overline{2}} = 1 + Qp_1p_{\overline{1}}.$$

In addition, by definition,

$$adj(g) = \begin{pmatrix} 1 + Qp_2p_{\overline{2}} & -Qp_1p_{\overline{2}} \\ -Qp_2p_{\overline{1}} & 1 + Qp_1p_{\overline{1}} \end{pmatrix},$$

and by Laplace's formula[5], $\det(g) = 1 + Qp_1p_{\overline{1}} + Qp_2p_{\overline{2}}$. Hence, the lemma holds for $n = 2$.

Now we assume that the statement holds for $1 < k \leq n - 1$ and prove the nth case. We will need to break this into three subcases.

When $\alpha = \beta$, the augmented matrix of g, obtained by deleting the α-row

[5]**Laplace's Formula**: Suppose that $B = (b_{ij})$ is an $n \times n$ matrix and fix any index i_0 or j_0. Then the determinant of B is given by

$$\det(B) = \sum_{j=1}^{n} b_{i_0,j}C_{i_0,j} = \sum_{i=1}^{n} b_{i,j_0}C_{i,j_0}.$$

Here, $C_{ij} = (-1)^{i+j}M_{ij}$ and M_{ij} is the (i,j)-minor of B.

and β-column, can be blocked off in the following manner:

$$\begin{aligned}
\tilde{g} &= g(\{\alpha\}', \{\overline{\beta}\}') \\
&= \left(\begin{array}{ccc|ccc}
1+Qp_1p_{\overline{1}} & \cdots & Qp_1p_{\overline{\alpha-1}} & Qp_1p_{\overline{\alpha+1}} & \cdots & Qp_1p_{\overline{n}} \\
\vdots & & \vdots & \vdots & & \vdots \\
Qp_{\alpha-1}p_{\overline{1}} & \cdots & 1+Qp_{\alpha-1}p_{\overline{\alpha-1}} & Qp_{\alpha-1}p_{\overline{\alpha+1}} & \cdots & Qp_{\alpha-1}p_{\overline{n}} \\
\hline
Qp_{\alpha+1}p_{\overline{1}} & \cdots & Qp_{\alpha+1}p_{\overline{\alpha-1}} & 1+Qp_{\alpha+1}p_{\overline{\alpha+1}} & \cdots & Qp_{\alpha+1}p_{\overline{n}} \\
\vdots & & \vdots & \vdots & & \vdots \\
Qp_np_{\overline{1}} & \cdots & Qp_np_{\overline{\alpha-1}} & Qp_np_{\overline{\alpha+1}} & \cdots & 1+Qp_np_{\overline{n}}
\end{array}\right) \\
&= \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).
\end{aligned}$$

By the induction hypothesis, A is invertible and $\det(A) = 1 + Q\Sigma_1^{\alpha-1}$. Hence, by block matrices,

$$\det(\tilde{g}) = \det(A)\det(D - CA^{-1}B).$$

In addition, by Cramer's rule[6],

$$\begin{aligned}
CA^{-1}B &= C\frac{adj(A)}{\det(A)}B \\
&= \frac{Q\Sigma_1^{\alpha-1}}{\det(A)}\begin{pmatrix} Qp_{\alpha+1}p_{\overline{\alpha+1}} & \cdots & Qp_{\alpha+1}p_{\overline{n}} \\ \vdots & & \vdots \\ Qp_np_{\overline{\alpha+1}} & \cdots & Qp_np_{\overline{n}} \end{pmatrix} \\
&= \frac{\det(A)-1}{\det(A)}E.
\end{aligned}$$

Note that $D = I_n^\alpha + E$. Hence, we have

$$D - CA^{-1}B = I_n^\alpha + E - \frac{\det(A)-1}{\det(A)}E$$

[6]**Cramer's Rule**: Let A be an $n \times n$ matrix. Then $adj(A) \cdot A = \det(A) \cdot I$, where $adj(A)$ denotes the adjugate matrix of A and I is the identity matrix. If A is invertible, then the inverse matrix of A satisfies

$$A^{-1} = \frac{1}{\det(A)}adj(A).$$

$$
\begin{aligned}
&= I_n^\alpha + \frac{1}{\det(A)} E \\
&= I_n^\alpha + E',
\end{aligned}
$$

where we set $E' = \frac{1}{\det(A)} E$ and $Q' = \frac{Q}{\det(A)}$. Let

$$
F = \begin{pmatrix} 1+Q' p_\alpha p_{\overline{\alpha}} \dots & Q' p_\alpha p_{\overline{n}} \\ \vdots & \vdots \\ Q' p_n p_{\overline{\alpha}} \quad \dots & 1+Q' p_n p_{\overline{n}} \end{pmatrix}.
$$

Then $D - CA^{-1}B = F(\{\alpha\}', \{\overline{\beta}\}')$, and by the induction hypothesis,

$$
\det(D - CA^{-1}B) = 1 + Q' \Sigma^n_{\alpha,\alpha}.
$$

Therefore,

$$
\begin{aligned}
\det(\tilde{g}) &= \det(A) \det(D - CA^{-1}B) \\
&= \det(A)(1 + Q' \Sigma^n_{\alpha,\alpha}) \\
&= \det(A) + Q \Sigma^n_{\alpha,\alpha} \\
&= 1 + Q \Sigma^n_{1,\alpha}.
\end{aligned}
$$

If $\alpha < \beta$, the augmented matrix of g, obtained by deleting the α-row and β-column, can be blocked off in the following manner:

$$
\begin{aligned}
\tilde{g} &= g(\{\alpha\}', \{\overline{\beta}\}') \\
&= \left(\begin{array}{cc|ccccc}
1+Q p_1 p_{\overline{1}} \dots & Q p_1 p_{\overline{\alpha-1}} & Q p_1 p_{\overline{\alpha}} & \dots & \widehat{Q p_1 p_{\overline{\beta}}} & \dots & Q p_1 p_{\overline{n}} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
Q p_{\alpha-1} p_{\overline{1}} \dots & 1+Q p_{\alpha-1} p_{\overline{\alpha-1}} & Q p_{\alpha-1} p_{\overline{\alpha}} & \dots & \widehat{Q p_{\alpha-1} p_{\overline{\beta}}} & \dots & Q p_{\alpha-1} p_{\overline{n}} \\
\hline
Q p_{\alpha+1} p_{\overline{1}} \dots & Q p_{\alpha+1} p_{\overline{\alpha-1}} & Q p_{\alpha+1} p_{\overline{\alpha}} & \dots & \widehat{Q p_{\alpha+1} p_{\overline{\beta}}} & \dots & Q p_{\alpha+1} p_{\overline{n}} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
Q p_n p_{\overline{1}} \quad \dots & Q p_n p_{\overline{\alpha-1}} & Q p_n p_{\overline{\alpha}} & \dots & \widehat{Q p_n p_{\overline{\beta}}} & \dots & 1+Q p_n p_{\overline{n}}
\end{array}\right)
\end{aligned}
$$

$$= \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

where $\widehat{\cdot}$ denotes the deleted entries. By the induction hypothesis, A is invertible and $\det(A) = 1 + Q\Sigma_1^{\alpha-1}$. Hence,

$$\det(\tilde{g}) = \det(A)\det(D - CA^{-1}B).$$

In addition, by Cramer's rule,

$$\begin{aligned} CA^{-1}B &= C\frac{adj(A)}{\det(A)}B \\ &= \frac{Q\Sigma_1^{\alpha-1}}{\det(A)}\begin{pmatrix} Qp_{\alpha+1}p_{\overline{\alpha}} & \dots & \widehat{Qp_{\alpha+1}p_{\overline{\beta}}} & \dots & Qp_{\alpha+1}p_{\overline{n}} \\ \vdots & & \vdots & & \vdots \\ Qp_n p_{\overline{\alpha}} & \dots & \widehat{Qp_n p_{\overline{\beta}}} & \dots & Qp_n p_{\overline{n}} \end{pmatrix} \\ &= \frac{\det(A) - 1}{\det(A)}E. \end{aligned}$$

Note that $D = I_n^{\alpha-1}(\{\alpha\}', \{\overline{\beta}\}') + E$. Hence, we have

$$\begin{aligned} D - CA^{-1}B &= I_n^{\alpha-1}(\{\alpha\}', \{\overline{\beta}\}') + E - \frac{\det(A) - 1}{\det(A)}E \\ &= I_n^{\alpha-1}(\{\alpha\}', \{\overline{\beta}\}') + \frac{1}{\det(A)}E \\ &= I_n^{\alpha-1}(\{\alpha\}', \{\overline{\beta}\}') + E', \end{aligned}$$

where we set $E' = \frac{1}{\det(A)}E$ and $Q' = \frac{Q}{\det(A)}$. Let

$$F = \begin{pmatrix} 1+Q'p_\alpha p_{\overline{\alpha}} & \dots & Q'p_\alpha p_{\overline{n}} \\ \vdots & & \vdots \\ Q'p_n p_{\overline{\alpha}} & \dots & 1+Q'p_n p_{\overline{n}} \end{pmatrix}.$$

Then $D - CA^{-1}B = F(\{\alpha\}', \{\overline{\beta}\}')$, and by the induction hypothesis,

$$\det(D - CA^{-1}B) = \begin{cases} -Q' p_\beta p_{\overline{\alpha}}, & \alpha \neq \beta, \alpha + \beta \text{ even}, \\ +Q' p_\beta p_{\overline{\alpha}}, & \alpha \neq \beta, \alpha + \beta \text{ odd}. \end{cases}$$

Therefore,

$$\begin{aligned} \det(\tilde{g}) &= \det(A) \det(D - CA^{-1}B) \\ &= \begin{cases} -Q p_\beta p_{\overline{\alpha}}, & \alpha \neq \beta, \alpha + \beta \text{ even}, \\ +Q p_\beta p_{\overline{\alpha}}, & \alpha \neq \beta, \alpha + \beta \text{ odd}. \end{cases} \end{aligned}$$

Lastly, if $\alpha > \beta$, the augmented matrix of g, obtained by deleting the α-row and β-column, can be blocked off in the following manner:

$$\begin{aligned} \tilde{g} &= g(\{\alpha\}', \{\overline{\beta}\}') \\ &= \left(\begin{array}{cc|cc} 1+Qp_1 p_{\overline{1}} \cdots & Qp_1 p_{\overline{\beta-1}} & Qp_1 p_{\overline{\beta+1}} \cdots & Qp_1 p_{\overline{n}} \\ \vdots & \vdots & \vdots & \vdots \\ Qp_{\beta-1} p_{\overline{1}} \cdots & 1+Qp_{\beta-1} p_{\overline{\beta-1}} & Qp_{\beta-1} p_{\overline{\beta+1}} \cdots & Qp_{\beta-1} p_{\overline{n}} \\ \hline Qp_\beta p_{\overline{1}} \cdots & Qp_\beta p_{\overline{\beta-1}} & Qp_\beta p_{\overline{\beta+1}} \cdots & Qp_\beta p_{\overline{n}} \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{Qp_\alpha p_{\overline{1}}} \cdots & \widehat{Qp_\alpha p_{\overline{\beta-1}}} & \widehat{Qp_\alpha p_{\overline{\beta+1}}} \cdots & \widehat{Qp_\alpha p_{\overline{n}}} \\ \vdots & \vdots & \vdots & \vdots \\ Qp_n p_{\overline{1}} \cdots & Qp_n p_{\overline{\beta-1}} & Qp_n p_{\overline{\beta+1}} \cdots & 1+Qp_n p_{\overline{n}} \end{array}\right) \\ &= \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \end{aligned}$$

where $\widehat{\cdot}$ denotes the deleted entry. By the induction hypothesis, A is invertible and $\det(A) = 1 + Q\Sigma_1^{\beta-1}$. Hence,

$$\det(\tilde{g}) = \det(A) \det(D - CA^{-1}B).$$

In addition, by Cramer's rule,

$$
\begin{aligned}
CA^{-1}B &= C\frac{adj(A)}{\det(A)}B \\
&= \frac{Q\Sigma_1^{\beta-1}}{\det(A)}\begin{pmatrix} Qp_\beta p_{\overline{\beta+1}} & \dots & Qp_\beta p_{\overline{n}} \\ \vdots & & \vdots \\ \widehat{Qp_\alpha p_{\overline{\beta+1}}} & \dots & \widehat{Qp_\alpha p_{\overline{n}}} \\ \vdots & & \vdots \\ Qp_n p_{\overline{\beta+1}} & \dots & Qp_n p_{\overline{n}} \end{pmatrix} \\
&= \frac{\det(A)-1}{\det(A)}E.
\end{aligned}
$$

Note that $D = I_n^{\beta-1}(\{\alpha\}', \{\overline{\beta}\}') + E$. Hence, we have

$$
\begin{aligned}
D - CA^{-1}B &= I_n^{\beta-1}(\{\alpha\}', \{\overline{\beta}\}') + E - \frac{\det(A)-1}{\det(A)}E \\
&= I_n^{\beta-1}(\{\alpha\}', \{\overline{\beta}\}') + \frac{1}{\det(A)}E \\
&= I_n^{\beta-1}(\{\alpha\}', \{\overline{\beta}\}') + E',
\end{aligned}
$$

where we set $E' = \frac{1}{\det(A)}E$ and $Q' = \frac{Q}{\det(A)}$. Let

$$
F = \begin{pmatrix} 1+Q'p_\beta p_{\overline{\beta}} & \dots & Q'p_\beta p_{\overline{n}} \\ \vdots & & \vdots \\ Q'p_n p_{\overline{\beta}} & \dots & 1+Q'p_n p_{\overline{n}} \end{pmatrix}.
$$

Then $D - CA^{-1}B = F(\{\alpha\}', \{\overline{\beta}\}')$, and by the induction hypothesis,

$$
\det(D - CA^{-1}B) = \begin{cases} -Q'p_\beta p_{\overline{\alpha}}, & \alpha \neq \beta, \alpha+\beta \text{ even}, \\ +Q'p_\beta p_{\overline{\alpha}}, & \alpha \neq \beta, \alpha+\beta \text{ odd}. \end{cases}
$$

Therefore,

$$
\det(\tilde{g}) = \det(A)\det(D - CA^{-1}B)
$$

$$= \begin{cases} -Qp_\beta p_{\overline{\alpha}}, & \alpha \neq \beta, \alpha+\beta \text{ even}, \\ +Qp_\beta p_{\overline{\alpha}}, & \alpha \neq \beta, \alpha+\beta \text{ odd}. \end{cases}$$

For the final statements, by definition of the adjugate matrix, we have that

$$adj(g) = \begin{pmatrix} 1+Q\Sigma^n_{1,1} & -Qp_1p_{\overline{2}} & \dots & -Qp_1p_{\overline{n}} \\ -Qp_2p_{\overline{2}} & 1+Q\Sigma^n_{1,2} & \dots & -Qp_2p_{\overline{n}} \\ \vdots & \vdots & & \vdots \\ -Qp_np_{\overline{1}} & -Qp_np_{\overline{2}} & \dots & 1+Q\Sigma^n_{1,n} \end{pmatrix}.$$

By Laplace's formula, $\det(g) = 1 + Q\Sigma^n_1$. □

Corollary 5.3. *Suppose that the Levi matrix g of the hypersurface M is given by*

$$g_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} + Qp_\alpha p_{\overline{\beta}}.$$

Then at every point of M,

$$g^{\alpha\overline{\beta}} p_\alpha p_{\overline{\beta}} = -\frac{q}{1-Qq},$$

where $(g^{\alpha\overline{\beta}})$ is the inverse of $(g_{\alpha\overline{\beta}})$.

Proof. Set

$$\hat{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}.$$

Then by Lemma 5.2 and Cramer's rule,

$$\begin{aligned} \hat{p}^* g^{-1} \hat{p} = \hat{p}^* \frac{adj(g)}{\det(g)} \hat{p} = \frac{\hat{p}^*\hat{p}}{\det(g)} &= \frac{\Sigma^n_1}{1+Q\Sigma^n_1} \\ = -\frac{q}{1-Qq}. \end{aligned}$$

Here we used the fact that $p(z,\overline{z}) + q(w,\overline{w}) = 0$ on M and $p(z,\overline{z}) = \Sigma^n_1$ when

$h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}$. □

Corollary 5.4. *Suppose that the Levi matrix g of the hypersurface M is given by*

$$g_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}} + Qp_\alpha p_{\overline{\beta}},$$

for some $n \times n$ hermitian positive definite matrix $(h_{\alpha\overline{\beta}})$. Then at every point of M,

$$g^{\alpha\overline{\beta}} p_\alpha p_{\overline{\beta}} = -\frac{q}{1 - Qq},$$

where $(g^{\alpha\overline{\beta}})$ is the inverse of $(g_{\alpha\overline{\beta}})$.

Proof. By Sylvester's law of inertia[7], there exists an invertible $n \times n$ matrix $S = (s_{\alpha\beta})$ such that $I_n = S(h_{\alpha\overline{\beta}})S^*$. Hence,

$$\begin{aligned} g &= (h_{\alpha\overline{\beta}} + Qp_\alpha p_{\overline{\beta}}) \\ &= S^{-1}(I_n + Q(p'_\alpha p'_{\overline{\beta}}))(S^{-1})^* \\ &= S^{-1} g' (S^{-1})^*. \end{aligned}$$

Here, $g' = I_n + Q(p'_\alpha p'_{\overline{\beta}})$ and $p'_\alpha = \sum_{\gamma=1}^n s_{\alpha\gamma} p_\gamma$. Let

$$\hat{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

and $\hat{p}' = S\hat{p} = (p'_1, \ldots, p'_n)^t$. By Corollary 5.3,

$$\hat{p}^* g^{-1} \hat{p} = \hat{p}^* (S^* (g')^{-1} S) \hat{p} = (S\hat{p})^* (g')^{-1} (S\hat{p})$$

[7]**Recall**: Two square matrices A and B are $*$-congruent if there is an invertible matrix S such that $SAS^* = B$. Also, the inertia of a Hermitian matrix A is defined to be the tuple (n_+, n_0, n_-), where n_+ is the number of positive eigenvalues of A, n_0 is the number of zero eigenvalues of A, and n_- is the number of negative eigenvalues of A.

Sylvester's law of inertia (Hermitian Version): Let A and B be Hermitian square matrices. Then A and B are $*$-congruent if and only if they have the same inertia.

$$= (\hat{p}')^*(g')^{-1}\hat{p}' = -\frac{q}{1-Qq}.$$

□

Lemma 5.5 (Webster; [17], [19]). *For the real hypersurface M with contact form $\theta = -i\partial r$, the pseudohermitian curvature tensor $R_{\alpha\overline{\beta}\mu\overline{\nu}}$ can be written as*

$$R_{\alpha\overline{\beta}\mu\overline{\nu}} = -A(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\mu\overline{\beta}}g_{\alpha\overline{\nu}}) - Bp_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}}, \tag{5.12}$$

where we set

$$A = -\frac{Q}{1-Qq} \tag{5.13}$$

and

$$\begin{aligned} B = \frac{Q_{w\overline{w}}}{q_w q_{\overline{w}}} &+ 2Q\left(\left(\frac{Q_w}{q_w}\right) + \left(\frac{Q_{\overline{w}}}{q_{\overline{w}}}\right)\right) \\ &+ 3Q^3 + \left|\left(\frac{Q_w}{q_w}\right) + Q^2\right|^2 \cdot \frac{q}{1-Qq}. \end{aligned} \tag{5.14}$$

Proof. Recall that this calculation holds only when $q_{\overline{w}} \neq 0$.

The dual frame $(X, X_\alpha, X_{\overline{\alpha}})$ of $(\theta, \theta^\alpha, \theta^{\overline{\alpha}})$ can be obtained from the differential of a function f on M as follows:

$$\begin{aligned} df &= f_\alpha dz^\alpha + f_w dw + f_{\overline{\beta}}dz^{\overline{\beta}} + f_{\overline{w}}d\overline{w} \\ &= f_\alpha dz^\alpha + f_w dw + f_{\overline{\beta}}dz^{\overline{\beta}} + f_{\overline{w}}d\overline{w} \\ &\quad + \frac{p_\alpha}{q_w}f_w dz^\alpha - \frac{p_\alpha}{q_w}f_w dz^\alpha + \frac{p_{\overline{\beta}}}{q_{\overline{w}}}f_{\overline{w}}dz^{\overline{\beta}} - \frac{p_{\overline{\beta}}}{q_{\overline{w}}}f_{\overline{w}}dz^{\overline{\beta}} \\ &= f_\alpha dz^\alpha - \frac{p_\alpha}{q_w}f_w dz^\alpha + f_{\overline{\beta}}dz^{\overline{\beta}} - \frac{p_{\overline{\beta}}}{q_{\overline{w}}}f_{\overline{w}}dz^{\overline{\beta}} \\ &\quad + \frac{f_w}{q_w}(p_\alpha dz^\alpha + q_w dw) - \frac{f_{\overline{w}}}{q_{\overline{w}}}\left(p_{\overline{\beta}}dz^{\overline{\beta}} + q_{\overline{w}}d\overline{w}\right) \\ &= f_\alpha dz^\alpha - \frac{p_\alpha}{q_w}f_w dz^\alpha + f_{\overline{\beta}}dz^{\overline{\beta}} - \frac{p_{\overline{\beta}}}{q_{\overline{w}}}f_{\overline{w}}dz^{\overline{\beta}} \end{aligned}$$

$$+\frac{i}{q_w}f_w\theta-\frac{i}{q_{\overline{w}}}f_{\overline{w}}\overline{\theta}. \tag{5.15}$$

Since $\theta=\overline{\theta}$ on M, (5.15) becomes

$$\begin{aligned}
df &= f_\alpha dz^\alpha + i\eta^\alpha f_\alpha\theta - \frac{p_\alpha}{q_w}f_w dz^\alpha - \frac{i}{q_w}p_\alpha\eta^\alpha f_w\theta + f_{\overline{\beta}}dz^{\overline{\beta}} - i\eta^{\overline{\beta}}f_{\overline{\beta}}\theta\\
&\quad - \frac{p_{\overline{\beta}}}{q_{\overline{w}}}f_{\overline{w}}dz^{\overline{\beta}} + \frac{i}{q_{\overline{w}}}p_{\overline{\beta}}\eta^{\overline{\beta}}f_{\overline{w}}\theta + \frac{i}{q_w}f_w\theta - \frac{i}{q_{\overline{w}}}f_{\overline{w}}\theta - i\eta^\alpha f_\alpha\theta + i\eta^{\overline{\beta}}f_{\overline{\beta}}\theta\\
&\quad + \frac{i}{q_w}p_\gamma\eta^\gamma f_w\theta - \frac{i}{q_{\overline{w}}}p_{\overline{\gamma}}\eta^{\overline{\gamma}}f_{\overline{w}}\theta\\
&= \left(f_\alpha - \frac{p_\alpha}{q_w}f_w\right)(dz^\alpha + i\eta^\alpha\theta)\left(f_{\overline{\beta}} - \frac{p_{\overline{\beta}}}{q_{\overline{w}}}f_{\overline{w}}\right)\left(dz^{\overline{\beta}} - i\eta^{\overline{\beta}}\theta\right)\\
&\quad + \left(-i\eta^\alpha f_\alpha + i\eta^{\overline{\beta}}f_{\overline{\beta}} + \frac{i}{q_w}(1+p_\gamma\eta^\gamma)f_w - \frac{i}{q_{\overline{w}}}(1+p_{\overline{\gamma}}\eta^{\overline{\gamma}})f_{\overline{w}}\right)\theta\\
&= X_\alpha f\theta^\alpha + X_{\overline{\beta}}f\theta^{\overline{\beta}} + Xf\theta,
\end{aligned} \tag{5.16}$$

where we set

$$X = -i\eta^\alpha\partial_\alpha + i\eta^{\overline{\alpha}}\partial_{\overline{\alpha}} + \frac{i}{q_w}(1+p_\gamma\eta^\gamma)\partial_w - \frac{i}{q_{\overline{w}}}(1+p_{\overline{\gamma}}\eta^{\overline{\gamma}})\partial_{\overline{w}} \tag{5.17}$$

and

$$\begin{aligned}
X_\alpha &= \partial_\alpha - \frac{p_\alpha}{q_w}\partial_w,\\
X_{\overline{\alpha}} &= \overline{X_\alpha}.
\end{aligned} \tag{5.18}$$

The procedure of [17] then shows that

$$\begin{aligned}
R_{\alpha\overline{\beta}\mu\overline{\nu}} &= -X_{\overline{\nu}}X_\mu g_{\alpha\overline{\beta}} + g^{\gamma\overline{\varepsilon}}X_\mu g_{\alpha\overline{\varepsilon}}X_{\overline{\nu}}g_{\gamma\overline{\beta}} + g_{\mu\overline{\nu}}\eta^\gamma\left(X_\alpha g_{\gamma\overline{\beta}} - X_\gamma g_{\alpha\overline{\beta}}\right)\\
&\quad - g_{\mu\overline{\beta}}X_{\overline{\nu}}\eta_\alpha - g_{\alpha\overline{\nu}}X_\mu\eta_{\overline{\beta}} - g_{\mu\overline{\nu}}X_\alpha\eta_{\overline{\beta}} - \eta_\alpha\eta_{\overline{\beta}}g_{\mu\overline{\nu}} - \eta_\gamma\eta^\gamma g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}.
\end{aligned} \tag{5.19}$$

By expanding (5.19), we can express $R_{\alpha\overline{\beta}\mu\overline{\nu}}$ in terms of the p_α, the Levi matrix

$g_{\alpha\overline{\beta}}$, and with coefficients only in terms of w and $\overline{w}$.

Recall that, by definition of the Levi matrix, we have that

$$p_{\alpha\overline{\beta}} = g_{\alpha\overline{\beta}} - Qp_\alpha p_{\overline{\beta}}.$$

By Corollary 5.4,

$$C = g^{\gamma\overline{\varepsilon}} p_\gamma p_{\overline{\varepsilon}} = -\frac{q}{1 - Qq}. \tag{5.20}$$

Here, $(g^{\gamma\overline{\varepsilon}})$ is the inverse of $(g_{\gamma\overline{\varepsilon}})$.

Since $p_{\alpha\overline{\beta}} = h_{\alpha\overline{\beta}}$ is constant, and by definition of the p_α and Q, we have that

$$\begin{aligned} X_\mu g_{\alpha\overline{\beta}} &= \left(\partial_\mu - \frac{p_\mu}{q_w}\partial_w\right)\left(p_{\alpha\overline{\beta}} + Qp_\alpha p_{\overline{\beta}}\right) \\ &= Qp_\alpha p_{\mu\overline{\beta}} - \frac{Q_w}{q_w} p_\alpha p_{\overline{\beta}} p_\mu. \end{aligned}$$

Hence, (5.19) can be expanded thusly:

$$\begin{aligned} -X_{\overline{\nu}} X_\mu g_{\alpha\overline{\beta}} &= -\left(\partial_{\overline{\nu}} - \frac{p_{\overline{\nu}}}{q_{\overline{w}}}\partial_{\overline{w}}\right)\left(Qp_\alpha p_{\mu\overline{\beta}} - \frac{Q_w}{q_w} p_\alpha p_{\overline{\beta}} p_\mu\right) \\ &= -Qg_{\alpha\overline{\nu}} g_{\mu\overline{\beta}} + Q^2 g_{\alpha\overline{\nu}} p_\mu p_{\overline{\beta}} + Q^2 g_{\mu\overline{\beta}} p_\alpha p_{\overline{\nu}} - Q^3 p_\alpha p_{\overline{\beta}} p_\mu p_{\overline{\nu}} \\ &\quad + \frac{Q_w}{q_w} g_{\alpha\overline{\nu}} p_{\overline{\beta}} p_\mu - Q\frac{Q_w}{q_w} p_\alpha p_{\overline{\beta}} p_\mu p_{\overline{\nu}} + \frac{Q_w}{q_w} g_{\mu\overline{\nu}} p_\alpha p_{\overline{\beta}} \\ &\quad + \frac{Q_{\overline{w}}}{q_{\overline{w}}} g_{\mu\overline{\beta}} p_\alpha p_{\overline{\nu}} - Q\frac{Q_{\overline{w}}}{q_{\overline{w}}} p_\alpha p_{\overline{\beta}} p_\mu p_{\overline{\nu}} - \frac{Q_{w\overline{w}}}{q_w q_{\overline{w}}} p_\alpha p_{\overline{\beta}} p_\mu p_{\overline{\nu}}; \end{aligned} \tag{5.21}$$

$$\begin{aligned} g^{\gamma\overline{\varepsilon}} X_\mu g_{\alpha\overline{\varepsilon}} X_{\overline{\nu}} g_{\gamma\overline{\beta}} &= g^{\gamma\overline{\varepsilon}}\left(Qp_\alpha p_{\mu\overline{\varepsilon}} - \frac{p_\mu}{q_w} Q_w p_\alpha p_{\overline{\varepsilon}}\right)\left(Qp_{\gamma\overline{\nu}} p_{\overline{\beta}} - \frac{p_{\overline{\nu}}}{q_{\overline{w}}} Q_{\overline{w}} p_\gamma p_{\overline{\beta}}\right) \\ &= Q^2 g_{\mu\overline{\nu}} p_\alpha p_{\overline{\beta}} - 2Q^3 p_\alpha p_{\overline{\beta}} p_\mu p_{\overline{\nu}} - Q\frac{Q_{\overline{w}}}{q_{\overline{w}}} p_\alpha p_{\overline{\beta}} p_\mu p_{\overline{\nu}} \\ &\quad - Q\frac{Q_w}{q_w} p_\alpha p_{\overline{\beta}} p_\mu p_{\overline{\nu}} + \left|\frac{Q_w}{q_w} + Q^2\right|^2 p_\alpha p_{\overline{\beta}} p_\mu p_{\overline{\nu}} C; \end{aligned} \tag{5.22}$$

$$g_{\mu\overline{\nu}}\eta^{\gamma}\left(X_{\alpha}g_{\gamma\overline{\beta}}-X_{\gamma}g_{\alpha\overline{\beta}}\right)=g_{\mu\overline{\nu}}\eta^{\gamma}\left(Qg_{\alpha\overline{\beta}}p_{\gamma}-Qg_{\gamma\overline{\beta}}p_{\alpha}\right)$$
$$=-g_{\mu\overline{\nu}}g^{\gamma\overline{\varepsilon}}Qp_{\overline{\varepsilon}}\left(Qg_{\alpha\overline{\beta}}p_{\gamma}-Qg_{\gamma\overline{\beta}}p_{\alpha}\right)$$
$$=-Q^{2}g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}}C+Q^{2}g_{\mu\overline{\nu}}p_{\alpha}p_{\overline{\beta}}; \tag{5.23}$$

$$-g_{\mu\overline{\beta}}X_{\overline{\nu}}\eta_{\alpha}=-g_{\mu\overline{\beta}}\left(-Qp_{\alpha\overline{\nu}}+\frac{p_{\overline{\nu}}}{q_{\overline{w}}}Q_{\overline{w}}p_{\alpha}\right)$$
$$=-g_{\mu\overline{\beta}}\left(-Qg_{\alpha\overline{\nu}}+Q^{2}p_{\alpha}p_{\overline{\nu}}+\frac{Q_{\overline{w}}}{q_{\overline{w}}}p_{\alpha}p_{\overline{\nu}}\right)$$
$$=Qg_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}-Q^{2}g_{\mu\overline{\beta}}p_{\alpha}p_{\overline{\nu}}-\frac{Q_{\overline{w}}}{q_{\overline{w}}}g_{\mu\overline{\beta}}p_{\alpha}p_{\overline{\nu}}; \tag{5.24}$$

$$-g_{\alpha\overline{\nu}}X_{\mu}\eta_{\overline{\beta}}=Qg_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}-Q^{2}g_{\alpha\overline{\nu}}p_{\mu}p_{\overline{\beta}}-\frac{Q_{w}}{q_{w}}g_{\alpha\overline{\nu}}p_{\overline{\beta}}p_{\mu}; \tag{5.25}$$

$$-g_{\mu\overline{\nu}}X_{\alpha}\eta_{\overline{\beta}}=Qg_{\alpha\overline{\beta}}g_{\mu\overline{\nu}}-Q^{2}g_{\mu\overline{\nu}}p_{\alpha}p_{\overline{\beta}}-\frac{Q_{w}}{q_{w}}g_{\mu\overline{\nu}}p_{\alpha}p_{\overline{\beta}}; \tag{5.26}$$

$$-\eta_{\alpha}\eta_{\overline{\beta}}g_{\mu\overline{\nu}}=-Q^{2}g_{\mu\overline{\nu}}p_{\alpha}p_{\overline{\beta}}; \tag{5.27}$$

$$-\eta_{\gamma}\eta^{\gamma}g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}=-Q^{2}g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}C. \tag{5.28}$$

By combining (5.21)-(5.28) and replacing the C's with (5.20), we obtain the following formula for the pseudohermitian curvature tensor:

$$\begin{aligned}
R_{\alpha\overline{\beta}\mu\overline{\nu}} &= -2Q\frac{Q_w}{q_w}p_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}} - 2Q\frac{Q_{\overline{w}}}{q_{\overline{w}}}p_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}} - \frac{Q_{w\overline{w}}}{q_w q_{\overline{w}}}p_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}} \\
&\quad - 3Q^3 p_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}} + \left|\frac{Q_w}{q_w} + Q^2\right|^2 p_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}}C - Q^2 g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}}C \\
&\quad + Q g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}} + Q g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} - Q^2 g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}C \\
&= \left(\frac{Q^2 q}{1-Qq} + Q\right)\left(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}\right) - \left(\frac{Q_{w\overline{w}}}{q_w q_{\overline{w}}} + 2Q\left(\frac{Q_w}{q_w} + \frac{Q_{\overline{w}}}{q_{\overline{w}}}\right)\right. \\
&\quad \left. +3Q^3 + \left|\frac{Q_w}{q_w} + Q^2\right|^2 \cdot \frac{q}{1-Qq}\right) p_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}} + \\
&= -\left(-\frac{Q}{1-Qq}\right)\left(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}\right) - \left(\frac{Q_{w\overline{w}}}{q_w q_{\overline{w}}} + 2Q\left(\frac{Q_w}{q_w} + \frac{Q_{\overline{w}}}{q_{\overline{w}}}\right)\right. \\
&\quad \left. +3Q^3 + \left|\frac{Q_w}{q_w} + Q^2\right|^2 \cdot \frac{q}{1-Qq}\right) p_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}} \\
&= -A\left(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}\right) - B p_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}},
\end{aligned} \tag{5.29}$$

where we set

$$A = -\frac{Q}{1-Qq}$$

and

$$\begin{aligned}
B = \frac{Q_{w\overline{w}}}{q_w q_{\overline{w}}} &+ 2Q\left(\left(\frac{Q_w}{q_w}\right) + \left(\frac{Q_{\overline{w}}}{q_{\overline{w}}}\right)\right) \\
&+ 3Q^3 + \left|\left(\frac{Q_w}{q_w}\right) + Q^2\right|^2 \frac{q}{1-Qq}.
\end{aligned}$$

□

Corollary 5.6 (Webster; [17], [19])**.** *For the real hypersurface M, the pseudohermitian Ricci curvature tensor $R_{\mu\overline{\nu}}$ and the pseudohermitian scalar curvature* *tensor R can be written, respectively, as*

$$R_{\mu\overline{\nu}} = -(n+1)Ag_{\mu\overline{\nu}} + Bp_{\mu}p_{\overline{\nu}}\frac{q}{1-Qq} \tag{5.30}$$

and

$$R = -n(n+1)A - B\left(\frac{q}{1-Qq}\right)^2. \tag{5.31}$$

Proof. By definition (2.21), in addition to (5.20) and (5.12), we have that

$$\begin{aligned}
R_{\mu\overline{\nu}} = R_{\alpha}{}^{\alpha}{}_{\mu\overline{\nu}} &= g^{\alpha\overline{\alpha}} R_{\alpha\overline{\alpha}\mu\overline{\nu}} \\
&= g^{\alpha\overline{\alpha}}\left(-A(g_{\alpha\overline{\alpha}}g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}}g_{\mu\overline{\alpha}}) - Bp_{\alpha}p_{\overline{\alpha}}p_{\mu}p_{\overline{\nu}}\right) \\
&= -A\delta^{\alpha}{}_{\alpha}g_{\mu\overline{\nu}} - A\delta^{\overline{\alpha}}{}_{\overline{\nu}}g_{\mu\overline{\alpha}} + Bp_{\mu}p_{\overline{\nu}}\frac{q}{1-Qq} \\
&= -(n+1)Ag_{\mu\overline{\nu}} + Bp_{\mu}p_{\overline{\nu}}\frac{q}{1-Qq}
\end{aligned}$$

and

$$\begin{aligned}
R = R_{\mu}{}^{\mu} &= g^{\mu\overline{\mu}}R_{\mu\overline{\mu}} \\
&= g^{\mu\overline{\mu}}\left(-(n+1)Ag_{\mu\overline{\mu}} + Bp_{\mu}p_{\overline{\mu}}\frac{q}{1-Qq}\right) \\
&= -(n+1)A\delta^{\mu}{}_{\mu} - B\left(\frac{q}{1-Qq}\right)^2 \\
&= -n(n+1)A - B\left(\frac{q}{1-Qq}\right)^2.
\end{aligned}$$

□

Corollary 5.7 (Webster; [19]). *For the real hypersurface M, the pseudoconformal curvature tensor $S_{\alpha\overline{\beta}\mu\overline{\nu}}$ can be written as*

$$S_{\alpha\overline{\beta}\mu\overline{\nu}} = -\frac{Bq^2}{(n+1)(n+2)(1-Qq)^2}\left(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\mu\overline{\beta}}g_{\alpha\overline{\nu}}\right) - Bp_{\alpha}p_{\overline{\beta}}p_{\mu}p_{\overline{\nu}}$$

$$-\frac{Bq}{(n+2)(1-Qq)}\left(g_{\alpha\overline{\beta}}p_\mu p_{\overline{\nu}} + g_{\mu\overline{\beta}}p_\alpha p_{\overline{\nu}} + g_{\alpha\overline{\nu}}p_\mu p_{\overline{\beta}} + g_{\mu\overline{\nu}}p_\alpha p_{\overline{\beta}}\right). \tag{5.32}$$

Proof. Recall from (2.20) that $S_{\alpha\overline{\beta}\mu\overline{\nu}}$ can be written as

$$S_{\alpha\bar{\beta}\mu\bar{\nu}} = R_{\alpha\bar{\beta}\mu\bar{\nu}} - \frac{R_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + R_{\mu\bar{\beta}}g_{\alpha\bar{\nu}} + R_{\alpha\bar{\nu}}g_{\mu\bar{\beta}} + R_{\mu\bar{\nu}}g_{\alpha\bar{\beta}}}{n+2} + \frac{R(g_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + g_{\alpha\bar{\nu}}g_{\mu\bar{\beta}})}{(n+1)(n+2)}.$$

By Lemma 5.5 and Corollary 5.6, we have that

$$R_{\mu\overline{\nu}}g_{\alpha\overline{\beta}} = -(n+1)Ag_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + Bg_{\alpha\overline{\beta}}p_\mu p_{\overline{\nu}}\frac{q}{1-Qq}.$$

Hence,

$$\begin{aligned}
S_{\alpha\overline{\beta}\mu\overline{\nu}} &= -A(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}) - Bp_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}} \\
&\quad + \frac{2(n+1)A}{(n+2)}(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}) - \frac{n(n+1)A}{(n+1)(n+2)}(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}) \\
&\quad - \frac{Bq}{(n+2)(1-Qq)}\left(g_{\alpha\overline{\beta}}p_\mu p_{\overline{\nu}} + g_{\mu\overline{\beta}}p_\alpha p_{\overline{\nu}} + g_{\alpha\overline{\nu}}p_\mu p_{\overline{\beta}} + g_{\mu\overline{\nu}}p_\alpha p_{\overline{\beta}}\right) \\
&\quad - \frac{B}{(n+1)(n+2)}\left(\frac{q}{1-Qq}\right)^2(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}) \\
&= -\frac{Bq^2}{(n+1)(n+2)(1-Qq)^2}(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}) - Bp_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}} \\
&\quad - \frac{Bq}{(n+2)(1-Qq)}\left(g_{\alpha\overline{\beta}}p_\mu p_{\overline{\nu}} + g_{\mu\overline{\beta}}p_\alpha p_{\overline{\nu}} + g_{\alpha\overline{\nu}}p_\mu p_{\overline{\beta}} + g_{\mu\overline{\nu}}p_\alpha p_{\overline{\beta}}\right) \\
&\quad + \left(-A + \frac{2(n+1)A}{n+2} - \frac{nA}{n+2}\right)(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}) \\
&= -\frac{Bq^2}{(n+1)(n+2)(1-Qq)^2}(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\alpha\overline{\nu}}g_{\mu\overline{\beta}}) - Bp_\alpha p_{\overline{\beta}}p_\mu p_{\overline{\nu}} \\
&\quad - \frac{Bq}{(n+2)(1-Qq)}\left(g_{\alpha\overline{\beta}}p_\mu p_{\overline{\nu}} + g_{\mu\overline{\beta}}p_\alpha p_{\overline{\nu}} + g_{\alpha\overline{\nu}}p_\mu p_{\overline{\beta}} + g_{\mu\overline{\nu}}p_\alpha p_{\overline{\beta}}\right).
\end{aligned}$$

□

5.4 Gaussian Curvature K

Now we will compute the Gaussian curvature of D_0 with respect to the hermitian metric h:

Lemma 5.8 (Webster, [19]). *The Gaussian curvature K of the metric $ds^2 = hdwd\overline{w}$ on D_0 is given by*

$$K = -2 + \frac{q^3}{k^3}\left(kq_{ww\overline{w}\overline{w}} + qq_{ww\overline{w}}q_{w\overline{w}\overline{w}} - q_w q_{\overline{w}\overline{w}}q_{ww\overline{w}}\right.$$
$$\left. - q_{\overline{w}}q_{ww}q_{w\overline{w}\overline{w}} + q_{ww}q_{w\overline{w}}q_{\overline{w}\overline{w}}\right), \tag{5.33}$$

where $k = q_w q_{\overline{w}} - qq_{w\overline{w}}$.

Proof. By definition,

$$K = \frac{K_0}{h},$$

where $\partial_{\overline{w}}\partial_w \log(h) = K_0 dw \wedge d\overline{w}$. Recall that the hermitian metric h on D_0 was defined by

$$h = \frac{q_w q_{\overline{w}} - qq_{w\overline{w}}}{q^2} = \frac{k}{q^2}.$$

By definition of the partial derivatives and direct calculation,

$$k_w = q_{\overline{w}}q_{ww} - qq_{ww\overline{w}},$$

$$k_{\overline{w}} = q_w q_{\overline{w}\overline{w}} - qq_{w\overline{w}\overline{w}},$$

$$k_{w\overline{w}} = q_{ww}q_{\overline{w}\overline{w}} - qq_{ww\overline{w}\overline{w}}.$$

Hence, we have that

$$\partial_{\overline{w}}\partial_w \log(h) = \partial_{\overline{w}}\left(\frac{k_w}{k}dw - \frac{2q_w}{q}dw\right)$$
$$= \left(kk_{w\overline{w}}q^2 - k_w k_{\overline{w}}q^2 - 2k^2 qq_{w\overline{w}} + 2k^2 q_w q_{\overline{w}}\right)\frac{d\overline{w}\wedge dw}{k^2q^2}$$

$$
\begin{aligned}
&= \left(-q^3 q_w q_{\overline{w}} q_{ww\overline{ww}} - q^3 q_{ww} q_{w\overline{w}} q_{\overline{ww}} + q^4 q_{w\overline{w}} q_{ww\overline{ww}}\right.\\
&\quad + q^3 q_{\overline{w}} q_{ww} q_{w\overline{ww}} + q^3 q_w q_{\overline{ww}} q_{ww\overline{w}} - q^4 q_{ww\overline{w}} q_{w\overline{ww}}\\
&\quad + 6q^2 q_w q_{\overline{w}} {q_{w\overline{w}}}^2 - 2q^3 {q_{w\overline{w}}}^3 + 2{q_w}^3 {q_{\overline{w}}}^3\\
&\quad \left. -6q{q_w}^2 {q_{\overline{w}}}^2 q_{w\overline{w}}\right) \frac{d\overline{w} \wedge dw}{k^2 q^2}\\
&= (-kq_{ww\overline{ww}} - q_{ww} q_{w\overline{w}} q_{\overline{ww}} + q_{\overline{w}} q_{ww} q_{w\overline{ww}}\\
&\quad + q_w q_{\overline{ww}} q_{ww\overline{w}} - q q_{ww\overline{w}} q_{w\overline{ww}}) \frac{q}{k^2} d\overline{w} \wedge dw\\
&\quad + \frac{2k}{q^2} d\overline{w} \wedge dw.
\end{aligned}
$$

Now we set

$$
\begin{aligned}
K_0 = (&kq_{ww\overline{ww}} + q_{ww} q_{w\overline{w}} q_{\overline{ww}} - q_{\overline{w}} q_{ww} q_{w\overline{ww}}\\
&- q_w q_{\overline{ww}} q_{ww\overline{w}} + q q_{ww\overline{w}} q_{w\overline{ww}}) \frac{q}{k^2} - \frac{2k}{q^2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
K &= \frac{K_0}{h} = K_0 \frac{q^2}{k}\\
&= -2 + \frac{q^3}{k^3} (kq_{ww\overline{ww}} + q_{ww} q_{w\overline{w}} q_{\overline{ww}} - q_{\overline{w}} q_{ww} q_{w\overline{ww}}\\
&\quad - q_w q_{\overline{ww}} q_{ww\overline{w}} + q q_{ww\overline{w}} q_{w\overline{ww}}).
\end{aligned}
$$

□

Corollary 5.9 (Webster, [19])**.**

$$
B = \frac{(K+2)k^2}{q^3 (q_w q_{\overline{w}})^2}. \tag{5.34}
$$

Proof. Recall that B was defined earlier to be

$$B = \frac{Q_{w\overline{w}}}{q_w q_{\overline{w}}} + 2Q\left(\frac{Q_w}{q_w} + \frac{Q_{\overline{w}}}{q_{\overline{w}}}\right) + 3Q^3 + \left|\frac{Q_w}{q_w} + Q^2\right|^2 \cdot \frac{q}{1 - Qq},$$

where

$$Q = \frac{q_{w\overline{w}}}{q_w q_{\overline{w}}}.$$

By definition, we have that

$$\begin{aligned} Q_w &= \frac{q_w q_{\overline{w}} q_{ww\overline{w}\overline{w}} - q_{\overline{w}} q_{ww} q_{w\overline{w}} - q_w q_{w\overline{w}}{}^2}{q_w{}^2 q_{\overline{w}}{}^2}, \\ Q_{\overline{w}} &= \frac{q_w q_{\overline{w}} q_{ww\overline{w}\overline{w}} - q_{\overline{w}} q_{w\overline{w}}{}^2 - q_w q_{w\overline{w}} q_{\overline{w}\overline{w}}}{q_w{}^2 q_{\overline{w}}{}^2}, \end{aligned}$$

and

$$\begin{aligned} Q_{w\overline{w}} = \big(&q_w{}^3 q_{\overline{w}}{}^3 q_{ww\overline{w}\overline{w}} - 2q_w{}^2 q_{\overline{w}}{}^3 q_{w\overline{w}} q_{ww\overline{w}} - q_w{}^3 q_{\overline{w}}{}^2 q_{\overline{w}\overline{w}} q_{ww\overline{w}} \\ &- q_w{}^2 q_{\overline{w}}{}^3 q_{ww} q_{w\overline{w}\overline{w}} + 2q_w q_{\overline{w}}{}^3 q_{ww} q_{w\overline{w}}{}^2 + q_w{}^2 q_{\overline{w}}{}^2 q_{ww} q_{w\overline{w}} q_{\overline{w}\overline{w}} \\ &-2q_w{}^3 q_{\overline{w}}{}^2 q_{w\overline{w}} q_{w\overline{w}\overline{w}} + q_w{}^2 q_{\overline{w}}{}^2 q_{w\overline{w}}{}^3 + 2q_w{}^3 q_{\overline{w}} q_{w\overline{w}}{}^2 q_{\overline{w}\overline{w}}\big) \frac{1}{q_w{}^4 q_{\overline{w}}{}^4}. \end{aligned}$$

Hence, we can expand each term of B as follows:

$$\begin{aligned} \frac{Q_{w\overline{w}}}{q_w q_{\overline{w}}} = \big(&q_w{}^3 q_{\overline{w}}{}^3 q_{ww\overline{w}\overline{w}} - 2q_w{}^2 q_{\overline{w}}{}^3 q_{w\overline{w}} q_{ww\overline{w}} - q_w{}^3 q_{\overline{w}}{}^2 q_{\overline{w}\overline{w}} q_{ww\overline{w}} \\ &- q_w{}^2 q_{\overline{w}}{}^3 q_{ww} q_{w\overline{w}\overline{w}} + 2q_w q_{\overline{w}}{}^3 q_{ww} q_{w\overline{w}}{}^2 + q_w{}^2 q_{\overline{w}}{}^2 q_{ww} q_{w\overline{w}} q_{\overline{w}\overline{w}} \\ &-2q_w{}^3 q_{\overline{w}}{}^2 q_{w\overline{w}} q_{w\overline{w}\overline{w}} + q_w{}^2 q_{\overline{w}}{}^2 q_{w\overline{w}}{}^3 + 2q_w{}^3 q_{\overline{w}} q_{w\overline{w}}{}^2 q_{\overline{w}\overline{w}}\big) \frac{1}{q_w{}^5 q_{\overline{w}}{}^5}; \end{aligned} \tag{5.35}$$

$$2Q\left(\frac{Q_w}{q_w} + \frac{Q_{\overline{w}}}{q_{\overline{w}}}\right) = \big(2q_w q_{\overline{w}}{}^2 q_{w\overline{w}} q_{ww\overline{w}} - 2q_{\overline{w}}{}^2 q_{ww} q_{w\overline{w}}{}^2 - 4q_w q_{\overline{w}} q_{w\overline{w}}{}^3$$

$$+2{q_w}^2 q_{\overline{w}} q_{w\overline{w}} q_{w\overline{w}\overline{w}} - 2{q_w}^2 {q_{w\overline{w}}}^2 q_{\overline{w}\overline{w}}\Big) \frac{1}{{q_w}^4 {q_{\overline{w}}}^4}; \tag{5.36}$$

$$3Q^3 = \frac{3q_{w\overline{w}}^3}{q_w^3 q_{\overline{w}}^3}; \tag{5.37}$$

$$\begin{aligned}\left|\frac{Q_w}{q_w} + Q^2\right|^2 \cdot \frac{q}{1-Qq} = \Big(&{q_w}^2 {q_{\overline{w}}}^2 q_{ww\overline{w}} q_{w\overline{w}\overline{w}} - {q_w}^2 q_{\overline{w}} q_{w\overline{w}} q_{\overline{w}\overline{w}} q_{ww\overline{w}} \\ &- q_w {q_{\overline{w}}}^2 q_{ww} q_{w\overline{w}} q_{w\overline{w}\overline{w}} + q_w q_{\overline{w}} q_{ww} {q_{w\overline{w}}}^2 q_{\overline{w}\overline{w}}\Big) \frac{q}{k{q_w}^4 {q_{\overline{w}}}^4}.\end{aligned} \tag{5.38}$$

Adding (5.35)-(5.38) together gives us

$$\begin{aligned}B &= \frac{Q_{w\overline{w}}}{q_w q_{\overline{w}}} + 2Q\left(\frac{Q_w}{q_w} + \frac{Q_{\overline{w}}}{q_{\overline{w}}}\right) + 3Q^3 + \left|\frac{Q_w}{q_w} + Q^2\right|^2 \cdot \frac{q}{1-Qq} \\
&= \Big(k{q_w}^3 {q_{\overline{w}}}^3 q_{ww\overline{w}\overline{w}} - k{q_w}^3 {q_{\overline{w}}}^2 q_{\overline{w}\overline{w}} q_{ww\overline{w}} - k{q_w}^2 {q_{\overline{w}}}^3 q_{ww} q_{w\overline{w}\overline{w}} \\
&\quad + k{q_w}^2 {q_{\overline{w}}}^2 q_{ww} q_{w\overline{w}} q_{\overline{w}\overline{w}} + q{q_w}^3 {q_{\overline{w}}}^3 q_{ww\overline{w}} q_{w\overline{w}\overline{w}} - q{q_w}^3 {q_{\overline{w}}}^2 q_{w\overline{w}} q_{\overline{w}\overline{w}} q_{ww\overline{w}} \\
&\quad - q{q_w}^2 {q_{\overline{w}}}^3 q_{ww} q_{w\overline{w}} q_{w\overline{w}\overline{w}} + q{q_w}^2 {q_{\overline{w}}}^2 q_{ww} {q_{w\overline{w}}}^2 q_{\overline{w}\overline{w}}\Big) \frac{1}{k{q_w}^5 {q_{\overline{w}}}^5} \\
&= \Big(k{q_w}^3 {q_{\overline{w}}}^3 q_{ww\overline{w}\overline{w}} - {q_w}^4 {q_{\overline{w}}}^3 q_{\overline{w}\overline{w}} q_{ww\overline{w}} - {q_w}^3 {q_{\overline{w}}}^4 q_{ww} q_{w\overline{w}\overline{w}} \\
&\quad + {q_w}^3 {q_{\overline{w}}}^3 q_{ww} q_{w\overline{w}} q_{\overline{w}\overline{w}} + q{q_w}^3 {q_{\overline{w}}}^3 q_{ww\overline{w}} q_{w\overline{w}\overline{w}}\Big) \frac{1}{k{q_w}^5 {q_{\overline{w}}}^5} \\
&= \left(\frac{(K+2){q_w}^3 {q_{\overline{w}}}^3 k^3}{q^3}\right) \frac{1}{k{q_w}^5 {q_{\overline{w}}}^5} \\
&= \frac{(K+2)k^2}{q^3 {q_w}^2 {q_{\overline{w}}}^2}.\end{aligned}$$

□

5.5 Proof of Theorem 1.2

With the above formulas, we can now prove Theorem 1.2:

Proof of Theorem 1.2. We prove the theorem by way of contradiction. Let $w \in$

D_0 and $(z,w) \in M$. Let us assume that $f : M \to \mathbb{S}^{\hat{n}}$ is a smooth CR embedding of M into $\mathbb{S}^{\hat{n}}$ such that $1 < n \leq \hat{n} < 2n-1$. We give D_0 the hermitian metric

$$h = \frac{q_w q_{\overline{w}} - q q_{w\overline{w}}}{q^2}.$$

and assume that the Gaussian curvature K of the metric h satisfies $K > -2$ at w.

Denote by $(\omega_\alpha{}^a{}_\mu)$ to be the second fundamental form matrix of f relative to an admissible coframe $(\hat{\theta}, \hat{\theta}^A, \hat{\theta}^{\overline{A}})$ on $\mathbb{S}^{\hat{n}}$ near $f(M)$ which is adapted to $(\theta, \theta^\alpha, \theta^{\overline{\alpha}})$ on M. We follow [5] by identifying $f(M) \cong M$ in $\mathbb{S}^{\hat{n}}$. In addition, we identify $T^{1,0}_{f(z,w)}\mathbb{S}^{\hat{n}}/f_* T^{1,0}_{(z,w)}M \cong (T^{1,0}_{(z,w)}M)^\perp$ with respect to the Levi form relative to $\hat{\theta}$ and we consider the second fundamental form as a $\mathbb{C}$-bilinear map

$$T^{1,0}_{(z,w)}M \times T^{1,0}_{(z,w)}M \to T^{1,0}_{f(z,w)}\mathbb{S}^{\hat{n}}/f_* T^{1,0}_{(z,w)}M.$$

By the pseudoconformal Gauss equation (4.7), we have that

$$[\hat{S}_{\alpha\overline{\beta}\mu\overline{\nu}}] = S_{\alpha\overline{\beta}\mu\overline{\nu}} + [g_{a\bar{b}}\omega_\alpha{}^a{}_\mu \omega_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}}], \tag{5.39}$$

where $\hat{S}_{\alpha\overline{\beta}\mu\overline{\nu}}$ denotes the pseudoconformal curvature tensor of $\mathbb{S}^{\hat{n}}$ restricted to M. The square bracket notation, $[\cdot]$, will again denote the traceless component of a tensor. Since $\mathbb{S}^{\hat{n}}$ is a sphere, by the results of Chern-Moser [2], $\hat{S}_{\alpha\overline{\beta}\mu\overline{\nu}} \equiv 0$, implying

$$S_{\alpha\overline{\beta}\mu\overline{\nu}} + [g_{a\bar{b}}\omega_\alpha{}^a{}_\mu \omega_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}}] = 0 \tag{5.40}$$

By the definition of the traceless component,

$$S_{\alpha\overline{\beta}\mu\overline{\nu}} + g_{a\bar{b}}\omega_\alpha{}^a{}_\mu \omega_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}} = H_{\alpha\bar{\beta}} g_{\mu\bar{\nu}} + \hat{H}_{\mu\bar{\beta}} g_{\alpha\bar{\nu}} + \tilde{H}_{\alpha\bar{\nu}} g_{\mu\bar{\beta}} + \check{H}_{\mu\bar{\nu}} g_{\alpha\bar{\beta}}, \tag{5.41}$$

for hermitian matrices $(H_{\alpha\bar{\beta}})$, $(\hat{H}_{\mu\bar{\beta}})$, $(\tilde{H}_{\alpha\bar{\nu}})$, and $(\check{H}_{\mu\bar{\nu}})$. Replacing $S_{\alpha\overline{\beta}\mu\overline{\nu}}$ with (5.32), we obtain

$$g_{a\bar{b}}\omega_{\alpha}{}^{a}{}_{\mu}\omega_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}} - Bp_{\alpha}p_{\overline{\beta}}p_{\mu}p_{\overline{\nu}} = T_{\alpha\overline{\beta}\mu\overline{\nu}}, \tag{5.42}$$

where we set

$$\begin{aligned} T_{\alpha\overline{\beta}\mu\overline{\nu}} = {} & H_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + \hat{H}_{\mu\bar{\beta}}g_{\alpha\bar{\nu}} + \tilde{H}_{\alpha\bar{\nu}}g_{\mu\bar{\beta}} + \check{H}_{\mu\bar{\nu}}g_{\alpha\bar{\beta}} \\ & + \frac{Bq^2}{(n+1)(n+2)(1-Qq)^2}\left(g_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + g_{\mu\overline{\beta}}g_{\alpha\overline{\nu}}\right) \\ & + \frac{Bq}{(n+2)(1-Qq)}\left(g_{\alpha\overline{\beta}}p_{\mu}p_{\overline{\nu}} + g_{\mu\overline{\beta}}p_{\alpha}p_{\overline{\nu}} + g_{\alpha\overline{\nu}}p_{\mu}p_{\overline{\beta}} + g_{\mu\overline{\nu}}p_{\alpha}p_{\overline{\beta}}\right). \end{aligned} \tag{5.43}$$

So, for all (non-zero) $X = X^{\alpha}\frac{\partial}{\partial z^{\alpha}} \in T^{1,0}_{(z,w)}M$,

$$\begin{aligned} T_{\alpha\overline{\beta}\mu\overline{\nu}}X^{\alpha}X^{\overline{\beta}}X^{\mu}X^{\overline{\nu}} = {} & g_{a\bar{b}}\left(\omega_{\alpha}{}^{a}{}_{\mu}X^{\alpha}X^{\mu}\right)\left(\omega_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}}X^{\overline{\beta}}X^{\overline{\nu}}\right) \\ & - Bp_{\alpha}p_{\overline{\beta}}p_{\mu}p_{\overline{\nu}}X^{\alpha}X^{\overline{\beta}}X^{\mu}X^{\overline{\nu}}. \end{aligned} \tag{5.44}$$

Note that the left-hand side is of the form $H(X,\overline{X})|X|^2$. Hence, by the restrictions on the dimensions n and $\hat{n}$, Corollary 4.4 implies

$$g_{a\bar{b}}\left(\omega_{\alpha}{}^{a}{}_{\mu}X^{\alpha}X^{\mu}\right)\left(\omega_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}}X^{\overline{\beta}}X^{\overline{\nu}}\right) - Bp_{\alpha}p_{\overline{\beta}}p_{\mu}p_{\overline{\nu}}X^{\alpha}X^{\overline{\beta}}X^{\mu}X^{\overline{\nu}} \equiv 0. \tag{5.45}$$

By definiteness,

$$g_{a\bar{b}}\left(\omega_{\alpha}{}^{a}{}_{\mu}X^{\alpha}X^{\mu}\right)\left(\omega_{\bar{\beta}}{}^{\bar{b}}{}_{\bar{\nu}}X^{\overline{\beta}}X^{\overline{\nu}}\right) > 0. \tag{5.46}$$

Since $w \in D_0$, we have that $q < 0$. Therefore, by (5.34), $K > -2$ implies that $B < 0$. Thus,

$$-Bp_{\alpha}p_{\overline{\beta}}p_{\mu}p_{\overline{\nu}}X^{\alpha}X^{\overline{\beta}}X^{\mu}X^{\overline{\nu}} > 0. \tag{5.47}$$

(5.45) combined with (5.46) and (5.47) is a contradiction. Hence, the assumption is false and no smooth CR embedding $f : M \to \mathbb{S}^{\hat{n}}$ exists. $\square$

6 Further Results

The techniques used in the previous section can be used to prove a Kähler version of Theorem 1.2.

Recall that a holomorphic mapping $f : (X, g_X) \to (Y, g_Y)$ between hermitian manifolds is called *conformal* if $f^*g_Y = hg_X$, where h is some positive function on X. Note that, when X and Y are both Kähler and $\dim X > 1$, the conformal coefficient h is a positive constant (which we will assume so in this case).

A tensor $T_{\alpha\overline{\beta}\mu\overline{\nu}}$ over a complex manifold is called *pseudoconformally flat* if, in any holomorphic chart, we have

$$T_{\alpha\overline{\beta}\mu\overline{\nu}} = H_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + \hat{H}_{\mu\overline{\beta}}g_{\alpha\overline{\nu}} + \tilde{H}_{\alpha\overline{\nu}}g_{\mu\overline{\beta}} + \check{H}_{\mu\overline{\nu}}g_{\alpha\overline{\beta}},$$

where $(H_{\alpha\overline{\beta}})$, $(\hat{H}_{\mu\overline{\beta}})$, $(\tilde{H}_{\alpha\overline{\nu}})$, and $(\check{H}_{\mu\overline{\nu}})$ are smoothly-varying hermitian matrices and $(g_{\alpha\overline{\beta}})$ is the local representation of the hermitian metric over the chart.

Proof of Theorem 1.4. Pick an arbitrary point $p \in X$ in a holomorphic coordinate neighborhood (U, ϕ) of X, where the coordinates are given by

$$\phi = (z^1, \ldots, z^n).$$

Let (V, ψ) be the holomorphic coordinate neighborhood of Y such that

$$f(U) \subset V,\ f(p) \in V, \text{ and } \psi = (w^1, \ldots, w^{\hat{n}}).$$

By the Gauss-Codazzi equation[8],

$$\hat{R}_{\alpha\overline{\beta}\mu\overline{\nu}} - R_{\alpha\overline{\beta}\mu\overline{\nu}} = g_{a\overline{b}}\omega_{\alpha}{}^{a}{}_{\mu}\omega_{\overline{\beta}}{}^{\overline{b}}{}_{\overline{\nu}}, \tag{6.1}$$

where $\hat{R}_{\alpha\overline{\beta}\mu\overline{\nu}}$ is the curvature tensor of Y restricted to $f(X)$, $R_{\alpha\overline{\beta}\mu\overline{\nu}}$ is the curvature tensor of X, $g_{a\overline{b}}$ is the local representation of the hermitian metric on Y, and $\omega_{\alpha}{}^{a}{}_{\mu}$ is the local representation of the second fundamental form of f. By assumption, Y is pseudoconformally flat, and hence,

$$\hat{R}_{\alpha\overline{\beta}\mu\overline{\nu}} = H_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + \hat{H}_{\mu\overline{\beta}}g_{\alpha\overline{\nu}} + \tilde{H}_{\alpha\overline{\nu}}g_{\mu\overline{\beta}} + \check{H}_{\mu\overline{\nu}}g_{\alpha\overline{\beta}}, \tag{6.2}$$

where $(H_{\alpha\overline{\beta}})$, $(\hat{H}_{\mu\overline{\beta}})$, $(\tilde{H}_{\alpha\overline{\nu}})$, and $(\check{H}_{\mu\overline{\nu}})$ are smoothly-varying hermitian matrices. Since f is conformal and X pseudoconformally flat, we also have that

$$R_{\alpha\overline{\beta}\mu\overline{\nu}} = J_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + \hat{J}_{\mu\overline{\beta}}g_{\alpha\overline{\nu}} + \tilde{J}_{\alpha\overline{\nu}}g_{\mu\overline{\beta}} + \check{J}_{\mu\overline{\nu}}g_{\alpha\overline{\beta}}, \tag{6.3}$$

where $(J_{\alpha\overline{\beta}})$, $(\hat{J}_{\mu\overline{\beta}})$, $(\tilde{J}_{\alpha\overline{\nu}})$, and $(\check{J}_{\mu\overline{\nu}})$ are smoothly-varying hermitian matrices.

For a non-zero $Z = Z^{\alpha}\frac{\partial}{\partial z^{\alpha}} \in T_p^{1,0}X$, (6.1)-(6.3) gives us

$$g_{a\overline{b}}\omega_{\alpha}{}^{a}{}_{\mu}\omega_{\overline{\beta}}{}^{\overline{b}}{}_{\overline{\nu}}Z^{\alpha}Z^{\overline{\beta}}Z^{\mu}Z^{\overline{\nu}} = T_{\alpha\overline{\beta}\mu\overline{\nu}}Z^{\alpha}Z^{\overline{\beta}}Z^{\mu}Z^{\overline{\nu}}, \tag{6.4}$$

where

$$\begin{aligned} T_{\alpha\overline{\beta}\mu\overline{\nu}} := {} & H_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} + \hat{H}_{\mu\overline{\beta}}g_{\alpha\overline{\nu}} + \tilde{H}_{\alpha\overline{\nu}}g_{\mu\overline{\beta}} + \check{H}_{\mu\overline{\nu}}g_{\alpha\overline{\beta}} \\ & - J_{\alpha\overline{\beta}}g_{\mu\overline{\nu}} - \hat{J}_{\mu\overline{\beta}}g_{\alpha\overline{\nu}} - \tilde{J}_{\alpha\overline{\nu}}g_{\mu\overline{\beta}} - \check{J}_{\mu\overline{\nu}}g_{\alpha\overline{\beta}}. \end{aligned}$$

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[8] **Gauss-Codazzi Equation**. From Riemannian geometry, let M be an n-dimensional submanifold of the $n+p$-dimensional manifold P. Then, for all $X, Y, Z, W \in TM$,

$$\langle R_P(X,Y)Z, W\rangle = \langle R_M(X,Y)Z, W\rangle + \langle II(X,Z), II(Y,W)\rangle - \langle II(Y,Z), II(X,W)\rangle,$$

where $R.$ is the Riemannian curvature tensor and II is the second fundamental form tensor of a mapping $M \to P$.

Since the right-hand side of (6.4) is of the form $|Z|^2 h$, for some hermitian function h, and by the assumption on the dimensions of X and Y, Corollary 4.4 implies that

$$g_{a\bar{b}}\omega_{\alpha}{}^{a}{}_{\mu}\omega_{\overline{\beta}}{}^{\overline{b}}{}_{\overline{\nu}}Z^{\alpha}Z^{\overline{\beta}}Z^{\mu}Z^{\overline{\nu}} \equiv 0.$$

Since $g_{a,\bar{b}}$ is positive definite, we must have that $\omega_{\alpha}{}^{a}{}_{\mu} \equiv 0$. Hence, f is geodesic at $p \in X$. Since p are chosen arbitrarily, f is totally geodesic on X. $\square$

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