UNCONSTRAINED VARIATIONAL PRINCIPLES AND MORSE INDICES FOR LINEAR ELLIPTIC EIGENPROBLEMS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

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December 2013

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Acknowledgements

I would like to express my deepest gratitude to my doctoral adviser Professor Giles Auchmuty for his encouragement, guidance, and inspiration throughout this work, and for giving me a wonderful introduction to the PDE applications of Morse theory. I feel very fortunate to have the opportunity to work with him and to experience the beauty of how he sees mathematics. I am also very thankful to my doctoral co-adviser Professor William Ott for his never-ending interest in our various ongoing investigations and for always taking the time to listen to my research ideas. I hope that in the future I will have the opportunity to continue to collaborate with both of my advisers.

I would like to thank my committee members Professor Yuliya Gorb from the University of Houston and Professor Robert Hardt from Rice University for taking the time to read my dissertation, and for their valuable comments and suggestions.

I would like to thank Professor Shanyu Ji for being a supportive graduate student adviser throughout all these years. I would like to give my immense thanks to Professor Brian M. Loft at Sam Houston State University, where I did my Master's studies, for introducing me to the wonderful theory of Morse and for sharing the joy that comes from working in topology and visualizing mathematics.

Special thanks go to my friend Ricky for sharing his enthusiasm for pure mathematics and for always being there for me when I'm down. I would also like to thank my friend Sacide for being a good Turkish teacher, and my friend Cihan for motivating me to continue to strive to be both a good mathematician and a good artist.

I dedicate this dissertation to my parents, whose love is a source of strength.

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Abstract

Variational principles for finding eigenvalues, and associated eigenvectors, for symmetric matrices and compact self-adjoint linear operators have been studied for some time now (see [2] or [3], for instance). Here we shall introduce and study unconstrained variational principles for the eigenproblem of a pair of bilinear forms (a, m) on a Hilbert space. Each functional in the one-parameter family of functionals has well-defined first and second variations.

First variations characterize the critical points as eigenvectors of (a, m) with associated eigenvalues given by specific formulae. Properties of the set of critical points, that depend on the parameter value of the family of functionals, are given and summarized by a bifurcation diagram. Second variations enable a Morse index theory that characterizes the critical point as being associated with the j^{th} eigenvalue.

The framework is quite general, but the assumption on (a, m) are appropriate for the study of second-order divergence form elliptic problems in Hilbert-Sobolev spaces, including problems with non-zero boundary data and indefinite weights. These problems include Robin, Steklov and general eigenvalue problems.

Contents

Chapter 1

Background and Motivation

1.1 Introduction

A primary purpose of eigenvalue analyses is to provide spectral representations of solutions of linear systems. When these systems are described by symmetric matrices, self-adjoint linear operators, or symmetric bilinear forms, the standard variational principles for eigenvalues of these forms have been based on minimizing or maximizing Rayleigh's quotient. This is Rayleigh's principle and it is essentially a constrained optimization problem.

However, some unconstrained variational principles for matrices and compact, self-adjoint, linear operators have been described in Auchmuty [2, 3, 4]. Following this direction, in this work we shall describe a one-parameter family of unconstrained variational principles for finding eigenvalues and eigenvectors of a pair (a, m) of continuous, symmetric, bilinear forms on a separable Hilbert space V. This framework is appropriate for elliptic eigenvalue problems set in Hilbert-Sobolev spaces on bounded regions, including problems with non-zero boundary data. The functionals involved in these principles have well-defined first and second variations which allow non-zero critical points to be characterized as certain eigenvectors of the pair (a, m) and to be characterized by their Morse index. More specifically, a non-degenerate critical point associated to the j^{th} eigenvalue of (a, m), counting multiplicities, will have Morse index $(j-1)$, provided the parameter is large enough. Thus, in this case, a minimizer of the functional will be a critical point associated to the least eigenvalue λ_1 with zero Morse index.

We begin in the next section with a discussion on bilinear forms and the (a, m)-eigenvalue problem that we address in this dissertation. We also summarize some of the spectral properties of the sequence of eigenvectors of the pair (a, m) defined by the iterative construction given in Auchmuty [7], and recall some results from the calculus of variations that we shall use throughout this work .

In Chapter 2 the Morse and null index of bilinear forms satisfying a Gårding type inequality are introduced along the lines outlined in Zeidler [10]. However, our work here uses bilinear forms exclusively as opposed to the use of the corresponding linear operators and associated dual spaces. The indices of these particular bilinear forms are then shown to be finite, and in a certain sense invariant; we thus provide an infinite-dimensional version of Sylvester's law of inertia. However, before showing all of these properties, we first give spectral representations of these special bilinear forms, and also of those bilinear forms that are weakly continuous. Lastly, the case where the bilinear form is a Hessian form (the second derivative) of a functional is considered and a splitting of the corresponding quadratic form given.

Chapter 3 introduces the one-parameter family of unconstrained variational principles and then properties of the functional $\mathscr{G}(.;\mu)$ are proved. The results enable the existence of global minimizers of the variational problem, and first variations characterize non-zero critical points as certain eigenvectors of the pair of bilinear forms (a, m), where the m-norm of these eigenvectors is a function of the parameter μ and the associated eigenvalue. We then describe the dependence of the set of critical points on the parameter value μ and provide a bifurcation diagram. Lastly, we show how the functional $\mathscr{G}(.;\mu)$ may be penalized to yield again unconstrained variational principles, but this time for finding the second smallest eigenvalue λ_2 of the pair (a, m). Though this penalization method may be generalized for higher eigenvalues, we end the chapter by providing constrained variational principles for finding higher eigenvalues.

We emphasize that up to this point only first variations of the functional $\mathscr{G}(.;\mu)$ have been used in the work. It is only until Chapter 4 that we show $\mathscr{G}(.;\mu)$ has well-defined second derivatives, which allows us to use the work from Chapter 2 to evaluate the Morse and null indices of critical points of the functional. These calculations in turn provide further information on the bifurcation results given in Chapter 3. In particular, we show that minimizers of the functional $\mathscr{G}(.;\mu)$ have zero Morse index and that non-minimizing critical points have Morse index being strictly positive. More specifically, we show the Morse index of a critical point associated to the j^{th}-distinct eigenvalue is equal to the sum of the multiplicities of the $(j-1)$ previous distinct eigenvalues.

The results proved up to Chapter 4 assume that the bilinear form m satisfies $m(v, v) \geq 0$, or that it be strictly positive for non-zero vectors $v \in V$. In Chapter 5 we consider the indefinite weighted eigenproblem, which is the problem where the bilinear form m is not necessarily positive. That is, we consider the case where there are vectors v_1, v_2 in V such that

$$m(v_1, v_1) \quad < \quad 0 \quad < \quad m(v_2, v_2).$$

We show that our previous work applies to this case as well.

Chapter 6 is devoted to showing how our results applies to linear elliptic eigenvalue problems; we consider three classes of problems. The first type of problems considered are Robin eigenvalue problems, and Steklov eigenproblems are studied - which are problems with homogeneous equations where the eigenparameter appears in the boundary condition instead of the differential equation. The last class of problems we consider are general problems where the eigenparameter appears both in the differential equation and in the boundary condition. We note that the condition on the boundary of the (bounded) region where the problems are defined is that it it be composed of finitely many Lipschitz surfaces.

Also, we again emphasize that in all this work we make exclusive use of bilinear forms and not of the associated linear operators. We point out too that some of the results given here are related to results on unconstrained variational principles for eigenproblems described by Auchmuty in [2, 3, 4, 5, 7] .

1.2 Terminology and Assumptions

1.2.1 Bilinear Forms and Eigenvalue Problems

In this work, V is a real, separable, infinite-dimensional Hilbert space with inner product and norm denoted $\langle ., . \rangle_V$ and $\|.\|_V$, respectively. The dual space of V is denoted V^* and is again a real, separable, infinite-dimensional Hilbert space with dual norm $\|.\|_*$. The dual pairing between V and V^* is denoted $\langle ., . \rangle$ so that the value of a functional $u^* \in V^*$ evaluated at a vector $u \in V$ is written $\langle u^*, u \rangle$.

A bilinear form $b : V \times V \to \mathbb{R}$ is said to be *symmetric* provided $b(u, v) = b(v, u)$ holds for all $u, v \in V$. The corresponding quadratic form $\mathscr{B}(u) := b(u, u)$ is said to be *positive* provided it satisfies $\mathscr{B}(u) \geq 0$ for all $u \in V$, and *strictly positive* if $\mathscr{B}(u) > 0$ for all non-zero $u \in V$; similarly, *negative* and *strictly negative* are defined with inequalities reversed. The bilinear form b is said to be V*-coercive* provided there exists a constant $k > 0$ such that $\mathscr{B}(u) \geq k\|u\|_V^2$ for all $u \in V$.

When the bilinear form b is symmetric, we also have the following. The *null space* of b is the subset $N(b)$ of V given by

$$N(b) := \{u \in V : b(u, v) = 0, \quad \forall v \in V\}.$$

If $N(b) = \{0\}$, then b is said to be *non-degenerate*; otherwise it is *degenerate*. Furthermore, the vectors $u, v \in V$ are said to be b*-orthogonal* if $b(u, v) = 0$. A subset $\mathcal{E}$ of V is said to be a *basis* of V when it is a maximal linearly independent set in V with respect to inclusion. A subset $\mathcal{E}$ of V is said to be a b*-orthogonal basis* of V provided it is a basis and any two vectors in $\mathcal{E}$ are b-orthogonal.

Moreover, we say that b has *finite rank* M, with $M \in \mathbb{N}$, provided there are M linearly independent functionals $u_1^*, \ldots, u_M^*$ in V^*, such that b has the following representation:

$$b = \sum_{j=1}^{M} u_j^* \otimes u_j^* \tag{1.1}$$

That is, b has finitie rank if $b(u, v) = \sum_{j=1}^{M} \langle u_j^*, u \rangle \langle u_j^*, v \rangle$ for all $u, v \in V$. It follows from definition that a finite rank form is weakly continuous.

In this work, we shall consider bilinear forms $a : V \times V \to \mathbb{R}$ and $m : V \times V \to \mathbb{R}$ subject to some of the following conditions:

(A1): $a(.,.)$ is a continuous, symmetric, bilinear form that is V-*coercive*. That is, there exist constants $0 < k_0 \leq k_1 < \infty$ such that

$$k_0\|u\|_V^2 \leq a(u,u) \leq k_1\|u\|_V^2 \qquad \text{for all } u \in V. \tag{1.2}$$

(A2): $m(.,.)$ is a weakly continuous, symmetric, bilinear form on V.

(A3): $m(u,u) \geq 0$ for all $u \in V$.

(A4): $m(u,u) > 0$ for all non-zero $u \in V$.

Define $\mathscr{A}$, $\mathscr{M}$ to be the quadratic forms on V associated to a, m, so that

$$\mathscr{A}(u) := a(u,u) \qquad \text{and} \qquad \mathscr{M}(u) := m(u,u).$$

When (A1) holds, the bilinear form $a(.,.)$ defines an inner product

$$[u,v]_a := a(u,v) \tag{1.3}$$

on V that is equivalent to the V inner product. When $m(.,.)$ satisfies (A2) and (A3), then $\|u\|_m := \sqrt{\mathscr{M}(u)}$ defines a semi-norm on V. A vector $u \in V$ is said to be *m-normalized* provided it satisfies $\|u\|_m = 1$.

We will study issues related to finding non-trivial solutions $(\lambda, u) \in \mathbb{R} \times V$ of

$$a(u,v) = \lambda m(u,v) \qquad \text{for all } v \in V. \tag{1.4}$$

This will be called the (a,m)-*eigenproblem*. The number λ is an *eigenvalue* of (a,m)

if there is a non-zero vector u in V satisfying (1.4) and the associated u is called an *eigenvector* of (a, m) corresponding to λ. When λ is an eigenvalue, let E_λ be the set of all $u \in V$ such that (1.4) holds. The number of linearly independent eigenvectors of (a, m) corresponding to the eigenvalue λ is called the *multiplicity* of λ. When the multiplicity of λ is one, then λ is said to be a *simple* eigenvalue.

When (A1)-(A3) hold, by taking $v = u$ an eigenvector in (1.4), we see that every eigenvalue λ of (a, m) must be strictly positive. The iterative construction given in Auchmuty [7] yields the following summary of spectral results about the (a, m)-eigenproblem.

Theorem 1.1. *Assume (a, m) satisfy (A1)-(A3). Then either there are*

(i) finitely many strictly positive eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$ of (a, m) and a corresponding m-orthonormal set of eigenvectors $\mathcal{E}_M := \{e_j : 1 \leq j \leq M\}$, or else

(ii) countably infinitely many strictly positive eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ of (a, m), with $\lim_{j\to\infty} \lambda_j = \infty$, and a corresponding m-orthonormal set of eigenvectors $\mathcal{E}_+ := \{e_j : j \geq 1\}$.

In both cases, these eigenvalues can be found iteratively and are repeated according to their multiplicities. When (A4) also holds, then (ii) holds and $\mathcal{E}_+$ is an m-orthonormal basis of V.

Proof. The assumption on (a, m) are the assumptions for the bilinear forms in Theorem 4.2 and 4.3 in Auchmuty [7]. From these theorems the desired conclusions are obtained with a-orthonormality instead of m-orthonormality. For an eigenvector e corresponding to an eigenvalue $\lambda > 0$ of (a, m), we note that

$$a(e, u) = 0 \qquad \text{if and only if} \qquad m(e, u) = 0,$$

which holds for all $u \in V$, allows a-orthogonality to be replaced by m-orthogonality. Moreover, m-normalized eigenvectors are obtained from a-normalized eigenvectors by taking $\tilde{e} := \lambda^{1/2} e$ for an a-normalized eigenvector e corresponding to the eigenvalue λ. $\square$

In view of this theorem, by taking W to be the a-orthogonal complement of the null space $N(m)$ of m, we see that when conditions (A1)-(A3) are satisfied, the following a-orthogonal decomposition that we shall use in our analysis holds

$$V = N(m) \oplus_a W \tag{1.5}$$

with the eigenvectors of (a, m) lying in W. The subspace W will be finite or infinite dimensional accordingly as (i) or (ii) in the theorem hold.

1.2.2 Tools from the Calculus of Variations

Various results from the calculus of variations will be used in this work. Background material may be found in Attouch, Buttazzo, Michaille [1], Blanchard and Brüning [8], or Zeidler [10].

Let $\mathscr{F} : V \to \mathbb{R}$ be a given functional. The *first variation* of $\mathscr{F}$ at the point $u \in V$ in the direction $v \in V$ is defined to be the following derivative

$$\delta\mathscr{F}(u; v) := \frac{d}{dt}\mathscr{F}(u + tv)\Big|_{t=0}$$

provided this derivative exists. When $\delta\mathscr{F}(u; v)$ exists for all $v \in V$ and is a continuous linear functional in v, then $\mathscr{F}$ is said to be *Gâteaux differentiable* at u and the linear functional $v \mapsto \delta\mathscr{F}(u; v)$ is the *Gâteaux derivative* of $\mathscr{F}$ at u. A point $u \in V$ is a *critical point* of $\mathscr{F}$ provided $\mathscr{F}$ is Gâteaux differentiable at u and

$$\delta\mathscr{F}(u;v) = 0 \qquad \text{for all } v \in V.$$

A number c is a *critical value* of $\mathscr{F}$ if there is a critical point u with $\mathscr{F}(u) = c$.

The *second variation* of $\mathscr{F}$ at u in the directions $v, w \in V$ is defined by

$$\delta^2\mathscr{F}(u;v,w) := \frac{\partial^2}{\partial t_2 \partial t_1}\mathscr{F}(u + t_1 v + t_2 w)\Big|_{t_1=t_2=0}$$

whenever this derivative exists. If $\delta^2\mathscr{F}(u;v,w)$ exists for all $v, w \in V$ and is a continuous bilinear form in (v,w), then $\mathscr{F}$ is said to be *twice Gâteaux differentiable* at u and the bilinear form $(v,w) \mapsto \delta^2\mathscr{F}(u;v,w)$ is called the *Gâteaux second derivative*, or *Hessian form*, of $\mathscr{F}$ at u.

We point out that in contrast with most references dealing with the calculus of variations (see, for instance, [1], [8], or [10]) which take the second variation of a functional $\mathscr{F}$ at a point u to be the derivative given by

$$\delta^2\mathscr{F}(u;v) = \frac{d^2}{dt^2}\mathscr{F}(u + tv)\Big|_{t=0},$$

for a given direction vector v in V, provided this derivative exists, we opt for the above definition of second variations since a goal of our analysis is to determine the type (degenerate or non-degenerate) of a critical point of $\mathscr{F}$, and not merely whether the critical point provides a local miniminum or not.

Chapter 2

Types and Morse Indices

Morse theory for infinite-dimensional spaces investigates the strong relations between variational problems and topology. In this work, however, since we shall prove results about the critical points of smooth unconstrained variational problems, our main interest is only in the type of a critical point determined by the Hessian form of the functional at the point.

We thus define the Morse index of bilinear forms satisfying a Gårding type inequality, and use the theory of the Morse index along the lines outlined in Zeidler [10], Section 37.27b. Here, in contrast to [10], the presentation and analysis uses bilinear forms directly rather than the associated linear operators between dual spaces.

2.1 Spectral Representation of Bilinear Forms

In order to define the Morse index of bilinear forms, we first provide the following spectral representations of two classes of bilinear forms. We show the Morse index of the second class of bilinear forms is finite and invariant in a certain sense.

2.1.1 Weakly Continuous Bilinear Forms

Consider first the class of weakly continuous bilinear forms on V whose corresponding quadratic forms are positive. That is, assume the bilinear form m satisfies $(A2)$ and $(A3)$, and let the bilinear form a satisfy (A1). Take W as in the decomposition (1.5), and let $\mathcal{E}_0 = \{e_j : j \in J_0\}$ be a maximal a-orthonormal set in the null space $N(m)$ of m. Define J_+ to be the indexing set of an m-orthonormal basis of eigenvectors for the subspace W. That is, J_+ is equal to $\{1, \ldots, M\}$ or $\mathbb{N}$ accordingly as (i) or (ii) of Theorem 1.1 holds. That $\mathcal{E} := \mathcal{E}_0 \cup \{e_j : j \in J_+\}$ is an a-orthogonal basis of V follows from Corollary 4.4 and 4.5 of Auchmuty [7]. We then have the following result.

Theorem 2.1. *Assume* (a, m) *satisfy* $(A1)$-$(A3)$. *With the subspace* W *as in the decomposition* (1.5)*, and* J_+ *as above, we have*

$$m(u, v) = \sum_{j \in J_+} \lambda_j^{-2} a(e_j, u) a(e_j, v) \qquad \text{for all } u, v \in V. \tag{2.1}$$

Proof. By taking $\tilde{e}_j := \lambda_j^{-1/2} e_j$ for an m-normalized eigenvector e_j of (a, m) corresponding to λ_j, we obtain an a-orthonormal basis $\tilde{\mathcal{E}} := \mathcal{E}_0 \cup \{\tilde{e}_j : j \in J_+\}$ for V. As $a(.,.)$ is equivalent to the V-inner product, an element u in V then has an expansion

$$u = \sum_{j \in J_+} c_j^+ \tilde{e}_j \; + \; \sum_{j \in J_0} c_j^0 e_j$$

with $c_j^+ = a(\tilde{e}_j, u)$ for $j \in J_+$, and $c_j^0 = a(e_j, u)$ for $j \in J_0$. Since $m(e_j, v) = 0$ for each $j \in J_0$ and all $v \in V$, we have

$$m(u, v) = \sum_{j \in J_+} c_j^+ m(\tilde{e}_j, v) = \sum_{j \in J_+} \lambda_j^{-2} a(e_j, u) a(e_j, v)$$

for all $u, v \in V$ as desired. $\square$

For $j \in J_+$, denote by ε_j the linear functional given by $v \mapsto a(e_j, v)$ for $v \in V$. Then ε_j is in V^* for each $j \in J_+$ by continuity of a, and the representation (2.1) for m in terms of the a-inner product may be written in terms of the linear functionals $\varepsilon_j \in V^*$ as

$$m = \sum_{j \in J_+} \lambda_j^{-2} \varepsilon_j \otimes \varepsilon_j \tag{2.2}$$

When (i) of Theorem 1.1 holds, then the cardinality of J_+ in the representation (2.2) is finite and thus m is a finite rank bilinear form on V. Also, for different a-inner products we have different representations of m.

2.1.2 Semi-coercive Bilinear Forms

The second class of bilinear forms that we shall now consider are those satisfying a Gårding-type inequality on V. Specifically, we consider bilinear forms b subject to the following conditions:

(M1): $b : V \times V \to \mathbb{R}$ is a symmetric, continuous, bilinear form on V, and

(M2): there is a bilinear form m satisfying (A2) and (A4), and there are constants $k_2, k_3 > 0$ such that

$$\mathscr{B}(v) \geq k_2 \|v\|_V^2 - k_3 \mathscr{M}(v) \qquad \text{for all } v, w \in V. \tag{2.3}$$

A bilinear form b is said to be *semi-coercive* on V provided it satisfies both (M1) and (M2). The following theorem provides a spectral representation of semi-coercive bilinear forms which will be used later when we evaluate Morse and null indices of such bilinear forms. We point out that the proof of Theorem 1.1 is based on a constructive algorithm found in [7] that determines successive eigenvalues.

Theorem 2.2. *Assume the pair* (b, m) *satisfies* $(M1)$ *and* $(M2)$. *Then there exists an* m*-orthonormal basis* $\mathcal{E} = \{e_j : j \in \mathbb{N}\}$ *of* V *consisting of eigenvectors corresponding to eigenvalues* $-\infty < \lambda_1 \leq \lambda_2 \leq \cdots$ *of the pair* (b, m), *with* $\lim_{j\to\infty} \lambda_j = \infty$, *and*

$$b(v, w) = \sum_{j=1}^{\infty} \lambda_j m(e_j, v)m(e_j, w) \qquad \textit{for all } v, w \in V. \tag{2.4}$$

Proof. Consider the eigenproblem of finding non-trivial solutions $(\lambda, v) \in \mathbb{R} \times V$ of

$$b(v, w) = \lambda m(v, w) \qquad \text{for all } w \in V \tag{2.5}$$

From (M2) we have $\mathscr{B}(v) + k_3\mathscr{M}(v) \geq k_2\|v\|_V^2$ for all $v \in V$, i.e., the bilinear form $\tilde{b} := b + k_3m$ is V-coercive and thus satisfies the conditions of the bilinear form in (A1). Since m satisfies (A2) and (A4), it follows from Theorem 1.1 that there is an m-orthonormal basis $\mathcal{E}_+ := \{e_j : j \in \mathbb{N}\}$ of V consisting of eigenvectors of $(\tilde{b}, m)$ corresponding to an increasing sequence of strictly positive eigenvalues $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$ with no accumulation point. That is,

$$\tilde{b}(e_j, w) = \tilde{\lambda}_j m(e_j, w) \qquad \text{for all } w \in V$$

with $m(e_j, e_k) = \delta_{jk}$ for all $j, k \in \mathbb{N}$.

For each $j \in \mathbb{N}$ set $\lambda_j := \widetilde{\lambda}_j - k_3$. Then

$$b(e_j, w) = \lambda_j m(e_j, w) \tag{2.6}$$

holds for all $w \in V$, so that taking $\mathcal{E} = \mathcal{E}_+$ gives an m-orthonormal basis of V

consisting of eigenvectors corresponding to an increasing sequence of eigenvalues $-\infty < \lambda_1 \leq \lambda_2 \leq \cdots$ for the pair (b, m) with no accumulation point. The first part of the theorem then holds.

Since $\mathcal{E}$ is a basis of V, an element v in V has an expansion

$$v = \sum_{j=1}^{\infty} m(e_j, v) e_j.$$

Since b is continuous on V, we have

$$b(v, w) = \sum_{j=1}^{\infty} m(e_j, v) b(e_j, w) = \sum_{j=1}^{\infty} \lambda_j m(e_j, v) m(e_j, w)$$

for all $w \in V$. □

As in the case of weakly continuous bilinear forms, the representation (2.4) for the semi-coercive bilinear form b in terms of the bilinear form m may be written in terms of functionals in V^* as follows. For each $j \in \mathbb{N}$, let ε_j denote the linear functional on V given by $v \mapsto m(e_j, v)$ for $v \in V$. Then ε_j is in V^* for each j as m is weakly continuous on V, so that (2.4) turns into

$$b = \sum_{j=1}^{\infty} \lambda_j \varepsilon_j \otimes \varepsilon_j. \tag{2.7}$$

2.2 Type and Morse Index of a Bilinear Form

Let $b : V \times V \to \mathbb{R}$ be a symmetric, continuous, bilinear form on V. That is, b satisfies assumption (M1). Recall b is said to be *non-degenerate* provided the null space $N(b)$ of b is equal to $\{0\}$; otherwise, b is said to be degenerate. The dimension of the null space of b on V is said to be the *null index* of b on V and denoted $i_0(b)$. It follows from definitions that the null index $i_0(b)$ of b is zero whenever b is non-degenerate.

Suppose the corresponding quadratic form $\mathscr{B}$ of b is strictly negative on subspaces W of V, with $\dim W \leq J$ for all such subspaces. Furthermore, suppose that among these subspaces W there is a closed subspace W_b with $\dim W_b = J$. Then we say that the *Morse index* of b on V is J, and denote this as $i(b) = J$. If no such subspace W exists, then $\mathscr{B}$ is positive on V and we say the Morse index of b is zero.

When b also satisfies (M2), the following results show that the Morse index and null index of b are finite and independent of the choice of the bilinear form m in (M2).

Theorem 2.3. *Assume the pair* (b, m) *satisfies* $(M1)$ *and* $(M2)$. *Then*
(i) the Morse index $i(b)$ *of* b *is finite and equal to the number of negative eigenvalues of* (b, m) *counting multiplicities,*
(ii) the null index $i_0(b)$ *of* b *is finite and equal to the multiplicity of* 0 *as an eigenvalue of* (b, m)*, and*
(iii) b *is non-degenerate if and only if* 0 *is not an eigenvalue of* (b, m).

Proof. By Theorem 2.2, let $\mathcal{E} = \{e_j : j \in \mathbb{N}\}$ be an m-orthonormal basis of V consisting of eigenvectors corresponding to an increasing sequence of eigenvalues $-\infty < \lambda_1 \leq \lambda_2 \leq \cdots$, with no accumulation point, of the pair (b, m). Let

$$W_- = \operatorname{span}\{e_j : \lambda_j < 0\} \qquad \text{and} \qquad W_0 = \operatorname{span}\{e_j : \lambda_j = 0\}.$$

As the λ_j's form an increasing sequence of real numbers with no finite accumulation point, the dimensions of W_- and W_0 are both finite. From (2.4), for each $v \in W_-$ we have

$$b(v,v) = \sum_{j=1}^{\infty} \lambda_j m(v,e_j)^2 = \sum_{\{j:\lambda_j<0\}} \lambda_j m(e_j,v)^2$$

by the m-orthogonality of the e_j's. Hence, the associated quadratic form $\mathscr{B}$ is strictly negative on W_-, and also from (2.4) we see that $\dim(W_-)$ is maximal for subspaces on which $\mathscr{B}$ is strictly negative. The first assertion then holds.

By the m-orthonormality of the eigenvectors e_j's it is easy to see that $v \in W_0$ if and only if $v \in N(m)$, so the second assertion also holds. The last statement is also direct.

□

2.3 Invariance of Indices

We show next that the Morse and null index of b are independent of the form m chosen in (M2). This may be regarded as an infinite-dimensional version of Sylvester's law of inertia in finite-dimensional, real, linear spaces.

Theorem 2.4. *Assume b satisfies $(M1)$, and m_1, m_2 both satisfy $(A2)$ and $(A4)$. Suppose both pairs (b, m_1) and (b, m_2) satisfy an inequality of the form* (2.3) *with m_1, m_2 in place of m. Then*

(i) the number of negative eigenvalues (counting multiplicities) of (b, m_1) is equal to the number of negative eigenvalues (counting multiplicities) of (b, m_2), and

(ii) the multiplicity of 0 as an eigenvalue of (b, m_1) is equal to the multiplicity of 0 as an eigenvalue of (b, m_2).

Proof. Eigenvalues are counted with multiplicities. Denote by $i(b; m_1)$ the number of negative eigenvalues of (b, m_1). By Theorem 2.3 $i(b; m_1)$ is finite, and the dimension of any subspace W on which $\mathscr{B}$ is strictly negative satisfies

$$\dim W \leq i(b; m_1).$$

Let W_- be the subspace of V generated by eigenvectors corresponding to negative eigenvalues of (b, m_2). By Theorem 2.3 again, W_- has finite dimension, denoted $i(b; m_2)$, equal to the number of negative eigenvalues of (b, m_2), so that

$$i(b; m_2) \leq i(b; m_1),$$

as $\mathscr{B}$ is strictly negative on W_-. Interchanging the roles of $i(b; m_1)$ and $i(b; m_2)$ gives the reverse inequality so that the first statement holds.

The second statement holds as any vector v in the null space $N(b)$ of b satisfies

$$b(v, w) = 0m(v, w) \qquad \text{for all } w \in V,$$

regardless of the form m satisfying (M1) and (M2). □

2.4 Indices and the Calculus of Variations

When $\mathscr{F} : V \to \mathbb{R}$ is a twice Gâteaux differentiable functional on V the definitions and terminology given in previous sections correspond as follows for critical points of $\mathscr{F}$. A critical point u of $\mathscr{F}$ is said to be *degenerate* or *non-degenerate* accordingly as the Hessian form $\delta^2\mathscr{F}(u; ., .)$ is degenerate or non-degenerate. Furthermore, the *Morse index* and *null index* of a critical point u of $\mathscr{F}$ are defined as the Morse index and null index of $\delta^2\mathscr{F}(u; ., .)$, and are denoted $i(u)$ and $i_0(u)$, respectively.

In Morse theory, the splitting of the quadratic form corresponding to the Hessian form of a functional at a non-degenerate critical point plays a crucial role. For this investigation we thus consider the following splitting of a (Hessian) bilinear form which yields the corresponding splitting of the associated quadratic form.

Corollary 2.5. *Assume b satisfies $(M1)$, $(M2)$ and is non-degenerate. Then there are bilinear forms b_- and b_+ on V with the following properties:*
(i) b_- and b_+ are continuous and symmetric, with b_- having finite rank equal to the Morse index of b, and satisfy

$$b(v,w) = b_+(v,w) - b_-(v,w) \qquad \text{for all } v,w \in V. \tag{2.8}$$

(ii) The corresponding quadratic forms $\mathscr{B}_-$ and $\mathscr{B}_+$ are both positive, convex functionals on V, with $\mathscr{B}_-$ weakly continuous, and satisfy

$$\mathscr{B}(v) = \mathscr{B}_+(v) - \mathscr{B}_-(v) \qquad \text{for all } v \in V. \tag{2.9}$$

Proof. Let m be a bilinear form satisfying (M2). From Theorem 2.3 we have that a subspace of maximal (finite) dimension on which $\mathscr{B}$ is strictly negative is spanned by $\mathcal{E}_- := \{e_j : 1 \le j \le i(b)\}$, where $i(b)$ is the Morse index of b and the e_j's are m-orthonormal eigenvectors corresponding to strictly negative eigenvalues λ_j of (b,m).

Define the bilinear form $b_- : V \times V \to \mathbb{R}$ by

$$b_-(v,w) = -\sum_{j=1}^{i(b)} \lambda_j m(v,e_j)m(w,e_j). \tag{2.10}$$

Then b_- has finite rank $i(b)$, and the identities (2.4) and (2.10) imply that the bilinear form $b_+ := b - b_-$ is continuous and symmetric on V, so that the first assertions holds.

From (2.10), we see that the quadratic form $\mathscr{B}_-$ corresponding to b_- is a positive, convex functional on V that is weakly continuous. It follows from (2.4) and (2.10) that the quadratic form $\mathscr{B}_+$ coresponding to b_+ also is a positive, convex functional on V. Therefore, the last statement holds. $\square$

Chapter 3

Weighted Eigenvalue Problems

In the previous chapters, eigenvalue problems and representation theorems for bilinear forms, and concepts of Morse theory to be used in this work were introduced, as well as related notation. In this chapter, our interest is in describing and analyzing certain unconstrained parametrized functionals whose critical points yield eigenvalues and eigenvectors of the pair (a, m) of bilinear forms. It is shown that the functionals are minimized precisely at eigenvectors of the pair (a, m) corresponding to the smallest (strictly) positive eigenvalue λ_1, when the parameter is strictly bigger than λ_1. In this case, the m semi-norm of the minimizers and the minimum value of the functional are related to λ_1.

3.1 Unconstrained Variational Principles for the Least Eigenvalue

Let (a, m) be a pair of bilinear forms satisfying $(A1)$, $(A2)$, and consider the functional $\mathscr{G} : V \times (0, \infty) \to \mathbb{R}$ given by

$$\mathscr{G}(u; \mu) := a(u, u) - \mu m(u, u) + \frac{1}{2} m(u, u)^2. \tag{3.1}$$

The variational principle here is the unconstrained problem $(\mathcal{P}_\mu)$ of minimizing $\mathscr{G}(.; \mu)$ on V and finding

$$\alpha(\mu) := \inf_{u \in V} \mathscr{G}(u; \mu). \tag{3.2}$$

The following theorem gives properties of the functional $\mathscr{G}(.; \mu)$ that are used to provide results for this variational principle.

Theorem 3.1. *Assume* $(A1)$, $(A2)$ *hold and* $\mathscr{G}(.; \mu)$ *is defined by* (3.1). *Then*

(i) $\mathscr{G}(.; \mu)$ *is continuous, coercive and weakly l.s.c. on* V, *and*

(ii) $\mathscr{G}(.; \mu)$ *is Gâteaux differentiable on* V *with first variation at* u *in the direction* v *given by*

$$\delta\mathscr{G}(u; v; \mu) = 2a(u, v) + 2[m(u, u) - \mu] m(u, v). \tag{3.3}$$

Proof. The quadratic form $\mathscr{A}(u) := a(u, u)$ is continuous on V as for fixed $u \in V$ and $u_n \in V$ with $\|u_n - u\|_V \longrightarrow 0$, the identity

$$a(u_n, u_n) - a(u, u) = a(u_n - u, u_n - u) + a(u_n - u, u) + a(u, u_n - u) \tag{3.4}$$

gives

$$|\mathscr{A}(u_n) - \mathscr{A}(u)| \leq k_1\|u_n - u\|_V^2 + 2k_1\|u\|_V\|u_n - u\|_V \quad \longrightarrow 0 \qquad \text{as } n \to \infty.$$

It follows that $\mathscr{G}(.;\mu)$ is continuous on V as m is weakly continuous on V.

The weak continuity of m and the continuity of the function $\psi(x) = -\mu x + \frac{1}{2}x^2$ defined on $\mathbb{R}$, implies the composition

$$-\mu m(u,u) + \frac{1}{2}m(u,u)^2$$

is weakly continuous on V, hence weakly $l.s.c.$ on V. The quadratic form $\mathscr{A}$ is weakly $l.s.c.$ on V as $\mathscr{A}$ continuous and convex on V, and we then get that $\mathscr{G}(.;\mu)$ is weakly $l.s.c.$ on V.

Since $\psi(x) = -\mu x + \frac{1}{2}x^2 \geq -\frac{\mu^2}{2}$ for all $x \in \mathbb{R}$, as $\widehat{x} = \mu$ is the unique minimizer for ψ on $\mathbb{R}$, and since the bilinear form $a(.,.)$ is coercive on V we obtain

$$\mathscr{G}(u;\mu) \geq k_0\|u\|_V^2 - \frac{\mu^2}{2},$$

which implies $\mathscr{G}(\cdot;\mu)$ is coercive on V. Thus (i) holds.

Fix $u, v \in V$. Then for $t \neq 0$, we have for the quadratic form $\mathscr{M}(u) := m(u,u)$

$$\mathscr{M}(u+tv) = \mathscr{M}(u) + 2tm(u,v) + t^2\mathscr{M}(v).$$

Thus $\lim_{t\to 0}\mathscr{M}(u+tv) = \mathscr{M}(u)$ and

$$\delta\mathscr{M}(u;v) = \frac{d}{dt}\mathscr{M}(u+tv)\Big|_{t=0} = 2m(u,v).$$

Similarly, $\delta\mathscr{A}(u;v) = 2a(u,v)$. As $\psi(x) = -\mu x + \frac{1}{2}x^2$ is a polynomial of degree two, the composition $\psi \circ \mathscr{M}(u+\varepsilon v)$ is a polynomial of degree four in t, so that we can apply the classical chain rule to compute the first variation of $\mathscr{G}(.;\mu)$ and obtain equation (3.3) for any $u, v \in V$.

For fixed $u \in V$ the first variation $\delta\mathscr{G}(u;.;\mu) = 2a(u,.) + 2[m(u,u) - \mu]m(u,.)$ is the sum of two continuous linear functionals on V. Hence $\delta\mathscr{G}(u;.;\mu)$ is continuous and linear on V for each u in V, showing that $\mathscr{G}(.;\mu)$ is Gâteaux differentiable on V, and therefore (ii) follows.

$\square$

From Theorem 1.1 there is a smallest strictly positive eigenvalue λ_1 of (a,m) when $(A1)$-$(A3)$ hold, so Theorem 3.1 now yields the following results about the unconstrained problem $(\mathcal{P}_\mu)$.

Theorem 3.2. *Assume* $(A1)$-$(A3)$ *hold and* $\mathscr{G}$ *is defined by* (3.1).

(i) 0 *is the unique critical point of* $\mathscr{G}(.;\mu)$ *when* $\mu \leq \lambda_1$, *and* 0 *and the points* $(\mu - \lambda_j)^{1/2}e$ *are the critical points of* $\mathscr{G}(.;\mu)$ *when* $\mu > \lambda_1$, *where* e *is an* m*-normalized eigenvector corresponding to the eigenvalue* λ_j *in the interval* $(0,\mu)$.

(ii) *The critical values of* $\mathscr{G}(.;\mu)$ *are* 0 *for any value of* $\mu > 0$, *and* 0 *and* $-\frac{1}{2}(\mu - \lambda_j)^2$ *when* $\mu > \lambda_1$ *and* λ_j *is in the interval* $(0,\mu)$.

(iii) *The minimizer(s) of* $\mathscr{G}(.;\mu)$ *on* V *are* 0 *when* $\mu \leq \lambda_1$, *and* $(\mu - \lambda_1)^{1/2}e$ *when* $\mu > \lambda_1$, *with* e *an* m*-normalized eigenvector of* (a,m) *corresponding to* λ_1.

(iv) *The value of the problem* $(\mathcal{P}_\mu)$ *is* $\alpha(\mu) = 0$ *when* $\mu \leq \lambda_1$, *and* $\alpha(\mu) = -\frac{1}{2}(\mu - \lambda_1)^2$ *when* $\mu > \lambda_1$.

Proof. By (ii) of Theorem 3.1, a critical point $\tilde{u} \in V$ must satisfy

$$a(\tilde{u}, v) = \big[\mu - m(\tilde{u}, \tilde{u})\big] m(\tilde{u}, v) \qquad \text{for all } v \in V. \tag{3.5}$$

The point 0 always is a solution to (3.5) for any value of μ. Equation (3.5) implies a non-zero critical point $\tilde{u}$ of $\mathscr{G}(.;\mu)$ is an eigenvector of the pair (a, m) corresponding to a strictly positive eigenvalue $\lambda_j = \mu - m(\tilde{u}, \tilde{u})$.

Substituting such a critical eigenvector $\tilde{u}$ for v in (3.5) gives

$$a(\tilde{u}, \tilde{u}) = \big[\mu - m(\tilde{u}, \tilde{u})\big] m(\tilde{u}, \tilde{u}) = \lambda_j(\mu - \lambda_j)$$

which implies $\lambda_j(\mu - \lambda_j) > 0$ as $a(.,.)$ is coercive. Thus $\lambda_j < \mu$, or that

$$0 < \lambda_j < \mu.$$

Therefore, the only critical points of $\mathscr{G}(.;\mu)$ are the point 0 and weighted eigenvectors of the pair (a, m) corresponding to eigenvalues in the interval $(0, \mu)$. Also, when $\mu \leq \lambda_1$, we see that 0 is the unique critical point of $\mathscr{G}(.;\mu)$, so that the first assertion follows.

From (i) of Theorem 3.1, $\mathscr{G}(.;\mu)$ attains a finite infimum on V which by (ii) of that theorem must occur at a critical point, making the infimum a critical value. For a non-zero critical point $\tilde{u}$ of $\mathscr{G}(.;\mu)$ corresponding to an eigenvalue $\lambda_j \in (0, \mu)$ we have that

$$\mathscr{G}(\tilde{u};\mu) = -(\mu - \lambda_j)^2 + \frac{1}{2}(\mu - \lambda_j)^2 = -\frac{1}{2}(\mu - \lambda_j)^2$$

is a critical value of $\mathscr{G}(.;\mu)$. Thus when $\mu \leq \lambda_1$, the infimum is given by $\alpha(\mu) = 0$ as 0 is the only critical point of $\mathscr{G}(.;\mu)$ in this case. When $\mu > \lambda_1$, then the last expression for $\mathscr{G}(\tilde{u};\mu)$ shows that the smallest critical value is $\alpha(\mu) = -\frac{1}{2}(\mu - \lambda_1)^2$ which occurs at any point $(\mu - \lambda_1)^{1/2}e$ with e an m-normalized eigenvector corresponding to λ_1. Assertions (ii), (iii) and (iv) then hold. $\square$

Given $\mu \in (0,\infty)$, Theorem 3.2 shows that the unconstrained variational problem $(\mathcal{P}_\mu)$ of minimizing $\mathscr{G}(.;\mu)$ can be used to find upper and lower bounds on λ_1 as stated in the following result.

Corollary 3.3. *Assume* (A1)-(A3) *hold and* $\mathscr{G}$ *is given by* (3.1)*.*

(i) *If* $\alpha(\mu) = 0$*, then* $\mu \leq \lambda_1$*.*

(ii) *If there exists* $\tilde{v} \in V$ *with* $\mathscr{G}(\tilde{v};\mu) < 0$*, then*

$$\lambda_1 = \inf_{\mathscr{G}(u;\mu)<0} \left[\mu - \sqrt{-2\mathscr{G}(u;\mu)}\right] \leq \mu - \sqrt{-2\mathscr{G}(\tilde{v};\mu)}. \tag{3.6}$$

Proof. The first assertion follows from (iii) of Theorem 3.2. When there exists $\tilde{v} \in V$ with $\mathscr{G}(\tilde{v};\mu) < 0$, then $-\frac{1}{2}(\mu - \lambda_1)^2 = \alpha(\mu) \leq \mathscr{G}(\tilde{v};\mu)$, which follows also from part (iii) of the previous theorem, gives the second assertion. $\square$

3.2 Birfurcation of Critical Points of $\mathscr{G}(.;\mu)$

Also from Theorem 3.2 we will see that as the parameter μ in $\mathscr{G}(.;\mu)$ increases past an eigenvalue of (a,m), the set of critical points of $\mathscr{G}(.;\mu)$ gains a new set of critical points which bifurcates from the origin. This will follow from the following theorems which also provide a number of topological properties of the sets of critical points. To make this more precise, we first we make the following definitions.

We shall now write the j^{th} distinct eigenvalue of (a, m) as $\tilde{\lambda}_j$ and the multiplicity of $\tilde{\lambda}_j$ as m_j, so that

$$\tilde{\lambda}_1 < \tilde{\lambda}_2 < \tilde{\lambda}_3 < \cdots$$

and $\tilde{\lambda}_1 = \lambda_1$ and $\tilde{\lambda}_2 > \lambda_2$ when λ_1 has multiplicity $m_1 \geq 2$.

For an eigenvalue $\tilde{\lambda}_j$, with $0 < \tilde{\lambda}_j < \mu$, let E_j be the eigenspace corresponding to $\tilde{\lambda}_j$ and define the set $C(\tilde{\lambda}_j;\mu)$ by

$$C(\tilde{\lambda}_j;\mu) := \{u \in E_j : \|u\|_m^2 = \mu - \tilde{\lambda}_j\}. \tag{3.7}$$

When (A1)-(A3) hold, Theorem 1.1 shows that each eigenspace E_j is finite-dimensional, and there are only finitely many eigenspaces corresponding to eigenvalues in a given interval $(0, \mu)$. It follows then that the sets $C(\tilde{\lambda}_j;\mu)$ lie in finite-dimensional subspaces of V. Moreover, the topological structure of the set $C(\tilde{\lambda}_j;\mu)$ is related to the multiplicity of $\tilde{\lambda}_j$ as described in the next theorem.

Theorem 3.4. *Assume (A1)-(A3) hold and that the sets $C(\tilde{\lambda}_j;\mu)$ are given by (3.7).*
(i) If $\tilde{\lambda}_j \in (0, \mu)$ is a simple eigenvalue of (a, m), then $C(\tilde{\lambda}_j;\mu)$ consists of exactly two points.
(ii) If $\tilde{\lambda}_j \in (0, \mu)$ is an eigenvalue of (a, m) of multiplicity $m_j \geq 2$, then $C(\tilde{\lambda}_j;\mu)$ is diffeomorphic to an $(m_j - 1)$-dimensional sphere.

Proof. If the eigenvalue $\tilde{\lambda}_j \in (0, \mu)$ is simple, then $C(\tilde{\lambda}_j;\mu) = \{\pm(\mu - \tilde{\lambda}_j)^{1/2} e\}$ where e is an m-normalized eigenvector of the pair (a, m) corresponding to $\tilde{\lambda}_j$, so (i) holds.

If $\tilde{\lambda}_j$ is an eigenvalue of (a, m) of multiplicity $m_j \geq 2$, let $\{e_j, e_{j+1}, \ldots, e_{j+m_j-1}\}$

be an m-orthonormal basis of the eigenspace E_j. A calculation then shows that

$$C(\tilde{\lambda}_j;\mu) = \left\{ u \in V : u = \sum_{k=1}^{m_j} c_k e_{j+k-1} \text{ with } \sum_{k=1}^{m_j} c_k^2 = \mu - \tilde{\lambda}_j \right\},$$

which is diffeomorphic to an $(m_j - 1)$-dimensional sphere. Thus (ii) holds. □

The next theorem shows the bilinear form m provides an additional orthogonality relationship between the sets $C(\tilde{\lambda}_j;\mu)$.

Theorem 3.5. *Assume* (*A*1)-(*A*3) *hold and that the sets* $C(\tilde{\lambda}_j;\mu)$ *are given by* (3.7).
(i) Two sets $C(\tilde{\lambda}_i;\mu)$, $C(\tilde{\lambda}_j;\mu)$ *corresponding to distinct eigenvalues* $\tilde{\lambda}_i, \tilde{\lambda}_j \in (0,\mu)$, *are m-orthogonal. That is, if* u_i *is in* $C(\tilde{\lambda}_i;\mu)$ *and* u_j *is in* $C(\tilde{\lambda}_j;\mu)$, *with* $i \neq j$, *then* $m(u_i, u_j) = 0$.
(ii) If $u_i \in C(\tilde{\lambda}_i;\mu)$ *and* $u_j \in C(\tilde{\lambda}_j;\mu)$, *with* $i \neq j$, *then*

$$\|u_i - u_j\|_m = \sqrt{2\mu - (\tilde{\lambda}_i + \tilde{\lambda}_j)}. \tag{3.8}$$

Proof. The m-orthogonality of the eigenspaces E_j gives the first assertion. To obtain the second one, we see that for $u_i \in C(\tilde{\lambda}_i;\mu)$ and $u_j \in C(\tilde{\lambda}_j;\mu)$ we have

$$u_i = (\mu - \lambda_i)^{1/2} e_i \qquad \text{and} \qquad u_j = (\mu - \lambda_j)^{1/2} e_j$$

for some m-normalized eigenvectors e_i, e_j corresponding to $\tilde{\lambda}_i$, $\tilde{\lambda}_j$, respectively. It then follows that

$$\|u_i - u_j\|_m^2 = m(u_i,u_i) - 2m(u_i,u_j) + m(u_j,u_j) = (\mu - \tilde{\lambda}_i) + (\mu - \tilde{\lambda}_j) = 2\mu - (\tilde{\lambda}_i + \tilde{\lambda}_j),$$

as desired. □

Now let

$$C(\mu) := \bigcup_{0<\tilde{\lambda}_j<\mu} C(\tilde{\lambda}_j;\mu). \tag{3.9}$$

When (A1)-(A3) hold, part (i) of Theorem 3.2 says $C(\mu)$ is the set of non-zero-critical points of $\mathscr{G}(.;\mu)$. Since each set $C(\tilde{\lambda}_j;\mu)$ lies in a finite dimensional subspace of V, as noted above, consequently so do the sets $C(\mu)$ for any value μ in $(0,\infty)$.

As a consequence of the above results, the set of critical points of $\mathscr{G}(.;\mu)$ may be described as follows.

Corollary 3.6. *Assume (A1)-(A3) hold, $\mathscr{G}$ is given by* (3.1)*, and the set $C(\mu)$ is given by* (3.9)*. Then the set $C(\mu)$ is closed and bounded, consisting of a finite number of connected components and finitely many discrete points. In particular, if each eigenvalue $\tilde{\lambda}_j$ in $(0,\mu)$ is simple, then $C(\mu)$ consists of $2M$ points in V, where M is the number of eigenvalues in $(0,\mu)$.*

Thus, for fixed μ in $(0,\infty)$, the above results show that $C(\mu)$, the set of non-zero critical points of $\mathscr{G}(.;\mu)$, is the disjoint union of finitely many finite-dimensional spheres $C(\tilde{\lambda}_j;\mu)$, the spheres being pairwise m-orthogonal.

When the parameter μ increases, the number of critical points may increase. The following Figure 3.1 is a schematic bifurcation diagram for the critical points of $\mathscr{G}(.;\mu)$. In the figure,

$$C_j = \{(\tilde{\lambda}_j + s, s^{1/2}e) : s \geq 0, e \in E_j\}$$

is the bifurcation branch of the set of critical points corresponding to the j^{th} distinct eigenvalue $\tilde{\lambda}_j$ of (a,m); the sphere $C(\tilde{\lambda}_j;\mu)$ is thought of as a point on the branch C_j.

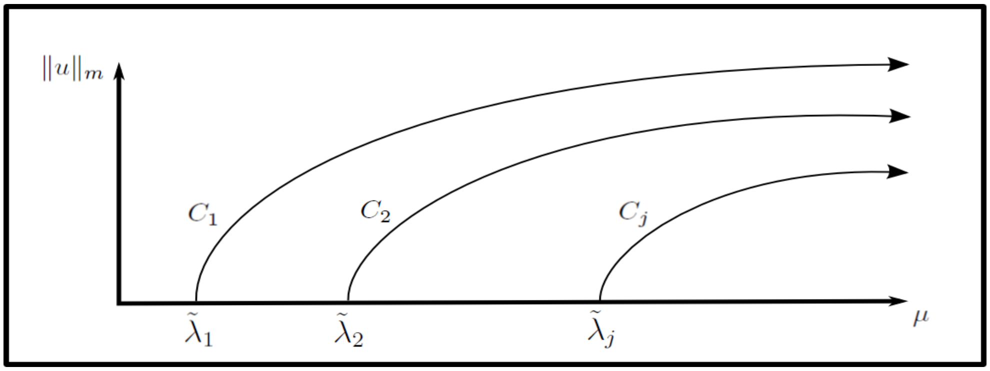

Figure 3.1: Bifurcation diagram for $\mathscr{G}(.;\mu)$

From Theorem 3.2 we see that 0 is a critical point for any value of μ, and it is the unique critical point when $\mu \leq \lambda_1$. As μ increases through an eigenvalue $\tilde{\lambda}_j$, a new sphere $C(\tilde{\lambda}_j;\mu)$ of critical points emanates from the origin and moves along the branch C_j. That is, each sphere of non-zero critical points, centered at the origin, persists and expands in V as the parameter μ increases, without any further bifurcations.

3.3 The Second and Higher Positive Eigenvalues

Theorem 3.2 of the previous section shows that when $\mu > \lambda_1$, the value $\alpha(\mu)$ of the unconstrained problem $(\mathcal{P}_\mu)$ yields the least eigenvalue λ_1 of (a, m) and the minimizers are associated eigenvectors. Our interest now is in obtaining variational principles for the second eigenvalue (consequently higher eigenvalues) and associated eigenvectors. In the first subsection, a penalization of the functional $\mathscr{G}(.;\mu)$ will be described which yields unconstrained variational principles for the second eigenvalue λ_2 of (a, m). This may be compared to the contrained variational principles given in the second subsection.

3.3.1 A Penalization Method for Finding λ_2

Suppose that we know an m-normalized eigenvector e_1 corresponding to the first eigenvalue λ_1 of (a, m). Let τ, called a *penalty parameter*, be a positive real number and let $\mathscr{G}_\tau : V \times (0, \infty) \to \mathbb{R}$ be the functional defined by

$$\mathscr{G}_\tau(u; \mu) := \mathscr{G}(u; \mu) + \tau m(e_1, u)^2 \tag{3.10}$$

where $\mathscr{G}$ is the functional given by (3.1).

Consider the problem $(\mathcal{P}_{\mu,\tau})$ of minimizing $\mathscr{G}_\tau(.; \mu)$ on V and finding

$$\alpha(\mu, \tau) := \inf_{u \in V} \mathscr{G}_\tau(u; \mu). \tag{3.11}$$

This is again an unconstrained variational problem.

A calculation similar to that as in the proof of Theorem 3.1 yields that this functional has first variation at u in the direction v given by

$$\delta\mathscr{G}_\tau(u; v; \mu) = 2\{a(u, v) + [m(u, u) - \mu]\, m(u, v) + \tau m(e_1, u) m(e_1, v)\} \tag{3.12}$$

Thus a vector $u \in V$ is a critical point of $\mathscr{G}_\tau(.; \mu)$ provided it satisfies

$$a(u, v) = [\mu - m(u, u)]\, m(u, v) - \tau m(e_1, u) m(e_1, v) \qquad \text{for all } v \in V. \tag{3.13}$$

Note that a critical point $\tilde{u}$ of $\mathscr{G}(.; \mu)$ will be a critical point of $\mathscr{G}_\tau(.; \mu)$ for all $\tau > 0$ provided it also satisfies $m(e_1, \tilde{u}) = 0$.

Theorem 3.2 shows that 0 is a minimizer of the functional $\mathscr{G}(.; \mu)$ for values μ in the interval $(0, \lambda_1]$. This will now translates into the following result for $\mathscr{G}_\tau$, which shows that when the least eigenvalue λ_1 is simple, the zero vector is always a critical point of this penalized functional and is a minimizer for a larger range of values of the parameter μ. Further results about the unconstrained variational principle $(\mathcal{P}_{\mu,\tau})$ are also summarized in the next theorem which show that the non-zero minimizers are eigenvectors associated to the second eigenvalue of the pair (a, m). First, let $\mu_c := \min\{\lambda_1 + 2\tau, \lambda_2\}$.

Theorem 3.7. *Assume* (A1)-(A3) *hold, the sets* $C(\tilde{\lambda}_j; \mu)$ *are given by* (3.7), *and* $\mathscr{G}_\tau$ *is defined by* (3.10).

(i) *If* $0 < \mu \leq \mu_c$ *and* $\tilde{\lambda}_1$ *is a simple eigenvalue of* (a, m), *then* 0 *is the unique minimizer of* $\mathscr{G}_\tau(.;\mu)$ *on* V *and the value of the problem* $(\mathcal{P}_{\mu,\tau})$ *is* $\alpha(\mu, \tau) = 0$.

(ii) *If* $\mu > \mu_c$, $\tau > \tilde{\lambda}_2 - \tilde{\lambda}_1$ *and* $\tilde{\lambda}_1$ *is a simple eigenvalue of* (a, m), *then the minimizers of* $\mathscr{G}_\tau(.;\mu)$ *on* V *are the vectors in the set* $C(\tilde{\lambda}_2; \mu)$ *and the value* $\alpha(\mu, \tau)$ *of the problem* $(\mathcal{P}_{\mu,\tau})$ *satisfies*

$$\alpha(\mu, \tau) = -\frac{1}{2}(\mu - \tilde{\lambda}_2)^2 > -\frac{1}{2}(\mu - \tilde{\lambda}_1)^2 = \alpha(\mu). \tag{3.14}$$

(iii) *If* $\mu > \mu_c$ *and* $\tilde{\lambda}_1$ *is an eigenvalue of multiplicity* $m_1 \geq 2$, *then the minimizers of* $\mathscr{G}_\tau(.;\mu)$ *on* V *are the vectors* u *in the set* $C(\tilde{\lambda}_1; \mu)$ *that also satisfy* $m(e_1, u) = 0$. *In this case, the value of the problems* $(\mathcal{P}_{\mu,\tau})$ *and* $(\mathcal{P}_\mu)$ *are the same and is given by*

$$\alpha(\mu, \tau) = \alpha(\mu) = -\frac{1}{2}(\mu - \tilde{\lambda}_1)^2. \tag{3.15}$$

Proof. Note that as the penalty parameter τ is positive, we have

$$\mathscr{G}_\tau(u; \mu) \geq \mathscr{G}(u; \mu) \qquad \text{for all } (u, \mu) \in V \times (0, \infty).$$

Evaluating $\mathscr{G}_\tau(.;\mu)$ at a critical point $u_j = (\mu - \tilde{\lambda}_j)^{1/2} e$ of $\mathscr{G}(.;\mu)$, with e an m-normalized eigenvector of (a, m) in E_j gives $\mathscr{G}_\tau(u_j;\mu) = -\frac{1}{2}(\mu-\tilde{\lambda}_j)^2+\tau(\mu-\tilde{\lambda}_j)m(e_1, e)^2$ which becomes

$$\mathscr{G}_\tau(u_j;\mu) = \begin{cases} -\frac{1}{2}(\mu - \tilde{\lambda}_1)^2 + \tau(\mu - \tilde{\lambda}_1) & \text{if} \quad m(e_1, e) = 1, \\ -\frac{1}{2}(\mu - \tilde{\lambda}_j)^2 & \text{if} \quad m(e_1, e) = 0. \end{cases} \tag{3.16}$$

When $\mu \leq \mu_c$, the only possible non-zero critical points u_j of $\mathscr{G}(.;\mu)$ are the points u_1 in E_1. From (3.16), we see that when $\tilde{\lambda}_1$ is simple, the corresponding critical value of $\mathscr{G}_\tau(.;\mu)$ at u_1 satisfies

$$\mathscr{G}_\tau(u_1;\mu) = -\frac{1}{2}(\mu - \tilde{\lambda}_1)^2 + \tau(\mu - \tilde{\lambda}_1) > 0$$

as $\mu < \tilde{\lambda}_1 + 2\tau$, so that $\alpha(\mu, \tau) = 0$ and (i) holds.

The inequalities $\mu > \mu_c$, $\tau > \tilde{\lambda}_2 - \tilde{\lambda}_1$ imply $\mu > \tilde{\lambda}_2$, when $\tilde{\lambda}_1$ is simple. Thus the points u_1, u_2 are critical points of $\mathscr{G}(.;\mu)$, and consequently of $\mathscr{G}_\tau(.;\mu)$, in this case. When $\tilde{\lambda}_1$ is simple, by adding $\mu - \tilde{\lambda}_1$ to both sides of the inequality $\mu - \tilde{\lambda}_2 < \mu - \tilde{\lambda}_1$, we get

$$[2\mu - (\tilde{\lambda}_1 + \tilde{\lambda}_2)] < 2(\mu - \tilde{\lambda}_1).$$

With $\tau > \tilde{\lambda}_2 - \tilde{\lambda}_1$, we then obtain

$$(\mu - \tilde{\lambda}_1)^2 - (\mu - \tilde{\lambda}_2)^2 = [\tilde{\lambda}_2 - \tilde{\lambda}_1][2\mu - (\tilde{\lambda}_1 + \tilde{\lambda}_2)] < 2\tau(\mu - \tilde{\lambda}_1),$$

Rearrangement then gives

$$\mathscr{G}_\tau(u_2;\mu) = -\frac{1}{2}(\mu - \tilde{\lambda}_2)^2 < -\frac{1}{2}(\mu - \tilde{\lambda}_1)^2 + \tau(\mu - \tilde{\lambda}_1) = \mathscr{G}_\tau(u_1;\mu),$$

from (3.16), so that the vectors u_2 in $C(\tilde{\lambda}_2;\mu)$ are minimizers of $\mathscr{G}_\tau(.;\mu)$, and (ii) holds.

When $\mu > \mu_c$ and $\tilde{\lambda}_1$ is non-simple, from (3.16) we see that the corresponding critical value of $\mathscr{G}_\tau(.;\mu)$ at any critical point $u_1 = (\mu - \tilde{\lambda}_1)^{1/2} e$ of $\mathscr{G}(.;\mu)$, with $e \in E_1$ and $m(e_1, u_1) = 0$, satisfies

$$\mathscr{G}_\tau(u_1;\mu) = -\frac{1}{2}(\mu - \tilde{\lambda}_1)^2 < -\frac{1}{2}(\mu - \tilde{\lambda}_j)^2 = \mathscr{G}_\tau(u_j;\mu)$$

for any critical point u_j in $C(\tilde{\lambda}_j;\mu)$, $j \geq 2$. Also from (3.16)

$$\mathscr{G}_\tau(u_1;\mu) < -\frac{1}{2}(\mu - \tilde{\lambda}_1)^2 + \tau(\mu - \tilde{\lambda}_1)$$

for any value of τ, so that (iii) now follows. $\square$

Theorem 3.7 shows that when λ_1 is a simple eigenvalue of (a, m), $\tau > \lambda_2 - \lambda_1$, and $\mu > \mu_c$, then the minimizers of $\mathscr{G}_\tau(.;\mu)$ are eigenvectors of (a, m) corresponding to the eigenvalue λ_2. That is, this penalized functional provides an unconstrained variational problem whose minimizers yield the second eigenvalue and associated eigenvectors, and we point out that the difference between the value of the penalized problem $(\mathcal{P}_{\mu,\tau})$ and that of $(\mathcal{P}_\mu)$ is a function of the difference $d := \lambda_2 - \lambda_1$ between the first two eigenvalues of (a, m), in the case λ_1 is simple.

3.3.2 Constrained Variational Principles

Suppose now that we know a finite sequence $\mathcal{E}_J := \{e_1, e_2, \ldots, e_J\}$ of m-normalized eigenvectors corresponding to the first J successive, smallest, strictly positive eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_J$ of (a, m), which obey

$$m(e_i, e_j) = \delta_{ij} \qquad \text{for } 1 \leq i, j \leq J. \tag{3.17}$$

Let $V_J := \text{span } \mathcal{E}_J$. Then there is a penalized functional similar to $\mathscr{G}_\tau$ with the property that when μ and τ are sufficiently large, the minimizers of the functional will be eigenvectors of (a, m) that are m-orthogonal to V_J. This yields unconstrained variational principles for higher eigenvalues. Instead of writing down such unconstrained problems, we consider the following constrained variational principles so that a comparison may be made.

Define W_J to be the m-orthogonal complement of V_J. That is, let

$$W_J = \{u \in V : m(u, e_j) = 0, \quad \forall\, 1 \leq j \leq J\}, \tag{3.18}$$

and consider the problem $(\mathcal{P}_\mu(J))$ of minimizing the functional $\mathscr{G}(.;\mu)$ given by (3.1) restricted to the subspace W_J and finding

$$\alpha_J(\mu) = \inf_{u \in W_J} \mathscr{G}(u; \mu). \tag{3.19}$$

Minimizers of this variational principle will be eigenvectors of the pair (a, m) corresponding to the next smallest strictly positive eigenvalue of (a, m), when these exist.

The results about this constrained variational principle are as follow.

Theorem 3.8. *Assume (A1)-(A3) hold, and* $\mathscr{G}$, W_J, $\alpha_J(\mu)$ *are given by* (3.1), (3.18), (3.19), *respectively. If there exists* $w \in W_J$ *with* $m(w,w) > 0$, *then there is another positive eigenvalue* λ_{J+1} *of* (a, m), *with* $\lambda_{J+1} \geq \lambda_J$, *and a corresponding eigenvector* e *in* W_J. *In this case,*

(i) the value of the problem $(\mathcal{P}_\mu(J))$ *is* $\alpha_J(\mu) = 0$ *when* $\mu \leq \lambda_{J+1}$, *and* $\alpha_J(\mu) = -\frac{1}{2}(\mu - \lambda_{J+1})^2$ *when* $\mu > \lambda_{J+1}$, *and*

(ii) the minimizers of $(\mathcal{P}_\mu(J))$ *are* 0 *when* $\mu \leq \lambda_{J+1}$, *and* $(\mu - \lambda_{J+1})^{1/2} e$ *when* $\mu > \lambda_{J+1}$, *with* e *an* m*-normalized eigenvector in* W_J *corresponding to* λ_{J+1}.

Proof. When such a $w \in W_J$ exists, Theorem 4.2 of Auchmuty [7] gives the existence of a next smallest eigenvalue $\lambda_{J+1} \geq \lambda_J$, and corresponding eigenvector e in W_J.

By (i) of Theorem 3.1, the functional $\mathscr{G}(.;\mu)$ is continuous, coercive and weakly *l.s.c.* on the closed subspace W_J. Hence, $\mathscr{G}(.;\mu)$ attains a finite infimum on W_J.

Let $\tilde{u} \in W_J$ be a minimizer of $\mathscr{G}(.;\mu)$ on W_J. For $v \in W_J$ we always have $\tilde{u} + tv \in W_J$ for any $t \in \mathbb{R}$ as W_J is a subspace of V, so part (ii) of Theorem 3.1 implies the function $\varphi(t) = \mathscr{G}(\tilde{u} + tv; \mu)$ is a differentiable function with $t = 0$ a critical point. By definition of first variation we then have

$$\delta\mathscr{G}(\tilde{u}; v; \mu) = \frac{d}{dt}\varphi(t)\Big|_{t=0} = 0$$

holding for each $v \in W_J$.

By the Lagrange multiplier rule, the minimizer $\tilde{u}$ of $\mathscr{G}(.;\mu)$ on W_J satisfies

$$\delta\mathscr{G}(\tilde{u}; v; \mu) = \sum_{j=1}^{J} \beta_j m(\tilde{u}, v) \qquad \text{for all } v \in V.$$

Thus, for $v \in V_J$ we obtain $\delta\mathscr{G}(\tilde{u}; v; \mu) = 0$ as $\tilde{u} \in W_J$. Hence,

$$\delta\mathscr{G}(\tilde{u}; v; \mu) = 0 \qquad \text{for all } v \in V,$$

so that $\tilde{u}$ is a critical point of $\mathscr{G}(.; \mu)$ on V. Since $\tilde{u} \in W_J$, the desired results follow as in the proof of Theorem 3.2 □

As in the end of the Section 3.1, the constrained variational principle $(\mathcal{P}_\mu(J))$ can be used to find both upper and lower bounds on λ_{J+1}; this is given in the next corollary whose proof is similar to that of Corollary 3.3.

Corollary 3.9. *Assume* (A1)-(A3) *hold, and* $\mathscr{G}$, W_J, *and* $\alpha_J(\mu)$ *are given by* (3.1), (3.18), *and* (3.19), *respectively.*

(i) *If* $\alpha_J(\mu) = 0$, *then* $\mu \leq \lambda_{J+1}$.

(ii) *If there exists* $\tilde{v} \in W_J$ *with* $\mathscr{G}(\tilde{v}; \mu) < 0$, *then*

$$\lambda_{J+1} = \inf_{\substack{\mathscr{G}(v;\mu)<0 \\ v \in W_J}} \left[\mu - \sqrt{-2\mathscr{G}(v; \mu)}\right] \leq \mu - \sqrt{-2\mathscr{G}(\tilde{v}; \mu)}. \tag{3.20}$$

Chapter 4

Morse and Null Indices of Critical Points of $\mathscr{G}(.;\mu)$

We point out that the results obtained in Chapter 3 were all based on an analysis which relied only on first variations of the functional $\mathscr{G}(.;\mu)$. In this chapter, the functional $\mathscr{G}(.;\mu)$ is shown to be twice Gâteaux differentiable on V and then an analysis based on second variations will enable a Morse index theory and the identification of the type of a critical point for $\mathscr{G}(.;\mu)$.

In particular, it is shown that a critical point of $\mathscr{G}(.;\mu)$ associated to an eigenvalue λ of (a,m) is non-degenerate if and only if λ is simple. Furthermore, the Morse index of the critical point is related to the number of eigenvalues that are less than λ. This may be compared to the Courant-Fischer-Weyl min-max results for variational methods associated with Rayleigh quotients.

Consequently, these results supplement those previously presented in Section 3.2 on the bifurcation of critical points of the functional $\mathscr{G}(.;\mu)$.

4.1 Morse and Null Indices

To enable a Morse index theory, the following formula for the Hessian form of $\mathscr{G}(.;\mu)$ will needed; the result complements Theorem 3.1.

Theorem 4.1. *Assume (A1), (A2) hold, and $\mathscr{G}(.;\mu)$ is given by (3.1). Then $\mathscr{G}(.;\mu)$ is twice Gâteaux differentiable on V with second variation at u in the directions v, w given by*

$$\delta^2\mathscr{G}(u;v,w;\mu) = 2a(v,w) + 2\big[m(u,u) - \mu\big]m(v,w) + 4m(u,v)m(u,w). \tag{4.1}$$

Proof. Fix $v \in V$ and consider the functionals $\mathscr{F}_1$, $\mathscr{F}_2$, and $\mathscr{F}_3$ defined on V by

$$\mathscr{F}_1(u) = 2a(u,v), \quad \mathscr{F}_2(u) = 2\big[m(u,u) - \mu\big], \quad \text{and} \quad \mathscr{F}_3(u) = m(u,v).$$

For $u, w \in V$, a computation then gives

$$\delta\mathscr{F}_1(u;w) = 2a(v,w), \qquad \delta\mathscr{F}_2(u;w) = 4m(u,w), \qquad \text{and} \qquad \delta\mathscr{F}_3(u;w) = 2m(v,w).$$

This, the classical product rule and definitions of first and second variations give

$$\begin{aligned}
\delta^2\mathscr{G}(u;v,w) =& \frac{\partial^2}{\partial t_2 dt_1}\mathscr{G}(u + t_1 v + t_2 w)\Big|_{t_1=t_2=0} \\
=& \frac{\partial}{\partial t_2}\Big[\delta\mathscr{G}(u + t_2 w; v)\Big]_{t_2=0} \\
=& \delta\mathscr{F}_1(u;w) + 2\big[m(u,u) - \mu\big]\delta\mathscr{F}_3(u;w) + \big[\delta\mathscr{F}_2(u;w)\big]m(u,v) \\
=& 2a(v,w) + 2\big[m(u,u) - \mu\big]m(v,w) + 4m(u,v)m(u,w),
\end{aligned}$$

as (3.3) holds, so that we obtain (4.1). Here we have suppressed the parameter μ at certain places. It then follows that $\mathscr{G}(.;\mu)$ is twice Gâteaux differentiable on V as each term of the Hessian $\delta^2\mathscr{G}(u;.,.)$ is a continuous bilinear form on V. $\square$

When (A1) and (A2) hold, Theorem 4.1 shows the Hessian $\delta\mathscr{G}(u;.,.;\mu)$ of $\mathscr{G}(.;\mu)$ in (4.1) satisfies (M1). The next result shows the Hessian also satisfies (M2) at critical points of $\mathscr{G}(.;\mu)$ provided (A1)-(A4) hold. Thus, in this case, the Hessian is a semi-coercive bilinear form on V, so the results of Theorem 2.3 may be used.

As we started doing so in Section 3.2, we denote the eigenvalues of (a,m) by λ_j when counting multiplicities, and the distinct eigenvalues by $\tilde{\lambda}_j$. The multiplicity of the j^{th} distinct eigenvalue $\tilde{\lambda}_j$ of (a,m) is denoted m_j. Moreover, we now let $\sigma(a,m)$ denote the set of distinct eigenvalues of (a,m) and is called the *spectrum* of (a,m).

Theorem 4.2. *Assume* (A1)-(A4) *hold, and* $\mathscr{G}$ *is given by* (3.1).

(i) 0 *is a non-degenerate critical point of* $\mathscr{G}(.;\mu)$ *if and only if* $\mu \notin \sigma(a,m)$. *The Morse and null index of* 0 *are, respectively,*

$$i(0;\mu) = \begin{cases} 0 & \text{if} \quad \mu \le \lambda_1, \\ \sum_{\tilde{\lambda}_j<\mu} m_j & \text{if} \quad \mu > \lambda_1, \end{cases} \qquad \text{and} \qquad i_0(0;\mu) = \begin{cases} 0 & \text{if} \quad \mu \notin \sigma(a,m), \\ m_j & \text{if} \quad \mu = \tilde{\lambda}_j. \end{cases} \tag{4.2}$$

(ii) When $u_k = (\mu - \tilde{\lambda}_k)^{1/2} e$ *is a non-zero critical point of* $\mathscr{G}(.;\mu)$, *with* e *an* m-*normalized eigenvector associated to* $\tilde{\lambda}_k \in (0,\mu)$, *then* u_k *is non-degenerate if and only if* $\tilde{\lambda}_k$ *is a simple eigenvalue of* (a,m). *The Morse and null index of* u_k *are, respectively,*

$$i(u_k;\mu) = \begin{cases} 0 & \text{if} \quad k = 1, \\ \sum_{j=1}^{k-1} m_j & \text{if} \quad k > 1, \end{cases} \qquad \text{and} \qquad i_0(u_k;\mu) = m_k - 1. \tag{4.3}$$

Proof. By Theorem 1.1 there is an m-orthonormal basis $\mathcal{E} := \{e_j : j \geq 1\}$ of V consisting of eigenvectors corresponding to strictly positive eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ of (a, m), counting multiplicities.

Taking $u = 0$ in (4.1) yields

$$\delta^2 \mathscr{G}(0; v, w; \mu) = 2a(v, w) - 2\mu m(v, w) \tag{4.4}$$

for all $v, w \in V$. Denote this Hessian form by $h_0(\mu)$, so then $h_0(\mu)$ is seen to satisfy (M1) in Section 2.1.2 as it is a sum of continuous, symmetric bilinear forms on V. When w is set to be equal to v in (4.4) we get

$$h_0(\mu)(v, v) \geq 2k_0 \|v\|_V^2 - 2\mu \mathscr{M}(v)$$

by the coercivity of $a(.,.)$, where $\mathscr{M}(v) := m(v, v)$. Thus the Hessian $h_0(\mu)$ of $\mathscr{G}(.;\mu)$ at the origin satisfies a Gårding type inequality of the form (2.3), i.e., it satisfies (M2) in Section 2.1.2. The results in Theorem 2.3 then apply to the pair of bilinear forms $(h_0(\mu), m)$.

By taking $v = e_j$ in (4.4), with $e_j \in \mathcal{E}$, we obtain

$$h_0(\mu)(e_j, w) = 2(\lambda_j - \mu) m(e_j, w)$$

for all $w \in V$, so that $\mathcal{E} := \{e_j : j \geq 1\}$ is an m-orthonormal set of eigenvectors for the pair $(h_0(\mu), m)$ corresponding to the eigenvalues $2(\lambda_j - \mu)$. By Theorem 2.3, the Hessian form $h_0(\mu)$ is non-degenerate if and only if $\lambda_j - \mu \neq 0$ for all $j \geq 1$. Hence, 0 is a non-degenerate critical point of $\mathscr{G}(.;\mu)$ if and only if μ is not an eigenvalue of (a, m). Also from Theorem 2.3, the Morse index of 0 is equal to the number of negative eigenvalues $2(\lambda_j - \mu)$, counting multiplicities, of $(h_0(\mu), m)$, and the null index of 0 is equal to the multiplicity of 0 as an eigenvalue of $(h_0(\mu), m)$. This yields the quantities in (4.2), so that (i) holds.

Taking u in (4.1) to be a non-zero critical point $u_k = (\mu - \tilde{\lambda}_k)^{1/2} e$ corresponding to an eigenvalue $\tilde{\lambda}_k \in (0, \mu)$ and e an associated m-normalized eigenvector gives

$$\delta^2 \mathscr{G}(u_k; v, w; \mu) = 2a(v, w) - 2\tilde{\lambda}_k m(v, w) + 4m(u_k, v) m(u_k, w) \tag{4.5}$$

for all $v, w \in V$. Denoting this Hessian form by $h_k(\mu)$, we thus have that $h_k(\mu)$ satisfies (M1) in Section 2.1.2. By setting $w = v$ in (4.5) we obtain

$$h_k(\mu)(v, v) \geq 2k_0 \|v\|_V^2 - 2\tilde{\lambda}_k \mathscr{M}(v)$$

by the coercivity of $a(.,.)$. The Hessian $h_k(\mu)$ of $\mathscr{G}(.;\mu)$ at a non-zero critical point u_k therefore satisfies (M2), so that the results in Theorem 2.3 may be applied to the pair $(h_k(\mu), m)$.

Without loss of generality we may assume the eigenvector e in u_k is $e = e_j$, for some eigenvector $e_j \in \mathcal{E}$ associated to the eigenvalue $\lambda_j = \tilde{\lambda}_k$. Taking $v = e_i$ in (4.5) gives

$$\begin{aligned} h_k(\mu)(e_i, w) &= 2(\lambda_i - \lambda_j) m(e_i, w) + 4(\mu - \lambda_j)\delta_{ij} m(e_j, w) \\ &= \begin{cases} 4(\mu - \lambda_j) m(e_i, w) & \text{if } i = j \\ 2(\lambda_i - \lambda_j) m(e_i, w) & \text{if } i \neq j \end{cases}. \end{aligned}$$

This shows $\mathcal{E}$ is an m-orthonormal set of eigenvectors for the pair $(h_k(\mu), m)$ corresponding to the eigenvalue $4(\mu - \lambda_j)$ plus the eigenvalues $2(\lambda_i - \lambda_j)$ for $i \neq j$. By Theorem 2.3, the Hessian form $h_k(\mu)$ is non-degenerate if and only if $\lambda_i - \lambda_j \neq 0$ for all $i \neq j$. Thus, u_k is a non-degenerate critical point of $\mathscr{G}(.;\mu)$ if and only if λ_j is a simple eigenvalue of (a, m). Also from Theorem 2.3 the Morse index of u_k is equal to the number of negative eigenvalues $2(\lambda_i - \lambda_j)$ of $(h_k(\mu), m)$, which is precisely the number of eigenvalues, counting multiplicies, of the pair (a, m) which are strictly less than $\lambda_j = \tilde{\lambda}_k$. Furthermore, the null index of u_k is equal to the multiplicity of 0 as an eigenvalue of $(h_k(\mu), m)$, which is precisely $(m_k - 1)$. Assertion (ii) then holds. $\square$

4.2 More on the Bifurcation of Critical Points

As we shall now see, for each distinct eigenvalue $\tilde{\lambda}_j$ of (a, m), Theorem 4.2 characterizes the points belonging to the set $C(\tilde{\lambda}_j; \mu)$.

From Theorems 3.2 and 4.2, when $\mu > \lambda_1$, we see that points u_1 in the set $C(\lambda_1; \mu)$ are minimizers of $\mathscr{G}(.;\mu)$ with Morse and null index, respectively,

$$i(u_1; \mu) = 0 \qquad \text{and} \qquad i_0(u_1; \mu) = m_1 - 1.$$

Considering the bifurcation diagram in Figure 3.1, we have that these indices are invariant as $C(\lambda_1; \mu)$ moves along the branch C_1. That is, for any value of $\mu > \lambda_1$, the Morse index is the same for all points in set $C(\lambda_1; \mu)$; the set $C(\lambda_1; \mu)$ is considered a point on the bifurcation branch C_1. The same is true for null indices. Thus, when λ_1 is a simple eigenvalue, i.e., $m_1 = 1$, then the set $C(\lambda_1; \mu)$ will consist of two non-degenerate critical points of $\mathscr{G}(.;\mu)$ with zero Morse index.

When $\mu > \tilde{\lambda}_j$, where $\tilde{\lambda}_j \in \sigma(a, m)$ and $j \geq 2$, each of the points u_j on the sphere

$C(\tilde{\lambda}_j; \mu)$ is a saddle point of $\mathscr{G}(.; \mu)$ with Morse and null index, respectively,

$$i(u_j; \mu) = \sum_{k=1}^{j-1} m_k \qquad \text{and} \qquad i_0(u_j; \mu) = m_j - 1.$$

These indices are invariant as $C(\tilde{\lambda}_j; \mu)$ moves along the branch C_j. That is, for any value $\mu > \tilde{\lambda}_j$, the Morse index and null index are the same for each of the points in the set $C(\tilde{\lambda}_j; \mu)$. As in the case for $C(\lambda_1; \mu)$, when $\tilde{\lambda}_j$ is simple we see that the set $C(\tilde{\lambda}_j; \mu)$ will consist of two non-degenerate critical points of $\mathscr{G}(.; \mu)$, but, in this case, with Morse index equal to the number of eigenvalues strictly less than $\tilde{\lambda}_j$, counting multiplicities.

For the trivial branch, part (i) of Theorem 4.2 shows that when $\mu = \tilde{\lambda}_j$ the null index of 0 is $i_0(0; \mu) = m_j$, so that the origin will be a degenerate critical point of $\mathscr{G}(.; \tilde{\lambda}_j)$ for any $\tilde{\lambda}_j \in \sigma(a, m)$. As μ passes through $\tilde{\lambda}_j$, that is, $\tilde{\lambda}_j < \mu < \tilde{\lambda}_{j+1}$, an $(m_j - 1)$-dimensional sphere in V bifurcates from 0 and the null index of 0 is zero, while the Morse index of 0 is equal to the number of eigenvalues strictly less than μ, counting multiplicities. In other words, as μ passes through a distinct eigenvalue $\tilde{\lambda}_j$, the Morse index of the origin 0 increases by m_j.

Chapter 5

Indefinite Weighted Eigenproblems

So far in this dissertation, we have considered the (a, m) eigenproblem in the case where the functional $\mathscr{M}$, that is, the quadratic form associated to m, is positive on the entire space V. In this chapter we shall show our analysis extends to the case where $\mathscr{M}$ is allowed to take both positive and negative values, so we will generally require the bilinear form m to satisfy the following condition:

(A5): There exist $v_1, v_2 \in V$ such that $\mathscr{M}(v_1) < 0 < \mathscr{M}(v_2)$.

An eigenproblem for (a, m) where a and m satisfies (A1), (A2), and (A5) will be called an *indefinite eigenproblem*.

5.1 Eigenvalues λ_1^+ and λ_1^- Closest to the Origin

Assume we have an indefinite eigenproblem for (a, m). Results in [7] show there are sets of positive and negative eigenvalues for the pair (a, m). Thus we let λ_1^+ be the least, strictly positive eigenvalue of (a, m) and λ_1^- the greatest, strictly negative eigenvalue of (a, m), so that the eigenvalues of (a, m) satisfy

$$\cdots \leq \lambda_2^- \leq \lambda_1^- < 0 < \lambda_1^+ \leq \lambda_2^+ \leq \cdots$$

These exist from Theorem 3.1 of [7].

Consider the problem $(\mathcal{P}(\mu))$ of minimizing the functional $\mathscr{G}(.;\mu)$ defined by (3.1) on V and finding the value $\alpha(\mu)$ given by (3.2). As the results in Theorem 3.1 hold when conditions (A1) and (A2) are satisfied, they remain valid when (A5) also holds, and the analogue of Theorem 3.2 is as follows for the unconstrained problem $(\mathcal{P}(\mu))$.

Theorem 5.1. *Assume* (A1), (A2), *and* (A5) *hold, and* $\mathscr{G}$ *is defined by* (3.1). *Let* λ_1^+ *be the least positive eigenvalue of* (a, m).

(i) 0 *is the unique critical point of* $\mathscr{G}(.;\mu)$ *when* $0 < \mu \leq \lambda_1^+$, *and* 0 *and the points* $(\mu - \lambda_j^+)^{1/2} e$ *are the critical points of* $\mathscr{G}(.;\mu)$ *when* $\mu > \lambda_1^+$, *where* e *is an* m*-normalized eigenvector corresponding to the eigenvalue* λ_j^+ *in the interval* $(0, \mu)$.

(ii) *The critical values of* $\mathscr{G}(.;\mu)$ *are* 0 *for any value of* $\mu > 0$, *and* 0 *and* $-\frac{1}{2}(\mu - \lambda_j^+)^2$ *when* $\mu > \lambda_1^+$ *and* λ_j^+ *is in the interval* $(0, \mu)$.

(iii) *The minimizers of* $\mathscr{G}(.;\mu)$ *on* V *are* 0 *when* $0 < \mu \leq \lambda_1^+$, *and* $(\mu - \lambda_1^+)^{1/2} e$ *when* $\mu > \lambda_1^+$, *with* e *an* m*-normalized eigenvector of* (a, m) *corresponding to* λ_1^+.

(iii) *The value* $\alpha(\mu)$ *of the problem* $(\mathcal{P}(\mu))$ *is of the same form as that of* $(\mathcal{P}_\mu)$ *for all values of* $\mu \in (0, \infty)$. *That is,* $\alpha(\mu) = 0$ *when* $0 < \mu \leq \lambda_1^+$, *and* $\alpha(\mu) = -\frac{1}{2}(\mu - \lambda_1^+)^2$ *when* $\mu > \lambda_1^+$.

Proof. As in the proofs of Theorem 3.2 we see that a critical point u of $\mathscr{G}(.;\mu)$ satisfies

$$a(u, v) = [\mu - m(u, u)] m(u, v) \qquad \text{for all } v \in V. \tag{5.1}$$

Thus 0 is always a critical point, and non-zero critical points u are eigenvectors corresponding to eigenvalues $\lambda = \mu - m(u,u)$. By the coercivity of $a(.,.)$, the eigenvalues of (a,m) are non-zero. If $\lambda = \lambda^+$ is a strictly positive eigenvalue of (a,m), then put $v = u$ in (5.1) to find

$$a(u,u) = [\mu - m(u,u)]m(u,u) = \lambda^+(\mu - \lambda^+)$$

which implies $\lambda^+(\mu - \lambda^+) > 0$ as $a(.,.)$ is coercive on V. This shows $\lambda^+ < \mu$. From (5.1), we also see that if u an eigenvector associated with a positive eigenvalue λ^+ of (a,m), then $m(u,u) > 0$.

If $\lambda = \lambda^-$ is a strictly negative eigenvalue of (a,m), then substituting a corresponding eigenvector u for v in (5.1) gives $\lambda^-(\mu - \lambda^-) > 0$, again, by the coercivity of $a(.,.)$. This implies $\mu < \lambda^-$, which then implies $\mu < \lambda^- < 0 < \mu$, a contradiction. Therefore, the only possible non-zero critical points of $\mathscr{G}(.;\mu)$ are eigenvectors of (a,m) corresponding to strictly positive eigenvalues λ_j that are strictly less than μ, and then (i) holds as in the proof of Theorem 3.2.

Also as in Theorem 3.2, minimizers of $\mathscr{G}(.;\mu)$ exist and are critical points. Therefore, when $\mu \in (0, \lambda_1^+]$, we conclude from the above work that 0 is the only critical point of $\mathscr{G}(.;\mu)$ with value zero. When $\mu > \lambda_1^+$, the non-zero critical points of $\mathscr{G}(.;\mu)$ are the points $(\mu - \lambda_j^+)^{1/2}e$ with e an m-normalized eigenvector corresponding to the j^{th} smallest, strictly positive eigenvalue λ_j^+ (counting multiplicities) of (a,m) less than μ. By considering the corresponding critical values, assertions (ii), (iii), and (iv) then follow. $\square$

For an indefinite eigenproblem, the negative eigenvalues λ_j^- of (a,m) are precisely the positive eigenvalues of $(a,-m)$. Thus negative eigenvalues of (a,m) may be found by considering the problem $(\mathcal{P}_-(\mu))$ of minimizing $\mathscr{G}_-(.;\mu)$, with $\mu > 0$, defined by

$$\mathscr{G}_-(u;\mu) := a(u,u) + \mu m(u,u) + \frac{1}{2} m(u,u)^2 \tag{5.2}$$

and finding

$$\alpha_-(\mu) := \inf_{u \in V} \mathscr{G}_-(u;\mu). \tag{5.3}$$

The following results about this unconstrained problem for finding the greatest negative eigenvalue of (a,m) follow from the previous theorem.

Corollary 5.2. *Assume (A1), (A2), and (A5) hold, and $\mathscr{G}_-$ is defined by (5.2). Let λ_1^- be the negative eigenvalue of (a,m) closest to zero.*

(i) 0 is the unique critical point of $\mathscr{G}_-(.;\mu)$ when $0 < \mu \leq -\lambda_1^-$, and 0 and the points $(\mu + \lambda_j^-)^{1/2} e$ are the critical points of $\mathscr{G}_-(.;\mu)$ when $\mu > -\lambda_1^-$, where e is an eigenvector of (a,m) corresponding to the eigenvalue λ_j^- in the interval $(-\mu, 0)$ satisfying $m(e,e) = -1$, .

(ii) The critical values of $\mathscr{G}_-(.;\mu)$ are 0 for any $\mu > 0$, and 0 and $-\frac{1}{2}(\mu + \lambda_j^-)^2$ when $\mu > -\lambda_1^-$ and λ_j^- is in the interval $(-\mu, 0)$.

(iii) The minimizer(s) of $\mathscr{G}_-(.;\mu)$ on V are 0 when $0 < \mu \leq -\lambda_1^-$, and the vectors $(\mu + \lambda_1^-)^{1/2} e$ when $\mu > -\lambda_1^-$, where e is an eigenvector of (a,m) corresponding to λ_1^- satisfying $m(e,e) = -1$.

(iv) The value $\alpha_-(\mu)$ of the problem $(\mathcal{P}_-(\mu))$ is $\alpha_-(\mu) = 0$ when $0 < \mu \leq -\lambda_1^-$, and $\alpha_-(\mu) = -\frac{1}{2}(\mu + \lambda_1^-)^2$, when $\mu > -\lambda_1^-$.

5.2 Morse and Null Indices for the Indefinite Case

For indefinite eigenproblems, we shall evaluate Morse and null indices of critical points of functionals associated to these problems by restricting the domain of these functionals to a certain subspace of V. In particular, we shall consider the restricted domain to be the closed subspace generated by the collection of eigenvectors of (a, m).

Consider the following decomposition

$$V = V_+ \oplus_a V_0 \oplus_a V_- \tag{5.4}$$

given by Corollary 4.5 of [7] in the case of an indefinite eigenproblem for (a, m), i.e., when conditions (A1), (A2) and (A5) hold. Here $\oplus_a$ indicates an a-orhogonal direct sum, V_+ is the closed subspace of V generated by the eigenvectors $\mathcal{E}_+ := \{e_j^+ : j \in J_+\}$ associated with strictly positive eigenvalues λ_j^+ of (a, m), V_- is the closed subspace of V generated by eigenvectors $\mathcal{E}_- := \{e_j^- : j \in J_-\}$ associated with strictly negative eigenvalues λ_j^- of (a, m), and $V_0 = N(m)$ is the null space of m. Now let

$$W := V_+ \oplus_a V_- \tag{5.5}$$

be the closed subspace generated by the eigenvectors of (a, m), and let $\sigma_+(a, m)$ and $\sigma_-(a, m)$ denote, respectively, the collection of distinct strictly positive and strictly negative eigenvalues $\tilde{\lambda}_j^+$ and $\tilde{\lambda}_j^-$ of (a, m). Thus the spectrum $\sigma(a, m)$ becomes

$$\sigma(a, m) = \sigma_+(a, m) \cup \sigma_-(a, m) \tag{5.6}$$

the union of its positive and negative parts. Also let m_j^+ and m_j^- denote, respectively, the multiplicity of the j^{th}-distinct strictly positive and strictly negative eigenvalue $\tilde{\lambda}_j^+$, $\tilde{\lambda}_j^-$ of (a, m). The analogue of Theorem 4.2 is as follows.

Theorem 5.3. *Assume* $(A1)$, $(A2)$, *and* $(A5)$ *hold, the subspace* W *is given by* (5.5), *and* $\mathscr{G} : W \times (0, \infty) \to \mathbb{R}$ *is defined by* (3.1).

(i) 0 *is a non-degenerate critical point of* $\mathscr{G}(.;\mu)$ *if and only if* $\mu \notin \sigma_+(a, m)$. *The Morse and null index of* 0 *are, respectively,*

$$i(0;\mu) = \begin{cases} 0 & \text{if} \quad \mu \leq \tilde{\lambda}_1^+ \\ \sum_{\tilde{\lambda}_j^+ < \mu} m_j^+ & \text{if} \quad \mu > \tilde{\lambda}_1^+ \end{cases} \quad \text{and} \quad i_0(0;\mu) = \begin{cases} 0 & \text{if} \quad \mu \notin \sigma_+(a, m) \\ m_j^+ & \text{if} \quad \mu = \tilde{\lambda}_j^+ \end{cases}. \tag{5.7}$$

(ii) *When* $u_k^+ = (\mu - \tilde{\lambda}_k^+)^{1/2} e$ *is a non-zero critical point of* $\mathscr{G}(.;\mu)$, *with* e *an* m *normalized eigenvector associated to* $\tilde{\lambda}_k^+ \in (0, \mu)$, *then* u_k^+ *is non-degenerate if and only if* $\tilde{\lambda}_k^+$ *is a simple eigenvalue of* (a, m). *The Morse and null index of* u_k^+ *are, respectively,*

$$i(u_k^+;\mu) = \begin{cases} 0 & \text{if} \quad k = 1 \\ \sum_{j=1}^{k-1} m_j^+ & \text{if} \quad k > 1 \end{cases} \quad \text{and} \quad i_0(u_k^+;\mu) = m_k^+ - 1. \tag{5.8}$$

Proof. Define the bilinear form $|m| : W \times W \to \mathbb{R}$, the absolute value of the bilinear form m, as follows. For an eigenvector e corresponding to the eigenvalue λ, let

$$|m|(e, e) := \begin{cases} m(e, e) & \text{if} \quad \lambda > 0 \\ -m(e, e) & \text{if} \quad \lambda < 0 \end{cases}$$

and extend the definition of $|m|$ to the entire space W using the bilinearity of m. We note that for all $w \in W$,

$$a(e,w) = \lambda m(e,w) = \begin{cases} \lambda |m|(e,w) & \text{if} \quad \lambda > 0, \\ |\lambda|\,|m|(e,w) & \text{if} \quad \lambda < 0. \end{cases}$$

Thus, the eigenvalues of $(a, |m|)$ are the eigenvalues of (a, m) in absolute value. Moreover, the quadratic form $|\mathscr{M}|$ associated to $|m|$ is strictly positive on W and

$$\mathscr{M}(u) \leq |\mathscr{M}|(u) \qquad \text{for all } u \in V.$$

Denote the Hessian form $\delta^2\mathscr{G}(0;.,.;\mu)$ of $\mathscr{G}(.;\mu) : W \to \mathbb{R}$ at 0 by $h_0(\mu)$. Then we have

$$h_0(\mu)(v,v) = 2a(v,v) - 2\mu m(v,v) \geq 2k_0\|v\|_V^2 - 2\mu|m|(v,v)$$

for all $v \in W$, so that $h_0(\mu)$ satisfies (M1)-(M2) on W with respect to the bilinear form $|m|$. Taking $e_j^+ \in \mathcal{E}_+$ for v in $h_0(\mu)(v,w)$ yields

$$h_0(\mu)(e_j^+, w) = 2(\lambda_j^+ - \mu)m(e_j^+, w)$$

and by taking $e_j^- \in \mathcal{E}_-$ for v instead we obtain

$$h_0(\mu)(e_j^-, w) = 2(|\lambda_j^-| + \mu)|m|(e_j^-, w).$$

A similar argument as in the proof of Theorem 3.2 now yields (i) as $\mathcal{E}_+ \cup \mathcal{E}_-$ is a set of eigenvectors for the pair $(h_0(\mu), |m|)$.

When $u_j^+ = (\mu - \tilde{\lambda}_j^+)^{1/2} e$ is a critical point of $\mathscr{G}(.;\mu)$ corresponding to $\tilde{\lambda}_j^+ \in (0, \mu)$, the Hessian form $h_j(\mu)(.,.) := \delta^2\mathscr{G}(u_j^+;.,.;\mu)$ of $\mathscr{G}(.;\mu)$ at u_j satisfies

$$h_j(\mu)(v,v) = 2a(v,v) - 2\tilde{\lambda}_j^+ m(v,v) \geq 2k_0\|v\|_V^2 - 2\tilde{\lambda}_j^+|m|(v,v)$$

for all $v \in W$. Thus $h_j(\mu)$ satisfies (M1)-(M2) with respect to the bilinear form $|m|$. Without loss of generality, suppose $e = e_k$ for some $e_k \in \mathcal{E}_+ \cap E_{\tilde{\lambda}_j^+}$ Taking $e_i^+ \in \mathcal{E}_+$ for v in $h_j(\mu)(v,w)$ yields

$$h_j(\mu)(e_j^+, w) = \begin{cases} 4(\mu - \tilde{\lambda}_j^+)|m|(e_i^+, w) & \text{if} \quad i = k \\ 2(\lambda_i^+ - \tilde{\lambda}_j^+)|m|(e_i^+, w) & \text{if} \quad i \neq k. \end{cases}$$

Taking $v = e_i^-$ instead results in

$$h_j(\mu)(e_i^-, w) = 2(|\lambda_j^-| + \tilde{\lambda}_j^+)|m|(e_i^-, w).$$

Hence, $\mathcal{E}_+ \cup \mathcal{E}_-$ is a set of eigenvectors for $(h_j(\mu), |m|)$, and a similar argument as in the proof of Theorem 3.2 now yields (ii). $\square$

With an appropriate substitution of positive signs with negative signs, one obtains a similar result for the functional $\mathscr{G}_-(.;\mu) : W \to \mathbb{R}$ defined by (5.2).

Moreover, higher positive, respectively negative, eigenvalues of (a,m) may be found using penalty methods for $\mathscr{G}(.;\mu)$, respectively $\mathscr{G}_-(.;\mu)$, as done at the end of Chapter 3.

Chapter 6

Applications to Linear Elliptic Eigenvalue Problems

In this chapter, the methods and analyses presented in this dissertation will first be used to obtain results for sequences of Robin eigenfunctions and eigenvalues of second-order, divergence form, elliptic systems. Then Steklov eigenproblems are considered, which are problems where the eigenparameter appears in the boundary equation instead of the differential equation. Finally, general linear elliptic eigenvalue problems, where the eigenparameter appears both in the differential equation as well as in the boundary equation, are discussed. The analysis of such problems demonstrates the advantages of using bilinear forms instead of the associated linear operators and dual spaces. Some of the eigenproblems considered here are problems as studied in Auchmuty [7] where the analysis there is based on constrained variational principles.

6.1 Notation for the Applications

In order to discuss the particular applications, we introduce the following notation and terminology. For the following sections, $\langle ., . \rangle$ will denote the usual Euclidean inner product on $\mathbb{R}^N$, and $|.|$ the corresponding Euclidean norm. A *region* Ω is a non-empty connected set in $\mathbb{R}^N$ and $\sigma, d\sigma$, respectively represent Hausdorff $(N-1)$-dimensional measure and integration with respect to this measure. The following conditions on the region Ω enables the use of trace results in Auchmuty [6].

(B1): Ω is a bounded region in $\mathbb{R}^N$ and its boundary $\partial\Omega$ is the union of a finite number of disjoint closed Lipschitz surfaces; each having finite surface area.

When (B1) holds, there is an outward unit normal ν at σ-a.e. point of the boundary $\partial\Omega$. Definitions and terminology of Evans and Gariepy [9] will be used here, except for surface measure σ as given above. Functions will take values in $\overline{\mathbb{R}} := [-\infty, \infty]$, and derivatives should be taken in a weak sense. The gradient of the function u is denoted ∇u.

Let $L^p(\Omega)$, $L^q(\partial\Omega, d\sigma)$ be the usual real Lebesgue spaces on Ω and $\partial\Omega$ with norm $\|.\|_p$ and $\|.\|_{q,\partial\Omega}$, respectively. Let $H^1(\Omega)$ be the real Sobolev space on Ω that is a real Hilbert space with the standard H^1-inner product

$$[u, v]_1 := \int_\Omega \Big[u(x)v(x) + \nabla u(x) \cdot \nabla v(x)\Big] dx.$$

The associated H^1-norm is denoted $\|.\|_{1,2}$. In the applications, we shall work exclusively in the case where V is the Hilbert-Sobolev space $H^1(\Omega)$.

The region Ω is said to satisfy the *Rellich-Kondrachov theorem* provided the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \leq p < p_S$ for $p_S = 2N/(N-2)$ when $N \geq 3$, or $p_S = \infty$ when $N = 2$.

The region Ω is said to satisfy the L^2-*compact trace theorem* provided the trace map of $H^1(\Omega)$ into $L^2(\partial\Omega, d\sigma)$ is compact. Here we shall always require that the region Ω satisfies

(B2): Ω is a region such that (B1), the Rellich-Kondrachov theorem, and the L^2-compact trace theorem hold.

6.2 Robin Eigenvalue Problems

Consider the problem of finding non-trivial solutions (λ, u) of

$$Lu(x) := -\mathrm{div}\big(A(x)\nabla u(x)\big) + c(x)u(x) = \lambda m_0(x)u(x) \qquad \text{on } \Omega \tag{6.1}$$

subject to

$$\big(A(x)\nabla u(x)\big) \cdot \nu(x) + b(x)u(x) = 0 \qquad \text{on } \partial\Omega, \tag{6.2}$$

where c, m_0, b are given functions and A a given matrix-valued field. Here we shall require the following conditions on these coefficients.

(B3): $A(x) := (a_{ij}(x))$ is a real, symmetric matrix whose components are bounded, Lebesgue-measurable functions on Ω and there exist constants $0 < k_2 \leq k_3$ such that

$$k_2|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq k_3|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^N, x \in \Omega. \tag{6.3}$$

(B4): $c \geq 0$ and $c \in L^p(\Omega)$ for some $p \geq N/2$ when $N \geq 3$, or $p > 1$ when $N = 2$.

(B5): $b \in L^\infty(\partial\Omega)$ with $b \geq 0$ σ-a.e. on $\partial\Omega$, and

$$\int_\Omega c dx + \int_{\partial\Omega} b d\sigma = b_0 > 0. \tag{6.4}$$

(B6): $m_0 \in L^q(\Omega)$ for some $q > N/2$ with m_0 positive on Ω and $\|m_0\|_1 > 0$.

When $b \equiv 0$, this is the *Neumann eigenproblem* for the formal operator L on Ω; otherwise, it is called the *Robin eigenproblem*.

We consider the weak form of the boundary value problem (6.1)-(6.2), which is the eigenproblem of finding non-trivial solutions $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ satisfying

$$\int_\Omega [(A\nabla u) \cdot \nabla v + cuv] dx + \int_{\partial\Omega} buv d\sigma = \lambda \int_\Omega m_0 uv dx \qquad \text{for all } v \in H^1(\Omega). \tag{6.5}$$

The associated bilinear forms for this eigenvalue problem are, with $u, v \in H^1(\Omega)$,

$$a(u,v) := \int_\Omega [(A\nabla u) \cdot \nabla v + cuv] dx + \int_{\partial\Omega} buv d\sigma \tag{6.6}$$

$$m(u,v) := \int_\Omega m_0 uv dx. \tag{6.7}$$

When (B2)-(B6) hold, Lemma 7.1 and Theorem 7.2 of [7] show the pair (a, m) above satisfies conditions (A1)-(A3) in Section 1.2.1 of this work, with V being the Hilbert-Sobolev space $H^1(\Omega)$. The cited theorem also gives the existence of an increasing sequence of strictly positive eigenvalues $\Lambda := \{\lambda_j : j \in \mathbb{N}\}$, counting multiplicities, and an associated sequence $\mathcal{E} := \{e_j : j \in \mathbb{N}\}$ of m-orthonormal eigenfunctions (after a renormalization) for the pair (a, m) in (6.6)-(6.7). Thus there is a least strictly positive eigenvalue λ_1 of (6.5), and a corresponding eigenvector.

The functional for this problem is $\mathscr{R} : H^1(\Omega) \times (0, \infty) \to \mathbb{R}$ with

$$\mathscr{R}(u; \mu) = \int_\Omega [(A\nabla u) \cdot \nabla u + (c - \mu m_0)u^2] dx + \int_{\partial\Omega} bu^2 d\sigma + \frac{1}{2}\left[\int_\Omega m_0 u^2 dx\right]^2. \quad (6.8)$$

The unconstrained variational principle here is the problem $(\mathcal{R}_\mu)$ of minimizing the functional $\mathscr{R}(.; \mu)$ on $H^1(\Omega)$ and finding the value

$$\alpha(\mu) = \inf_{u \in H^1(\Omega)} \mathscr{R}(u; \mu). \quad (6.9)$$

Theorem 4.1 in our Section 3.1 now turns into the following result that provides the properties of $\mathscr{R}(.; \mu)$ needed for the analysis of this unconstrained problem.

Theorem 6.1. *Assume* (B2)-(B6) *hold, and* $\mathscr{R}(.; \mu)$ *is given by* (6.8). *Then*
(i) $\mathscr{R}(.; \mu)$ *is continuous, coercive and weakly l.s.c. on* $H^1(\Omega)$,
(ii) $\mathscr{R}(.; \mu)$ *is Gâteaux differentiable on* $H^1(\Omega)$ *with first variation at* u *in the direction* v *given by*

$$\delta\mathscr{R}(u; v; \mu) = 2\int_\Omega [(A\nabla u)\cdot\nabla v + cuv] dx + 2\int_{\partial\Omega} buv d\sigma + 2\left[\int_\Omega m_0 u^2 dx - \mu\right] \int_\Omega m_0 uv dx. \quad (6.10)$$

Proof. As discussed above, the pair of bilinear forms (a, m) satisfy (A1)-(A3). Hence assertion (i) follows just as in the proof of Theorem 3.1, and a computation shows the first variation of $\mathscr{R}(.; \mu)$ at u is given by (6.10). □

This yields the following results about the unconstrained problem $(\mathcal{R}_\mu)$, whose proof is similar to the proof of Theorem 3.2.

Theorem 6.2. *Assume* $(B2)$-$(B6)$ *hold, and* $\mathscr{R}$ *is given by* (6.8).

(*i*) 0 *is the unique critical point of* $\mathscr{R}(.;\mu)$ *when* $\mu \le \lambda_1$, *and* 0 *and eigenfunctions* u_j *corresponding to the eigenvalue* $\lambda_j \in (0,\mu)$ *of* (6.5) *that satisfy* $\int_\Omega m_0 u_j^2 dx = \mu - \lambda_j$, *are the critical points of* $\mathscr{R}(.;\mu)$ *when* $\mu > \lambda_1$.

(*ii*) *The critical values of* $\mathscr{R}(.;\mu)$ *are* 0 *for any value of* $\mu > 0$, *and* 0 *and* $-\frac{1}{2}(\mu-\lambda_j)^2$ *when* $\mu > \lambda_1$ *and* λ_j *is in the interval* $(0,\mu)$.

(*iii*) *The minimizer(s) of* $\mathscr{R}(.;\mu)$ *on* $H^1(\Omega)$ *are* 0 *when* $\mu \le \lambda_1$, *and the eigenfunctions* u_1 *corresponding to* λ_1 *that satisfy* $\int_\Omega m_0 u_1^2 dx = \mu - \lambda_1$ *when* $\mu > \lambda_1$.

(*iv*) *The value of the problem* $(\mathcal{R}_\mu)$ *is* $\alpha(\mu) = 0$ *when* $\mu \le \lambda_1$, *and the value is* $\alpha(\mu) = -\frac{1}{2}(\mu-\lambda_1)^2$ *when* $\mu > \lambda_1$.

As in Section 3.3, the following unconstrained variational principles can be used to obtain the next successive smallest eigenvalue λ_2 and corresponding eigenfunctions of the Robin eigenproblem (6.5).

Suppose that we know an eigenfunction e_1 corresponding to the first eigenvalue λ_1 of (6.5) that also satisfies

$$\int_\Omega m_0 e_1^2 dx = 1$$

That is, suppose we know u_1 as in the theorem and consider $e_1 = (\mu-\lambda_1)^{-1/2} u_1$. Consider the problem of $(\mathcal{R}_{\mu,\tau})$ of minimizing, over $H^1(\Omega)$, the penalty functional $\mathscr{R}_\tau(.;\mu)$ given by

$$\mathscr{R}_\tau(u;\mu) := \mathscr{R}(u;\mu) + \tau\left[\int_\Omega m_0 e_1 u dx\right]^2, \tag{6.11}$$

where $\mathscr{R}$ is as in (6.8), and of finding

$$\alpha(\mu,\tau) = \inf_{u \in H^1(\Omega)} \mathscr{R}_\tau(u;\mu). \tag{6.12}$$

Again, this is an unconstrained problem, and a calculation similar to that taken for obtaining first variations of the functional $\mathscr{G}(.;\mu)$ in Section 3.1 yields that the first variation of $\mathscr{R}_\tau(.;\mu)$ at u in the direction v is given by

$$\delta\mathscr{R}_\tau(u;v;\mu) = \delta\mathscr{R}(u;v;\mu) + 2\tau \int_\Omega m_0 e_1 u dx \int_\Omega m_0 e_1 v dx, \tag{6.13}$$

where $\delta\mathscr{R}(u;v;\mu)$ is as in (6.10).

Let $\mu_c := \min\{\lambda_1 + \tau, \lambda_2\}$. Denote the distinct eigenvalues of (6.5) by $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots$, and let m_j denote the multiplicity of the j^{th} distinct eigenvalue of (6.5). The results from Theorem 3.2 now translate into the following for the functional $\mathscr{R}_\tau$.

Theorem 6.3. *Assume* (*B*2)-(*B*6) *hold, and* $\mathscr{R}_\tau$ *is defined by* (6.11).

(i) *If* $0 < \mu \le \mu_c$ *and* $\tilde{\lambda}_1$ *is a simple eigenvalue of* (6.5), *then* 0 *is the unique minimizer of* $\mathscr{R}_\tau(.;\mu)$ *on* $H^1(\Omega)$ *and the value of the problem* $(\mathcal{R}_{\mu,\tau})$ *is* $\alpha(\mu,\tau) = 0$.

(ii) *If* $\mu > \mu_c$, $\tau > \tilde{\lambda}_2 - \tilde{\lambda}_1$ *and* $\tilde{\lambda}_1$ *is a simple eigenvalue of* (6.5), *then the minimizers of* $\mathscr{R}_\tau(.;\mu)$ *on* $H^1(\Omega)$ *are eigenfunctions* $\tilde{u}_2$ *corresponding to* $\tilde{\lambda}_2$ *that also satisfy* $\int_\Omega m_0 \tilde{u}_2^2 dx = \mu - \tilde{\lambda}_2$. *In this case, the value* $\alpha(\mu,\tau)$ *of the problem* $(\mathcal{R}_{\mu,\tau})$ *satisfies*

$$\alpha(\mu,\tau) = -\frac{1}{2}(\mu - \tilde{\lambda}_2)^2 > \alpha(\mu). \tag{6.14}$$

(iii) *If* $\mu > \mu_c$ *and* $\tilde{\lambda}_1$ *is an eigenvalue of multiplicity* $m_1 \ge 2$, *then the minimizers of* $\mathscr{R}_\tau(.;\mu)$ *on* $H^1(\Omega)$ *are the eigenfunctions* u_2 *corresponding to* $\tilde{\lambda}_1$ *that satisfy*

$$\int_\Omega m_0 u_2^2 dx = \mu - \tilde{\lambda}_1 \qquad \text{and} \qquad \int_\Omega m_0 e_1 u_2 dx = 0. \tag{6.15}$$

In this case, the value of the problems $(\mathcal{R}_{\mu,\tau})$ *and* $(\mathcal{R}_\mu)$ *are the same:* $\alpha(\mu,\tau) = \alpha(\mu)$.

Proof. Lemma 7.1 in [7] shows m is strictly positive on an infinite dimensional subspace of $H^1(\Omega)$, so that the proof is similar to that of Theorem 3.7 as the pair of bilinear forms (a, m) satisfy (A1)-(A3). □

As described in Section 3.3.2, there is a penalized functional similar to $\mathscr{R}_\tau$ for finding higher eigenvalues and eigenfunctions when one knows a finite sequence of this data, or alternatively, one may use contrained variational principles such as those provided in that section for finding such eigenvalues.

Up to now, the analysis of the Robin eigenproblem has relied merely on first variations of the Robin functional $\mathscr{R}(.;\mu)$ in (6.8). To initiate a Morse index theory for $\mathscr{R}(.;\mu)$ the next lemma shows that the functional has a well-defined second derivative.

Lemma 6.4. *Assume* $(B2)$*-*$(B6)$ *hold, and* $\mathscr{R}(.;\mu)$ *is given by* (6.8). *Then* $\mathscr{R}(.;\mu)$ *is twice Gâteaux differentiable on* $H^1(\Omega)$ *with second variation at* u *in the directions* $v, w \in H^1(\Omega)$ *given by*

$$\delta^2\mathscr{R}(u; v, w; \mu) = 2a(v, w) + 2\left[\int_\Omega m_0 u^2 dx - \mu\right]\int_\Omega m_0 vw dx + 4\int_\Omega m_0 uv dx \int_\Omega m_0 uw dx \tag{6.16}$$

where $a(.,.)$ *is defined by* (6.6).

Proof. The steps taken in the proof of Theorem 4.1 give (6.16) so that $\mathscr{R}(.;\mu)$ is twice Gâteaux differentiable on $H^1(\Omega)$ as each term in (6.16) is a symmetric continuous bilinear form on $H^1(\Omega)$. □

In order to describe the degeneracy of critical points of $\mathscr{R}(.;\mu)$ and to compute their Morse and null index, we shall require the weight function m_0 in (B6) to also obey

$$\int_\Omega m_0 u^2 dx > 0 \qquad \text{for all non-zero } u \in H^1(\Omega). \tag{6.17}$$

That is, we shall require the bilinear form m in (6.7) to also satisfy (A4). When this holds, taking b in (M1)-(M2) of Section 2.1.2 to be equal to the Hessian form $(v, w) \mapsto \delta^2\mathscr{R}(u; v, w; \mu)$ defined by (6.16) allows us to use Theorem 2.3 to obtain the following results. Recall we are denoting the distinct strictly positive eigenvalues of (6.5) by $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots$, and now the collection of all such eigenvalues is called the spectrum of the pair (a, m) and denoted $\sigma(a, m)$. Also, the multiplicity of the j^{th} such eigenvalue $\tilde{\lambda}_j$ is denoted by m_j.

Theorem 6.5. *Assume (B2)-(B6), (6.17) hold, and $\mathscr{R}(.;\mu)$ is given by (6.8).*
(i) 0 is a non-degenerate critical point of $\mathscr{R}(.;\mu)$ if and only if $\mu \notin \sigma(a, m)$. The Morse and null index of 0 are, respectively,

$$i(0;\mu) = \begin{cases} 0 & \text{if} \quad \mu \le \tilde{\lambda}_1, \\ \sum_{\tilde{\lambda}_j < \mu} m_j & \text{if} \quad \mu > \tilde{\lambda}_1, \end{cases} \qquad \text{and} \qquad i_0(0;\mu) = \begin{cases} 0 & \text{if} \quad \mu \notin \sigma(a, m), \\ m_j & \text{if} \quad \mu = \tilde{\lambda}_j. \end{cases} \tag{6.18}$$

(ii) When $\tilde{u}_k$ is a critical point of $\mathscr{R}(.;\mu)$ associated to $\tilde{\lambda}_k$, that is, $\tilde{u}_k$ is an eigenfunction of (6.5) associated to $\tilde{\lambda}_k$ that satisfies $\int_\Omega m_0 \tilde{u}_k^2 dx = \mu - \tilde{\lambda}_k$, then $\widetilde{u}_k$ is non-degenerate if and only if $\tilde{\lambda}_k$ is a simple eigenvalue of (6.5). The Morse and null index of $\tilde{u}_k$ are, respectively,

$$i(\tilde{u}_k;\mu) = \begin{cases} 0 & \text{if} \quad k = 1, \\ \sum_{j=1}^{k-1} m_j & \text{if} \quad k > 1, \end{cases} \qquad \text{and} \qquad i_0(\tilde{u}_k;\mu) = m_k - 1. \tag{6.19}$$

Proof. As mentioned above, when the function m_0 also satisfies (6.17) the Hessian form $\delta^2\mathscr{R}(u; ., .; \mu)$ in Lemma 6.4 satisfies (M1) and (M2), and the proof of Theorem 4.2 gives the desired results. $\square$

If we instead assume that the coefficient function m_0 in (6.7) satisfies

(B7): $m_0 \in L^\infty(\Omega)$ and there is a constant k_4 such that $m_0(x) \geq k_4 > 0$ for all points x in Ω.

then the results in Theorem 6.5 also follow as the bilinear form m again satisfies (A2) and (A4) in this case. We point out that Theorem 7.2 in [7] shows that in this case the sequence of eigenfunctions of (6.5) forms a basis of $L^2(\Omega)$.

From the above results, the bifurcation diagram in Figure 3.1 can be used for the Robin eigenproblem to obtain a bifurcation description of the functional $\mathscr{R}(.;\mu)$ in (6.8) as discussed in Section 3.2 and also in Section 4.2.

We point out that results obtained here for Robin eigenproblems parallel with most results obtained in [4] Section 8 for linear second-order elliptic boundary value eigenproblems with homogeneous Dirichlet condition.

6.3 Steklov Eigenvalue Problems

In this section, results analogous to those obtained in the previous section will be described for Steklov eigenproblems, where the Steklov eigenproblem is that of finding non-trivial solutions (λ, u) of

$$Lu(x) := -\mathrm{div}\big(A(x)\nabla u(x)\big) + c(x)u(x) = 0 \qquad \text{on } \Omega \tag{6.20}$$

subject to

$$\big(A(x)\nabla u(x)\big)\cdot\nu(x) + b(x)u(x) = \lambda\rho(x)u(x) \qquad \text{on } \partial\Omega. \tag{6.21}$$

Here L is a formal operator on Ω, and we shall require the function $\rho : \partial\Omega \to (0, \infty]$ and the boundary $\partial\Omega$ of Ω to satisfy

(B8): $\rho \in L^q(\partial\Omega, d\sigma)$ with $\rho \geq \rho_0 > 0$ σ-a.e. and $q > N-1$, $\partial\Omega$ satisfies (B2) and the trace map of $H^1(\Omega)$ into $L^p(\partial\Omega, d\sigma)$ is compact for $p < \frac{2(N-1)}{N-2}$ when $N \geq 3$ ($p < \infty$ when $N = 2$).

We consider the weak form of the boundary value problem (6.20)-(6.21), which is the eigenproblem of finding non-trivial solutions $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ satisfying

$$\int_\Omega [(A\nabla u) \cdot \nabla v + cuv]dx + \int_{\partial\Omega} buvd\sigma = \lambda \int_{\partial\Omega} \rho uvd\sigma \qquad \text{for all } v \in H^1(\Omega). \quad (6.22)$$

The corresponding bilinear forms are $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by (6.6) and $m : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given

$$m(u, v) := \int_{\partial\Omega} \rho uvd\sigma. \quad (6.23)$$

When (B8) holds, Lemma 8.1 and Theorem 8.2 of [7] show the bilinear form m in (6.23) satisfies (B2) and (B3), and they also give the existence of an increasing sequence of strictly positive Steklov eigenvalues $\{\lambda_j : j \in \mathbb{N}\}$, with $\lim_{j\to\infty} \lambda_j = \infty$, and a corresponding sequence $\mathcal{E} := \{e_j : j \in \mathbb{N}\}$ of eigenfunctions of (6.22). Thus, there is a least positive eigenvalue λ_1 and a corresponding eigenvector of (6.22).

The functional for this problem is $\mathscr{S} : H^1(\Omega) \times (0, \infty) \to \mathbb{R}$ with

$$\mathscr{S}(u; \mu) = \int_\Omega [(A\nabla u) \cdot \nabla u + cu^2]dx + \int_{\partial\Omega} (b - \mu\rho)u^2 d\sigma + \frac{1}{2}\left[\int_{\partial\Omega} \rho u^2 d\sigma\right]^2 \quad (6.24)$$

The variational principle here is the unconstrained problem $(\mathcal{S}_\mu)$ of minimizing $\mathscr{S}(.;\mu)$ on $H^1(\Omega)$ and finding

$$\alpha(\mu) = \inf_{u \in H^1(\Omega)} \mathscr{S}(u;\mu). \tag{6.25}$$

The following theorem summarizes the properties of this unconstrained variational problem and functional $\mathscr{S}(.;\mu)$.

Theorem 6.6. *Assume* $(B3)$-$(B5)$ *and* $(B8)$ *hold, and* $\mathscr{S}(.;\mu)$ *is given by* (6.24). *Then the following hold.*

(i) $\mathscr{S}(.;\mu)$ *is continuous, coercive and weakly l.s.c. on* $H^1(\Omega)$.

(ii) $\mathscr{S}(.;\mu)$ *is Gâteaux differentiable on* $H^1(\Omega)$ *with first variation at* u *in the direction* v *given by*

$$\delta\mathscr{S}(u;v;\mu) = 2\int_\Omega [(A\nabla u)\cdot\nabla v + cuv]dx + 2\int_{\partial\Omega} buvd\sigma + 2\left[\int_{\partial\Omega}\rho u^2 d\sigma - \mu\right]\int_{\partial\Omega}\rho uvd\sigma. \tag{6.26}$$

(iii) *The value of the problem* $(\mathcal{S}_\mu)$ *is given by* $\alpha(\mu) = \begin{cases} 0 & \text{if} \quad \mu \le \lambda_1 \\ -\frac{1}{2}(\mu-\lambda_1)^2 & \text{if} \quad \mu > \lambda_1 \end{cases}.$

(iv) $\mathscr{S}(.;\mu)$ *attains its infimum on* $H^1(\Omega)$. *When* $\mu \le \lambda_1$, *the minimizer of* $\mathscr{S}(.;\mu)$ *is* 0, *and when* $\mu > \lambda_1$, *the minimizers of* $\mathscr{S}(.;\mu)$ *are eigenfunctions* $\tilde{u}$ *corresponding to* λ_1 *with* $\int_{\partial\Omega}\rho\tilde{u}^2 d\sigma = \mu - \lambda_1$.

Proof. When (B8) holds, the bilinear form m in (6.23) satisfies (A2) and (A3) from Lemma 8.1 in [7], and the assumptions here imply the bilinear form a in (6.6) satisfies (A1). The results follow as in Theorem 3.2 $\square$

Successive Steklov eigenvalues and eigenfunctions may be found using variational principles as explained for the case of the Robin eigenproblem. Here we show how some constrained variational principles look like.

Assume that the first k eigenvalues are $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k$ and that $\mathcal{E}_k := \{e_1, e_2, \ldots, e_k\}$ is a corresponding family of ρ-orthonormal eigenfunctions of (6.22), i.e.,

$$\int_{\partial\Omega} \rho e_i e_j d\sigma = \delta_{ij} \qquad \text{for } 1 \le i, j \le k. \tag{6.27}$$

To find λ_{k+1}, let

$$W_k = \left\{ u \in H^1(\Omega) : \int_{\partial\Omega} \rho e_j u d\sigma = 0,\ 1 \le j \le k \right\} \tag{6.28}$$

Consider the variational problem $(\mathcal{S}_{\mu,k})$ of minimizing $\mathscr{S}(.;\mu)$ on W_k and finding

$$\alpha_k(\mu) = \inf_{u \in W_k} \mathscr{S}(u;\mu). \tag{6.29}$$

Theorem 6.7. *Assume* $(B3)$*-*$(B5)$*,* $(B8)$ *hold, and* $\mathscr{S}(.;\mu)$ *is given by* (6.24)*. Then*
(i) *the value of the problem* $(\mathcal{S}_{\mu,k})$ *is given by*

$$\alpha_k(\mu) = \begin{cases} 0 & \text{if} \quad \mu \le \lambda_{k+1} \\ -\frac{1}{2}(\mu - \lambda_{k+1})^2 & \text{if} \quad \mu > \lambda_{k+1} \end{cases}$$

(ii) *the minimizers* $\tilde{u}$ *for the problem* $(\mathcal{S}_{\mu,k})$ *are*

$$\tilde{u} = \begin{cases} 0 & \text{if} \quad \mu \le \lambda_{k+1}, \\ (\mu - \lambda_{k+1})^{1/2} e & \text{if} \quad \mu > \lambda_{k+1}, \end{cases}$$

with $e \in W_k$ *an eigenfunction of* (6.22) *corresponding to* λ_{k+1} *with* $\int_{\partial\Omega} \rho e^2 d\sigma = 1$.

Proof. Lemma 8.1 in [7] shows the bilinear form m is strictly positive on an infinite dimensional subspace of $H^1(\Omega)$, so there is $w \in W_k$ with $\int_{\partial\Omega} \rho w d\sigma > 0$. The proof is then just as in the proof of Theorem 3.8. $\square$

The next lemma, whose proof is similar to that of Theorem 4.1, allows for a Morse index theory to be developed for the functional $\mathscr{S}(.;\mu)$ in (6.24).

Lemma 6.8. *Assume* $(B3)$*-*$(B5)$ *and* $(B8)$ *hold, and* $\mathscr{S}(.;\mu)$ *is given by* (6.24)*. Then* $\mathscr{S}(.;\mu)$ *is twice Gâteaux differentiable on* $H^1(\Omega)$ *with second variation at* u *in the directions* $v, w \in H^1(\Omega)$ *given by*

$$\delta^2\mathscr{S}(u;v,w;\mu) = 2a(v,w) + 2\Big[\int_{\partial\Omega} \rho u^2 d\sigma - \mu\Big] \int_{\partial\Omega} \rho v w d\sigma + 4\int_{\partial\Omega} \rho u v d\sigma \int_{\partial\Omega} \rho u w d\sigma \tag{6.30}$$

where $a(\cdot,\cdot)$ *is defined by* (6.6)*.*

When (B8) holds, we have the a-orthogonal decomposition

$$H^1(\Omega) = H_0^1(\Omega) \oplus_a N(L),$$

where $N(L)$ is the subspace of all H^1-solutions of the equation $Lu = 0$ on Ω, i.e., $u \in N(L)$ provided

$$a(u,v) = 0 \qquad \text{for all } v \in H_0^1(\Omega).$$

From Theorem 8.2 in Auchmuty [7], the Steklov eigenfunctions of (6.22) form a ρ-orthonormal basis of $N(L)$, the a-orthogonal complement of $H_0^1(\Omega)$. To provide a Morse index theory for the Steklov eigenproblem, we therefore restrict the domain of the pair of bilinear forms (a, m) to the closed subspace $N(L)$ of $H^1(\Omega)$.

As we have done so previously, the distinct strictly positive eigenvalues of (6.22) are denoted $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots$, the collection of all such eigenvalues is the spectrum of

(a, m) and denoted $\sigma(a, m)$, and the multiplicity of the j^{th} distinct eigenvalue is m_j.

Theorem 6.9. *Assume* $(B3)$-$(B5)$ *and* $(B8)$ *hold, and* $\mathscr{S}(.;\mu) : N(L) \to \mathbb{R}$ *is given by* (6.24)*.*

(i) *Then* 0 *is a non-degenerate critical point of* $\mathscr{S}(.;\mu) : N(L) \to \mathbb{R}$ *if and only if* $\mu \notin \sigma(a, m)$*. The Morse and null index of* 0 *are, respectively,*

$$i(0;\mu) = \begin{cases} 0 & \text{if} \quad \mu \le \tilde{\lambda}_1 \\ \sum_{\tilde{\lambda}_j < \mu} m_j & \text{if} \quad \mu > \tilde{\lambda}_1 \end{cases} \qquad \text{and} \qquad i_0(0;\mu) = \begin{cases} 0 & \text{if} \quad \mu \notin \sigma(a,m) \\ m_j & \text{if} \quad \mu = \tilde{\lambda}_j \end{cases}. \tag{6.31}$$

(ii) *When* $\tilde{u}_k$ *is a critical point of* $\mathscr{S}(.;\mu) : N(L) \to \mathbb{R}$ *associated to* $\tilde{\lambda}_k$*, that is,* $\tilde{u}_k$ *is an eigenfunction of* (6.22) *associated to* $\tilde{\lambda}_k$ *that satisfies* $\int_{\partial\Omega} \rho \tilde{u}_k^2 d\sigma = \mu - \tilde{\lambda}_k$*, then* $\tilde{u}$ *is non-degenerate if and only if* $\tilde{\lambda}_k$ *is a simple eigenvalue of* (6.22)*. The Morse and null index of* $\tilde{u}_k$ *are, respectively,*

$$i(\tilde{u}_k;\mu) = \begin{cases} 0 & \text{if} \quad k = 1 \\ \sum_{j=1}^{k-1} m_j & \text{if} \quad k > 1 \end{cases} \qquad \text{and} \qquad i_0(\tilde{u}_k;\mu) = m_k - 1. \tag{6.32}$$

Proof. When $H^1(\Omega)$ is replaced by $N(L)$, the results of Theorem 6.6, Theorem 6.7 and Lemma 6.8 hold for the functional $\mathscr{S}(\cdot;\mu) : N(L) \to \mathbb{R}$ defined by (6.24) as $N(L)$ is a closed subspace of $H^1(\Omega)$ and the Steklov eigenfunctions of (6.22) are in $N(L)$. Moreover, the bilinear form m in (6.23) is strictly positive on $N(L)$, so that Theorem 2.3 gives the desired results as in the proof of Theorem 4.2. $\square$

As in the Robin eigenproblem, Figure 3.1 and the bifurcation results at the end of Section 3.2 and Section 4.2 apply to the Steklov eigenproblem, where the schematic diagram is on the space $(0, \infty) \times N(L)$ instead of $(0, \infty) \times H^1(\Omega)$.

6.4 General Eigenvalue Problems

The general eigenproblem to be studied in this section is that of finding non-trivial solutions (λ, u) of

$$-\text{div}(A(x)\nabla u(x)) + c(x)u(x) = \lambda m_0(x) \qquad \text{on } \Omega \tag{6.33}$$

subject to

$$(A(x)\nabla u(x)) \cdot \nu(x) + b(x)u(x) = \lambda \rho(x)u(x) \qquad \text{on } \partial\Omega. \tag{6.34}$$

Here the eigenparameter is in both the differential equation and boundary condition. For discussions, analysis, and applications of these problems see Auchmuty [7] and references cited there.

The weak form of (6.33)-(6.34) is the eigenproblem of finding non-trivial solutions $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ satisfying

$$\int_\Omega [(A\nabla u)\cdot\nabla v + cuv]dx + \int_{\partial\Omega} buvd\sigma = \lambda\Big[\int_\Omega m_0 uvdx + \int_{\partial\Omega} \rho uvd\sigma\Big] \qquad \text{for all } v \in H^1(\Omega). \tag{6.35}$$

The associated bilinear forms are $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ as in equation (6.6) and $m : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$m(u, v) = \int_\Omega m_0 uvdx + \int_{\partial\Omega} \rho uvd\sigma. \tag{6.36}$$

When (B7) and (B8) hold, the bilinear form m satisfies (A2) and (A4) as seen in Lemma 9.1 of [7]. Then Theorem 9.2 of [7] gives the existence of an m-orthonormal basis of $H^1(\Omega)$ (after a renormalization) consisting of eigenfunctions $\mathcal{E} = \{e_j : j \in \mathbb{N}\}$

of (6.35), and an associated increasing sequence of strictly positive eigenvalues $\{\lambda_j : j \in \mathbb{N}\}$ when the assumptions of that theorem are satisfied.

The functional for this problem is $\mathscr{L} : H^1(\Omega) \times (0, \infty) \to \mathbb{R}$ given by

$$\mathscr{L}(u;\mu) = \int_\Omega [(A\nabla u)\cdot\nabla u + (c - \mu m_0)u^2]dx + \int_{\partial\Omega} (b - \mu\rho)u^2 d\sigma + \frac{1}{2}\left[\int_\Omega m_0 u^2 dx + \int_{\partial\Omega} \rho u^2 d\sigma\right]^2 \tag{6.37}$$

The variational principle $(\mathcal{L}_\mu)$ is to minimize $\mathscr{L}(.;\mu)$ on $H^1(\Omega)$ and to find

$$\alpha(\mu) = \inf_{u \in H^1(\Omega)} \mathscr{L}(u;\mu). \tag{6.38}$$

Results for this unconstrained variational principle and properties of the functional $\mathscr{L}(\cdot;\mu)$ are summarized as follow.

Theorem 6.10. *Assume* $(B2)$-$(B8)$ *hold, and* $\mathscr{L}(.;\mu)$ *is given by* (6.37). *Then the following hold.*

(i) $\mathscr{L}(.;\mu)$ *is continuous, coercive, and weakly l.s.c. on* $H^1(\Omega)$. *(ii)* $\mathscr{L}(.;\mu)$ *is Gâteaux differentiable on* $H^1(\Omega)$ *with first variation at* u *in the direction* v *given by*

$$\delta\mathscr{L}(u;v;\mu) = 2a(u,v) + 2\left[\int_\Omega m_0 u^2 dx + \int_{\partial\Omega} \rho u^2 d\sigma - \mu\right]\left[\int_\Omega m_0 uv dx + \int_{\partial\Omega} \rho uv d\sigma\right] \tag{6.39}$$

where $a(.,.)$ *is defined by* (6.6).

(iii) *The value of the problem* $(\mathcal{L}_\mu)$ *is given by* $\alpha(\mu) = \begin{cases} 0 & \text{if} \quad \mu \le \lambda_1 \\ -\frac{1}{2}(\mu - \lambda_1)^2 & \text{if} \quad \mu > \lambda_1 \end{cases}$.

(iv) $\mathscr{L}(.;\mu)$ *attains its infimum on* $H^1(\Omega)$. *When* $\mu \le \lambda_1$, *the minimizer of* $\mathscr{L}(.;\mu)$ *is* 0, *and when* $\mu > \lambda_1$, *the minimizers of* $\mathscr{L}(.;\mu)$ *are eigenfunctions* $\tilde{u}$ *corresponding to the smallest eigenvalue* λ_1 *with* $\int_\Omega m_0 \tilde{u}^2 dx + \int_{\partial\Omega} \rho \tilde{u}^2 d\sigma = \mu - \lambda_1$.

Proof. As the bilinear forms a and m satisfy (A1) and (B2), (B4), the proof is similar to that of Theorem 3.2. $\square$

As in the case for Robin and Steklov eigenproblems, the iterative construction of Section 3.3.2 yields successive eigenvalues and eigenfunctions for the general eigenproblem studied in this section.

Let $\mathcal{E}_k = \{e_j : 1 \le j \le k\}$ be a set of eigenfunctions of (6.35) corresponding to the successive strictly positive eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k$ which obey

$$\int_\Omega m_0 e_i e_j dx + \int_{\partial\Omega} \rho e_i e_j d\sigma = \delta_{ij} \qquad \text{for } 1 \le i, j \le k. \tag{6.40}$$

To find λ_{k+1} define

$$W_k = \left\{ u \in H^1(\Omega) : \int_\Omega m_0 e_j u dx + \int_{\partial\Omega} \rho e_j u d\sigma = 0, \text{ for } 1 \le j \le k \right\}. \tag{6.41}$$

Consider the problem $(\mathcal{L}_{\mu,k})$ of minimizing $\mathscr{L}(.;\mu)$ on W_k and finding

$$\alpha_k(\mu) = \inf_{u \in W_k} \mathscr{L}(u;\mu). \tag{6.42}$$

The next theorem describes the minimizers and the value of this variational principle.

Theorem 6.11. *Assume* $(B2)$*-*$(B8)$ *hold, and* $\mathscr{L}(.;\mu)$ *is given by* (6.37)*. Then*

(i) *the value of the problem* $(\mathcal{L}_{\mu,k})$ *is given by* $\alpha_k(\mu) = \begin{cases} 0 & \textit{if} \quad \mu \le \lambda_{k+1} \\ -\frac{1}{2}(\mu - \lambda_{k+1})^2 & \textit{if} \quad \mu > \lambda_{k+1} \end{cases}$,

and

(ii) *the minimizers* $\tilde{u}$ *for the problem* $(\mathcal{L}_{\mu,k})$ *are* $\tilde{u} = \begin{cases} 0 & \textit{if} \quad \mu \le \lambda_{k+1} \\ (\mu - \lambda_{k+1})^{1/2} e & \textit{if} \quad \mu > \lambda_{k+1} \end{cases}$, *where* $e \in W_k$ *is an eigenfunction of* (6.35) *corresponding to* λ_{k+1} *that obeys* $\int_\Omega m_0 e^2 dx + \int_{\partial\Omega} \rho e^2 d\sigma = 1$.

Proof. Since the bilinear form m in (6.36) satisfies (A2) and (A4), there exists $w \in W_k$ with

$$\int_\Omega m_0 w dx + \int_{\partial\Omega} \rho w d\sigma > 0,$$

so the results follow as in the proof of Theorem 3.8. □

That $\mathscr{L}(.;\mu)$ defined by (6.37) has a well-defined second derivative is given by the next lemma.

Lemma 6.12. *Assume* (*B*2)-(*B*8) *hold, and* $\mathscr{L}(.;\mu)$ *is given by* (6.37). *Then* $\mathscr{L}(.;\mu)$ *is twice Gâteaux differentiable on* $H^1(\Omega)$ *with second variation at* u *in the directions* $v, w \in H^1(\Omega)$ *given by*

$$\begin{aligned} \delta^2\mathscr{L}(u;v,w;\mu) = \;& 2a(v,w) + 2\Big[\int_\Omega m_0 u^2 dx + \int_{\partial\Omega} \rho u^2 d\sigma - \mu\Big]\Big[\int_\Omega m_0 vw dx + \int_{\partial\Omega} \rho vw d\sigma\Big] \\ & +4\Big[\int_\Omega m_0 uv dx + \int_{\partial\Omega} \rho uv d\sigma\Big]\Big[\int_\Omega m_0 uw dx + \int_{\partial\Omega} \rho uw d\sigma\Big] \end{aligned} \tag{6.43}$$

where $a(.,.)$ *is defined by* (6.6).

Proof. The proof is similar to that of Theorem 4.1. □

This result enables a Morse index theory for the functional $\mathscr{L}(.;\mu)$. In the next theorem, the distinct strictly positive eigenvalues of (6.35) are denoted $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots$, the collection of all such eigenvalues is the spectrum of (a, m) and denoted $\sigma(a, m)$, and the multiplicity of the j^{th} distinct eigenvalue is m_j.

Theorem 6.13. *Assume* (*B*2)-(*B*8) *hold, and* $\mathscr{L}(.;\mu)$ *is given by* (6.37).

(*i*) 0 *is a nondegenerate critical point of* $\mathscr{L}(.;\mu)$ *if and only if* $\mu \notin \sigma(a, m)$. *The*

Morse and null index of 0 *are, respectively,*

$$i(0;\mu) = \begin{cases} 0 & \text{if} \quad \mu \le \tilde{\lambda}_1 \\ \sum_{\tilde{\lambda}_j<\mu} m_j & \text{if} \quad \mu > \tilde{\lambda}_1 \end{cases} \qquad \text{and} \qquad i_0(0;\mu) = \begin{cases} 0 & \text{if} \quad \mu \notin \sigma(a,m) \\ m_j & \text{if} \quad \mu = \tilde{\lambda}_j \end{cases} \tag{6.44}$$

(ii) When $\tilde{u}_k$ *is a critical point of* $\mathscr{L}(.;\mu)$ *associated to* $\tilde{\lambda}_k$*, that is,* $\tilde{u}_k$ *is an eigenfunction of* (6.35) *associated to* $\tilde{\lambda}_k$ *that satisfies* $\int_\Omega m_0\tilde{u}_k^2dx+\int_{\partial\Omega}\rho\tilde{u}_kd\sigma = \mu-\tilde{\lambda}_k$*, then* $\tilde{u}_k$ *is non-degenerate if and only if* $\tilde{\lambda}_k$ *is a simple eigenvalue of* (6.35)*. The Morse and null index of* $\tilde{u}_k$ *are, respectively,*

$$i(\tilde{u}_k;\mu) = \begin{cases} 0 & \text{if} \quad k=1 \\ \sum_{j=1}^{k-1} m_j & \text{if} \quad k>1 \end{cases} \qquad \text{and} \qquad i_0(\tilde{u}_k;\mu) = m_k - 1. \tag{6.45}$$

Proof. Since the bilinear form m in (6.36) satisfies (A2) and (A4), it implies m satisfies (M2), and since the Hessian form $(v,w) \mapsto \delta^2\mathscr{L}(u;v,w;\mu)$ given by (6.43) satisfies (M1), the results follow as in the proof of Theorem 4.2. □

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