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Abstract Microemulsions can be modeled by an initial-boundary value problem for a sixth order Cahn-Hilliard equation. Introducing the chemical potential as a dual variable, a Ciarlet-Raviart type mixed formulation yields a system consisting of a linear second order evolutionary equation and a nonlinear fourth order equation. The spatial discretization is done by a C^0 Interior Penalty Discontinuous Galerkin (C^0IPDG) approximation with respect to a geometrically conforming simplicial triangulation of the computational domain. The DG trial spaces are constructed by C^0 conforming Lagrangian finite elements of polynomial degree $p \geq 2$. For the semidiscretized problem we derive quasi-optimal a priori error estimates for the global discretization error in a mesh-dependent C0IPDG norm. The semidiscretized problem represents an index 1 Differential Algebraic Equation (DAE) which is further discretized in time by an s-stage Diagonally Implicit Runge-Kutta (DIRK) method of order $q \geq 2$. Numerical results show the formation of microemulsions in an oil/water system and confirm the theoretically derived convergence rates.

1 Introduction

Microemulsions are thermodynamically stable colloidal dispersions of an oil/water system that typically occur as oil-in-water, water-in-oil, or water/oil droplets with a diameter up to 200 nm. They are thus considerably smaller than ordinary emulsions

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(macroemulsions). Moreover, in contrast to macroemulsions whose generation requires strong shear forces, microemulsions can be created by simple mixing. Due to their efficient drug solubilization capacity and bioavailability, microemulsions have significant applications in pharmacology as drug carriers for the delivery of hydrophilic as well as lipophilic drugs. Other applications include cleaning and polishing processes, food processing, and cutting oils (cf. [14, 21, 23, 24, 27, 28]).
As far as the mathematical modeling is concerned, for ternary oil-water-microemulsions Gompper et al. [15, 16, 17, 18] have considered a second order Ginzburg-Landau free energy so that the dynamics of the microemulsification process can be described by an initial-boundary value problem for a sixth order Cahn-Hilliard equation. The existence and uniqueness of strong and weak solutions has been investigated analytically by Pawlow et al. [25, 26, 29].
For the numerical simulation of the microemulsification process, we introduce the chemical potential as a dual variable and consider a Ciarlet-Raviart type mixed formulation as a system consisting of a linear second order evolutionary equation and a nonlinear fourth order elliptic equation. The spatial discretization is taken care of by a C^0 Interior Penalty Discontinuous Galerkin (C^0IPDG) approximation with respect to a geometrically conforming simplicial triangulation of the computational domain. The DG trial spaces are constructed by C^0 conforming Lagrangian finite elements of polynomial degree $p \geq 2$. We note that IPDG methods for the standard fourth order Cahn-Hilliard equation have been studied in [32] based on IPDG approximations of fourth order problems including the biharmonic equation considered in [5, 10] (cf. also [3, 11, 12, 13]). The semidiscretized problem represents an initial value problem for an index 1 Differential Algebraic Equation (DAE) which is discretized in time by an s-stage Diagonally Implicit Runge-Kutta method of order $q \geq 2$ with respect to a partitioning of the time interval (cf., e.g., [1, 7, 19]). The resulting parameter dependent nonlinear algebraic system is numerically solved by a predictor-corrector continuation strategy with the time step size as the continuation parameter featuring constant continuation as a predictor and Newton's method as corrector.
The paper is organized as follows: After some notations and preliminaries in section 2, in section 3 we present the initial-boundary value problem for the sixth order Cahn-Hilliard equation based on a Ginzburg-Landau free energy and introduce a Ciarlet-Raviart type mixed formulation as a system consisting of a linear second order evolutionary equation and a nonlinear fourth order elliptic equation. Then, section 4 is devoted to the semidiscretization in space by the C^0IPDG method. Quasi-optimal a priori error estimates for the global discretization error both in the primal and in the dual variable are derived in section 5. In section 6, very briefly we discuss the discretization in time by an s-stage DIRK method of order q and the numerical solution of the resulting parameter dependent nonlinear algebraic system by a predictor-corrector continuation strategy. In the final section 7, we present numerical results which show the formation of water-in-oil and oil-in-water droplets in a ternary water-oil-microemulsion system and confirm to some extent the theoretically derived convergence rates.

2 Notations and preliminaries

We use standard nontation from Lebesgue and Sobolev space theory (cf., e.g., [30]). In particular, for a bounded domain $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, we refer to $L^p(\Omega), 1 \le p < \infty$, as the Banach space of p-th power Lebesgue integrable functions on Ω with norm $\|\cdot\|_{0,p,\Omega}$ and to $L^\infty(\Omega)$ as the Banach space of essentially bounded functions on Ω with norm $\|\cdot\|_{0,\infty,\Omega}$. For functions $v_i \in L^{p_i}(\Omega), 1 \le i \le 3$, where $p_i \in \mathbb{R}_+, \sum_{i=1}^3 1/p_i = 1$, the generalized Hölder inequality

$$\int_\Omega \prod_{i=1}^3 |v_i| \, dx \le \prod_{i=1}^3 \|v_i\|_{0,p_i,\Omega}. \tag{1}$$

holds true. Further, we denote by $W^{s,p}(\Omega), s \in \mathbb{R}_+, 1 \le p \le \infty$, the Sobolev spaces with norms $\|\cdot\|_{s,p,\Omega}$. We note that for $p = 2$ the spaces $L^2(\Omega)$ and $W^{s,2}(\Omega) = H^s(\Omega)$ are Hilbert spaces with inner products $(\cdot,\cdot)_{0,2,\Omega}$ and $(\cdot,\cdot)_{s,2,\Omega}$. In the sequel, we will suppress the subindex 2 and write $(\cdot,\cdot)_{0,\Omega}, (\cdot,\cdot)_{s,\Omega}$ and $\|\cdot\|_{0,\Omega}, \|\cdot\|_{s,\Omega}$ instead of $(\cdot,\cdot)_{0,2,\Omega}, (\cdot,\cdot)_{s,2,\Omega}$ and $\|\cdot\|_{0,2,\Omega}, \|\cdot\|_{s,2,\Omega}$.
For $T > 0$ and a Banach space V with norm $\|\cdot\|_V$ the space $L^p((0,T),V), 1 \le p \le \infty$, refers to the Banach space of all functions v such that $v(t) \in V$ for almost all $t \in (0,T)$ with norm

$$\|v\|_{L^p((0,T),V)} := \begin{cases} (\int\limits_0^T \|v(t)\|_V^p \, dt)^{1/p} \ , 1 \le p < \infty \\ \operatorname*{ess\ sup}\limits_{t\in(0,T)} \|v(t)\|_V \ , p = \infty \end{cases}.$$

The spaces $W^{s,p}((0,T),V), s \in \mathbb{R}_+, 1 \le p \le \infty$, are defined analogously. Finally, $C([0,T],V)$ denotes the Banach space of functions v such that $v(t) \in V$ for all $t \in [0,T]$ with norm

$$\|v\|_{C([0,T],V)} := \max_{t\in[0,T]} \|v(t)\|_V.$$

3 The sixth order Cahn-Hilliard equation

Given a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$ and exterior unit normal vector $\mathbf{n}_\Gamma$, denoting by $T > 0$ the final time, and setting $Q := \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, we consider the following sixth order Cahn-Hilliard equation

$$\sigma \frac{\partial c}{\partial t} - M\Delta\Big(\kappa\Delta^2 c - a(c)\Delta c - \frac{1}{2}a'(c)|\nabla c|^2 + f_0(c)\Big) = 0 \quad \text{in } Q, \tag{2a}$$

with the boundary conditions

$$\mathbf{n}_\Gamma \cdot \nabla c = \mathbf{n}_\Gamma \cdot \nabla \mu(c) = \mathbf{n}_\Gamma \cdot \nabla \Delta c = 0 \quad \text{on } \Sigma, \tag{2b}$$

and the initial condition

$$c(\cdot,0) = c_0 \quad \text{in } \Omega. \tag{2c}$$

Here, σ is a surface energy density, M stands for the mobility which in the sequel will be assumed to be a positive constant, κ is a positive constant as well, and the coefficient function $a(c)$ is assumed to be of the form

$$a(c) = a_0 + a_2 c^2, \quad a_0 \in \mathbb{R},\ a_2 > 0. \tag{3}$$

The function $f_0(c) = \delta F_0(c)/\delta c$ is the variational derivative of the multiwell free energy

$$F_0(c) = \int\limits_\Omega \frac{\beta}{2}\,(c+1)^2(c^2+h_0)(c-1)^2, \quad h_0 \in \mathbb{R},$$

where β is another surface energy density and $h_0 \in \mathbb{R}$ measures the deviation from the oil-water-microemulsion coexistence. Moreover, $\mu(c)$ denotes the chemical potential which is the variational derivative

$$\mu(c) = \frac{\delta F(c)}{\delta c}$$

of the total free energy

$$F(c) = F_0(c) + \int\limits_\Omega \left(\frac{1}{2} a(c)|\nabla c|^2 + \frac{1}{2}\kappa|\Delta c|^2\right)\,dx, \tag{4}$$

and c_0 is a given initial condition.

Remark 1 *The initial-boundary value problem* (2a)-(2c) *describes the dynamics of ternary oil-water-microemulsion systems where the solution c is an order parameter representing the local difference between the oil and water concentrations. We note that the Ginzburg-Landau free energy* (4) *for such systems has been suggested in* [16, 17] *and* [15, 18].
For bounded convex domains with boundary Γ of class C^6 and initial data c_0 such that $c_0 \in H^5(\Omega)$ with spatial mean

$$c_m := \frac{1}{|\Omega|} \int\limits_\Omega c_0\,dx$$

satisfying the compatibility conditions

$$\mathbf{n}_\Gamma \cdot \nabla c_0 = \mathbf{n}_\Gamma \cdot \nabla \Delta c_0 = 0 \quad \text{on } \Gamma, \tag{5}$$

it has been shown in [25] *that the initial-boundary value problem for the sixth order Cahn-Hilliard equation* (2a)-(2c) *has a unique solution global in time such that*

$$c \in L^2((0,T),H^6(\Omega)) \cap H^1((0,T),H^4(\Omega)),$$
$$c(\cdot,0) = c_0, \quad \frac{1}{|\Omega|} \int_\Omega c(t)\,dx = c_m \quad \textit{for all } t \in \mathbb{R}_+.$$

Introducing the chemical potential $\mu(c)$ as an additional unknown $w := \mu(c)$, the sixth order Cahn-Hilliard equation (2a) can be equivalently formulated as a system of a linear second order evolutionary equation and a nonlinear fourth order elliptic equation in (c,w) according to

$$\sigma \frac{\partial c}{\partial t} - M\Delta w = 0 \quad \text{in } Q, \tag{6a}$$
$$\kappa\Delta^2 c - a(c)\Delta c - a_2 c|\nabla c|^2 + f_0(c) - w = 0 \quad \text{in } Q, \tag{6b}$$

with the boundary conditions

$$\mathbf{n}_\Gamma \cdot \nabla c = \mathbf{n}_\Gamma \cdot \nabla w = \mathbf{n}_\Gamma \cdot \nabla\Delta c = 0 \quad \text{on } \Sigma, \tag{6c}$$

and the initial condition

$$c(\cdot,0) = c_0 \quad \text{in } \Omega. \tag{6d}$$

We set

$$V := H^1(\Omega), \quad Z := \{z \in H^2(\Omega) \mid \mathbf{n}_\Gamma \cdot \nabla z = 0 \text{ on } \Gamma\}. \tag{7}$$

Observing

$$\nabla \cdot (a(c)\nabla c) = a(c)\Delta c + 2a_2|\nabla c|^2,$$

we define

$$(g(c),v)_{0,\Omega} := -(a(c)\Delta c, v)_{0,\Omega} - (a_2 c\,|\nabla c|^2, v)_{0,\Omega} + (f_0(c),v)_{0,\Omega}, \quad v \in Z. \tag{8}$$

A pair (c,w) is said to be a weak solution of (6a)-(6d), if for all $v \in V$ and $z \in Z$ it holds

$$\sigma \langle \frac{\partial c}{\partial t}, v\rangle_{V^*,V} + M\,(\nabla w, \nabla v)_{0,\Omega} = 0, \tag{9a}$$
$$\kappa\,(\Delta c, \Delta z)_{0,\Omega} + (g(c),z)_{0,\Omega} - (w,z)_{0,\Omega} = 0, \tag{9b}$$

and if the initial condition

$$c(\cdot,0) = c_0. \tag{9c}$$

is satisfied.

Remark 2 *The existence and uniqueness of a weak solution satisfying*

$$c \in H^1((0,T),V^*) \cap L^\infty((0,T),Z) \cap L^2((0,T),H^3(\Omega)), \qquad (10)$$
$$w \in L^2((0,T),V)$$

has been shown in [29].

4 C^0 Interior Penalty Discontinuous Galerkin approximation

For semidiscretization in space of the coupled system (6a)-(6d) we will use the C^0IPDG method with respect to a simplicial triangulation of the computational domain. Due to the convexity of the computational domain, we can use the Ciarlet-Raviart mixed formulation of (6b) by introducing $z = \Delta c$ as an additional unknown so that (6b) can be written as the following system of two second order equations

$$z = \Delta c, \qquad (11a)$$
$$\kappa \Delta z - a(c)\Delta c - a_2 c|\nabla c|^2 + f_0(c) = w. \qquad (11b)$$

Multiplying (11a) by a test function $\varphi \in H^1(\Omega)$ and (11b) by a test function $\psi \in H^2(\Omega)$ and integrating over Ω, integration by parts and observing (6c),(8) yields the weak formulation

$$(z,\varphi)_{0,\Omega} = -(\nabla c, \nabla \varphi)_{0,\Omega}, \qquad (12a)$$
$$(\kappa z, \Delta \psi)_{0,\Omega} - (\kappa z, \mathbf{n} \cdot \nabla \psi)_{0,\Gamma} + (g(c),\psi)_{0,\Omega} = (w,\psi)_{0,\Omega}. \qquad (12b)$$

We assume $\mathscr{T}_h(\Omega)$ to be a shape-regular simplicial triangulation of Ω. For $D \subseteq \overline{\Omega}$, we denote by $\mathscr{E}_h(D)$ the sets of nodal points of $\mathscr{T}_h$ in D. For $K \in \mathscr{T}_h(\Omega)$ and $E \in \mathscr{E}_h(\overline{\Omega})$ we further refer to h_K and h_E as the diameter of K and the length of E. We set $h := \max\{h_K \mid K \in \mathscr{T}_h(\Omega)$. For two quantities $A, B \in \mathbb{R}_+$ we use the notation $A \lesssim B$, if there exists a constant $C > 0$, independent of h, such that $A \leq CB$.
Denoting by $P_p(K), p \in \mathbb{N}$, the linear space of polynomials of degree $\leq p$ on K, for $p \geq 2$ we set

$$Q_h^{(p)} := \{v_h \in L^2(\Omega) \mid v_h|_K \in P_p(T),\ K \in \mathscr{T}_h\} \qquad (13)$$

and refer to

$$V_h^{(p)} := Q_h^{(p)} \cap H^1(\Omega) \qquad (14)$$

as the finite element space of Lagrangian finite elements of type p (cf., e.g., [4, 8]). We refer to $\mathscr{N}_h(\Omega)$ as the set of nodal points such that any $v_h \in V_h^{(p)}$ is uniquely de-

termined by its degrees of freedom $v_h(a), a \in \mathscr{N}_h(\Omega)$ and to $I_h : H^s(\Omega) \to V_h^{(p)}, s \geq 2$, as the nodal interpolation operator.
In the sequel, we will use the inverse inequalities [31]

$$\|\nabla v_h\|_{0,K} \leq C_{Inv}^{(1)} p^2 \, h^{-1} \, \|v_h\|_{0,K}, \quad v_h \in V_h^{(p)}, \tag{15a}$$

$$\|\Delta v_h\|_{0,K} \leq C_{Inv}^{(2)} (p-1)^2 \, h^{-1} \, \|\nabla v_h\|_{0,K}, \quad v_h \in V_h^{(p)}, \tag{15b}$$

and the trace inequality [31]

$$\|v_h\|_{0,\partial K} \leq C_{Tr} \, p \, h^{-1/2} \, \|v_h\|_{0,K}, \quad v_h \in V_h^{(p)}. \tag{15c}$$

We note that $V_h^{(p)} \not\subset H^2(\Omega)$ and hence, $V_h^{(p)}$ is a nonconforming finite element space for the approximation of the fourth order equation (6b). In particular, for a function z_h on $\overline{\Omega}$ that is elementwise polynomial, we define averages and jumps according to

$$\{z_h\}_E := \begin{cases} \frac{1}{2}\Big(z_h|_{E\cap T_+} + z_h|_{E\cap T_-}\Big) \, , \, E \in \mathscr{E}_h(\Omega), \\ z_h|_E \, , \, E \in \mathscr{E}_h(\Gamma), \end{cases} \tag{16a}$$

$$[z_h]_E := \begin{cases} z_h|_{E\cap T_+} - z_h|_{E\cap T_-} \, , \, E \in \mathscr{E}_h(\Omega), \\ z_h|_E \, , \, E \in \mathscr{E}_h(\Gamma), \end{cases} . \tag{16b}$$

The general C^{0}DG approximation of (12a),(12b) reads: Given $w_h \in V_h^{(p)}$, find $(c_h, z_h) \in V_h^{(p)} \times Q_h^{(p)}$ such that for all $(\varphi_h, v_h) \in Q_h^{(p)} \times V_h^{(p)}$ it holds

$$\sum_{K\in\mathscr{T}_h(\Omega)} \Big((z_h, \varphi_h)_{0,K} + (\nabla c_h, \nabla \varphi_h)_{0,K}\Big) - \sum_{E\in\mathscr{E}_h(\bar{\Omega})} (\mathbf{n}_E \cdot \hat{\mathbf{c}}_E, \varphi_h)_{0,\partial K}\Big) = 0, \tag{17a}$$

$$\sum_{K\in\mathscr{T}_h(\Omega)} \Big((\kappa z_h, \Delta v_h)_{0,T} + (g(c_h), v_h)_{0,K}\Big) - \sum_{E\in\mathscr{E}_h(\bar{\Omega})} (\hat{\mathbf{z}}_E, \nabla v_h)_{0,E} - (w_h, v_h)_{0,K}\Big) = 0, \tag{17b}$$

where $\hat{\mathbf{c}}_E$ and $\hat{\mathbf{z}}_E$ are suitably chosen numerical flux functions that determine the type of C^{0}DG approximation. In particular, for the C^{0}IPDG approximation we choose

$$\hat{\mathbf{c}}_E := \begin{cases} \{\nabla c_h\}_E \, , \quad E \in \mathscr{E}_h(\Omega) \\ 0 \, , \, E \in \mathscr{E}_h(\Gamma) \end{cases} , \tag{17c}$$

$$\hat{\mathbf{z}}_E := \Big(\{\Delta c_h\}_E - \frac{\alpha}{h_E}[\frac{\partial c_h}{\partial n}]_E\Big)\mathbf{n}_E, \quad E \in \mathscr{E}_h(\bar{\Omega}), \tag{17d}$$

where $\alpha > 0$ is a penalization parameter. The choice (17c),(17d) has the advantage that for $\varphi_h = \kappa \Delta v_h$ in (17a) we may eliminate the dual variable z_h from the system and thus arrive at the following primal variational formulation of the C^{0}IPDG approximation: Find $c_h \in V_h^{(p)}$ such that for all $v_h \in V_h^{(p)}$ it holds

$$a_h^{DG}(c_h, v_h) + \sum_{K \in \mathscr{T}_h(\Omega)} (g(c_h), v_h)_{0,K} = (w_h, v_h)_{0,\Omega}, \tag{18}$$

where $a_h^{DG}(\cdot,\cdot) : V_h^{(p)} \times V_h^{(p)} \to \mathbb{R}$ stands for the C^{0}IPDG bilinear form

$$a_h^{DG}(c_h, v_h) := \sum_{K \in \mathscr{T}_h(\Omega)} (\kappa \Delta c_h, \Delta v_h)_{0,K} - \sum_{E \in \mathscr{E}_h(\overline{\Omega})} \Big((\kappa \mathbf{n}_E \cdot \{\nabla c_h\}_E, [\Delta v_h]_E)_{0,E} \tag{19}$$
$$+ (\kappa [\Delta c_h]_E, \mathbf{n}_E \cdot \{\nabla v_h\}_E)_{0,E} \Big) + \sum_{E \in \mathscr{E}_h(\overline{\Omega})} \frac{\alpha}{h_E} (\mathbf{n}_E \cdot [\nabla c_h]_E, \mathbf{n}_E \cdot [\nabla v_h]_E)_{0,E}.$$

We note that the C^{0}IPDG bilinear form is not well-defined for functions $c \in Z$, since $\Delta c|_E, E \in \mathscr{E}_h(\overline{\Omega}$, does not live in $L^2(E)$. This can be cured by means of a lifting operator

$$L : V_h^{(p)} + Z \to V_h^{(p)}$$

which is defined according to

$$\int_\Omega L(c)\, v_h \, dx = - \sum_{E \in \mathscr{E}_h(\bar{\Omega})} \int_E \mathbf{n}_E \cdot [\nabla c]_E \, v_h \, ds.$$

We define an extension $\tilde{a}_h^{DG}(\cdot,\cdot) : (V_h^{(p)} + Z) \times (V_h^{(p)} + Z) \to \mathbb{R}$ as follows:

$$\tilde{a}_h^{DG}(c, v) := \sum_{K \in \mathscr{T}_h(\Omega)} \int_K \Big(\Delta c \, \Delta v + L(c)\, \Delta v + \Delta c \, L(v) \Big) dx + \tag{20}$$
$$\sum_{E \in \mathscr{E}_h(\bar{\Omega})} \frac{\alpha}{h_E} \, \mathbf{n}_E \cdot [\nabla c]_E \, \mathbf{n}_E \cdot [\nabla v]_E \, ds.$$

On $V_h^{(p)} + Z$ we introduce the mesh-dependent IPDG semi-norm

$$|c|_{2,h,\Omega} := \Big(\sum_{K \in \mathscr{T}_h(\Omega)} \|\Delta c\|_{0,K}^2 + \sum_{E \in \mathscr{E}_h(\bar{\Omega})} \frac{\alpha}{h_E} \, \|\mathbf{n}_E \cdot [\nabla c]_E\|_{0,E}^2 \Big)^{1/2} \tag{21}$$

and the mesh-dependent IPDG norm

$$\|c\|_{2,h,\Omega} := \Big(|c|_{2,h,\Omega}^2 + \|c\|_{0,\Omega}^2 \Big)^{1/2}. \tag{22}$$

From the Poincaré-Friedrichs inequality for piecewise H^2 functions (cf., e.g., [6]) we deduce that there exists a constant $C_{PF} > 0$ such that

$$\|\nabla v\|_{0,\Omega}^2 \le C_{PF} \, |v|_{2,h,\Omega}^2, \quad v \in V_h^{(p)} + Z. \tag{23}$$

It is not difficult to show that for sufficiently large penalty parameter α there exist constants $\gamma > 0$ and $\beta > 0$ such that the C^{0}IPDG bilinear form $\tilde{a}_h^{DG}$ satisfies the Gårding-type inequality

$$\tilde{a}_h^{DG}(c,c) \geq \gamma \, \|c\|_{2,h,\Omega}^2 - \beta \, \|c\|_{0,\Omega}^2, \quad c \in V_h^{(p)} + Z. \tag{24}$$

Moreover, there exists a constant $\Gamma > 0$ such that

$$|\tilde{a}_h^{DG}(c,v)| \leq \Gamma \, \|c\|_{2,h,\Omega} \, \|v\|_{2,h,\Omega}, \quad c,v \in V_h^{(p)} + Z. \tag{25}$$

The C^{0}IPDG method for the nonlinear fourth order elliptic equation has the advantage that we may approximate the dual variable w in the linear second order evolutionary equation by a function in $V_h^{(p)}$ as well. Hence, the C^{0}IPDG approximation of the initial-boundary value problem (6a)-(6d) for the sixth order Cahn-Hilliard equation reads:
Find $(c_h, w_h) \in H^1((0,T), V_h^{(p)}) \times L^2((0,T), V_h^{(p)})$ such that for all $v_h \in V_h^{(p)}$ it holds

$$(\sigma \, \frac{\partial c_h}{\partial t}, v_h)_{0,\Omega} - M \, (\nabla w_h, \nabla v_h)_{0,\Omega} = 0, \tag{26a}$$

$$a_h^{DG}(c_h, v_h) + \sum_{K \in \mathscr{T}_h(\Omega)} (g(c_h), v_h)_{0,K} - (w_h, v_h)_{0,\Omega} = 0, \tag{26b}$$

$$c_h(\cdot, 0) = I_h c_0. \tag{26c}$$

Remark 3 *(i) The unique solvability of* (26a)-(26c) *can be shown by similar arguments as in* [29].
(ii) The C^{0}IPDG approximation (26a)-(26c) *is consistent with the weak formulation* (9a)-(9c) *of the initial-boundary value problem* (6a)-(6d) *in the sense that for all $v_h \in V_h^{(p)}$ it holds (cf., e.g.,* [5]*)*

$$\langle \sigma \, \frac{\partial c}{\partial t}, v_h \rangle_{V,V^*} - M \, (\nabla w, \nabla v_h)_{0,\Omega} = 0, \tag{27a}$$

$$\tilde{a}_h^{DG}(c, v_h) + \sum_{K \in \mathscr{T}_h(\Omega)} (g(c), v_h)_{0,K} - (w, v_h)_{0,\Omega} = 0. \tag{27b}$$

5 Quasi-optimal a priori error estimates

We suppose that for some $r \geq 5$ the domain Ω has a boundary Γ of class C^{r+1}, the initial data satisfy $c_0 \in H^r(\Omega)$ as well as the compatibility condition (5) and that the unique solution (c,w) of (9a)-(9c) satisfies the regularity assumptions

$$c \in L^2((0,T),H^{r+1}(\Omega)) \cap H^1((0,T),H^{r-1}(\Omega)) \cap H^2((0,T),H^{r-3}(\Omega)), \quad (28a)$$
$$w \in L^2((0,T),H^{r-1}(\Omega)) \cap H^1((0,T),H^{r-3}(\Omega)) \cap H^2((0,T),H^{r-5}(\Omega)). \quad (28b)$$

Remark 4 *It follows from* (28a),(28b) *that the pair* (c,w) *satisfies*

$$c \in C([0,T],H^r(\Omega)) \cap C^1([0,T],H^{r-2}(\Omega)), \quad (29a)$$
$$w \in C([0,T],H^{r-2}(\Omega)) \cap C^1([0,T],H^{r-4}(\Omega)). \quad (29b)$$

The regularity assumptions (28a),(28b) imply the following interpolation estimates (cf., e.g., [4, 8])

$$\int_0^t \|c - I_h c\|_{m,\Omega}^2 \, d\tau \lesssim h^{2(min(p+1,r+1)-m)} \int_0^t |c|_{min(p+1,r+1),\Omega}^2 \, ds, \quad (30a)$$
$$\int_0^t \|\frac{\partial c}{\partial s} - I_h \frac{\partial c}{\partial s}\|_{0,\Omega}^2 \, ds \lesssim h^{2min(p+1,r-1)} \int_0^t |\frac{\partial c}{\partial s}|_{min(p+1,r-1),\Omega}^2 \, ds, \quad (30b)$$
$$\|(c - I_h c)(\cdot,t)\|_{m,\Omega}^2 \lesssim h^{2(min(p+1,r)-m)} \, |c(\cdot,t)|_{min(p+1,r),\Omega}^2, \quad (30c)$$
$$\int_0^t \|w - I_h w\|_{m,\Omega}^2 \, ds \lesssim h^{2(min(p+1,r-1)-m)} \int_0^t |w|_{min(p+1,r-1),\Omega}^2 \, ds, \quad (30d)$$
$$\int_0^t \|\frac{\partial w}{\partial s} - I_h \frac{\partial w}{\partial s}\|_{0,\Omega}^2 \, ds \lesssim h^{2min(p+1,r-3)} \int_0^t |\frac{\partial w}{\partial s}|_{min(p+1,r-3),\Omega}^2 \, ds, \quad (30e)$$
$$\|(w - I_h w)(\cdot,t)\|_{m,\Omega}^2 \lesssim h^{2(min(p+1,r-2)-m)} \, |w(\cdot,t)|_{min(p+1,r-2),\Omega}^2. \quad (30f)$$

For the interpolation error in the mesh-dependent IPDG-norm it follows from (30) that

$$\int_0^t \|c - I_h c\|_{2,h,\Omega}^2 \, d\tau \lesssim h^{2(min(p+1,r+1)-2)} \int_0^t |c|_{min(p+1,r+1),\Omega}^2 \, d\tau, \quad (31a)$$
$$\|(c - I_h c)(\cdot,t)\|_{2,h,\Omega}^2 \lesssim h^{2(min(p+1,r)-2)} \, |c(\cdot,t)|_{min(p+1,r),\Omega}^2,. \quad (31b)$$

Theorem 5. *Let* (c,w) *and* (c_h,w_h) *be the solutions of* (9a)-(9c) *and* (26a)-(26c). *Under the regularity assumptions* (28a),(28b), *and* (29a),(29b) *there exists a constant* $C > 0$, *independent of h, such that for all* $0 < t \leq T$ *it holds*

$$\|(c-c_h)(\cdot,t)\|^2_{2,h,\Omega} + \int_0^t \|c-c_h\|^2_{2,h,\Omega}\, ds + \int_0^t \|\nabla(w-w_h)\|^2_{0,\Omega}\, ds \lesssim \tag{32}$$
$$h^{2(p_{r+1}-2)} \int_0^t |c|^2_{p_{r+1},\Omega}\, ds + h^{2(p_{r-1}-2)} \int_0^t |\frac{\partial c}{\partial s}|^2_{p_{r-1},\Omega}\, ds +$$
$$h^{2(p_{r-1}-1)} \int_0^t |w|^2_{p_{r-1},\Omega}\, ds + h^{2p_{r-3}} \int_0^t |\frac{\partial w}{\partial s}|^2_{min(p+1,r-3),\Omega}\, ds +$$
$$h^{2(p_r-2)}\ |c_0|^2_{min(p+1,r),\Omega} + h^{2p_{r-2}}\ |w_0|^2_{p_{r-2},\Omega},$$

where $p_\ell := min(p+1,\ell)$.

The proof of Theorem 5 will be given by a series of lemmas and propositions.

First of all, recalling that $\tilde{a}_h^{DG}(\cdot,\cdot)$ satisfies the Gårding-type inequality (24), we perform a scaling of the primal variable c and the dual variable w according to

$$c(x,t) := \exp(\tau t)\ \hat{c}(x,t), \quad w(x,t) := \exp(\tau t)\ \hat{w}(x,t), \quad \tau > 0. \tag{33}$$

In the new variables $(\hat{c},\hat{w})$, the system (6a)-(6d) reads

$$\sigma\ \frac{\partial \hat{c}}{\partial t} + \sigma\tau\hat{c} - M\Delta\hat{w} = 0 \quad \text{in } Q, \tag{34a}$$
$$\kappa\Delta^2\hat{c} + \hat{g}(\hat{c}) - \hat{w} = 0 \quad \text{in } Q, \tag{34b}$$

with the boundary conditions

$$\mathbf{n}\cdot\nabla\hat{c} = \mathbf{n}\cdot\nabla\hat{w} = \mathbf{n}\cdot\nabla\Delta\hat{c} = 0 \quad \text{on } \Sigma, \tag{34c}$$

and the initial condition

$$\hat{c}(\cdot,0) = c_0 \quad \text{in } \Omega, \tag{34d}$$

where

$$\hat{g}(\hat{c}) := -\hat{a}(\hat{c})\ \Delta\hat{c} - a_2\ \exp(2\tau t)\ \hat{c}\ |\nabla\hat{c}|^2 + \hat{f}_0(\hat{c}), \tag{34e}$$
$$\hat{a}(\hat{c}) := a_0 + a_2\exp(2\tau t)\ \hat{c}^2, \tag{34f}$$
$$\hat{f}_0(\hat{c}) := \beta\ (\exp(\tau t)\ \hat{c}+1)(\exp(\tau t)\ \hat{c}-1)(\exp(2\tau t)\ \hat{c}^3 - (1-2h_0)\ \hat{c}). \tag{34g}$$

A pair (c,w) is said to be a weak solution of (6a)-(6d), if for all $v \in Z$ it holds

$$\sigma\ \langle\frac{\partial\hat{c}}{\partial t}, v\rangle_{V^*,V} + \sigma\tau\ (\hat{c},v)_{0,\Omega} + M\ (\nabla\hat{w},\nabla v)_{0,\Omega} = 0, \tag{35a}$$
$$\kappa\ (\Delta\hat{c},\Delta v)_{0,\Omega} + (\hat{g}(\hat{c}),v)_{0,\Omega} - (\hat{w},v)_{0,\Omega} = 0, \tag{35b}$$

and if the initial condition

$$\hat{c}(\cdot,0) = c_0. \tag{35c}$$

is satisfied. The semidiscrete variables (c_h, w_h) are scaled in the same way and hence, the semidiscrete approximation requires the computation of $(\hat{c}_h, \hat{w}_h) \in V_h^{(p)} \times V_h^{(p)}$ such that for all $v_h \in V_h^{(p)}$ it holds

$$(\sigma\ \frac{\partial \hat{c}_h}{\partial t}, v_h)_{0,\Omega} + \sigma\tau\ (\hat{c}_h, v_h)_{0,\Omega} - M\ (\nabla \hat{w}_h, \nabla v_h)_{0,\Omega} = 0, \tag{36a}$$

$$a_h^{DG}(\hat{c}_h, v_h) + \sum_{K\in\mathscr{T}_h(\Omega)} (\hat{g}(\hat{c}_h), v_h)_{0,K} - (\hat{w}_h, v_h)_{0,\Omega} = 0, \tag{36b}$$

$$\hat{c}_h(\cdot,0) = c_{h,0}. \tag{36c}$$

Remark 6 *If the regularity assumptions* (28a),(28b) *hold true for* (c,w)*, they also apply to* $(\hat{c},\hat{w})$ *and the interpolation estimates* (30) *are satisfied for* $(\hat{c},\hat{w})$ *as well.*

We will prove Theorem 5 based on an implicit time discretization of (34a)-(34d) and (36a)-(36c) by the backward Euler scheme with respect to an equidistant partition $\{t_m = m\,\Delta t, 0 \le m \le M\}, M \in \mathbb{N}$, of the time interval $[0,T]$ with step size $\Delta t = T/M$. Denoting by $(\hat{c}^m, \hat{w}^m)$ and $(\hat{c}_h^m, \hat{w}_h^m)$ approximations of $(\hat{c},\hat{w})$ and $(\hat{c}_h,\hat{w}_h)$ at time $t_m, 0 \le m \le M$, with $\hat{c}^0 = \hat{c}_0$ and $\hat{c}_h^0 = c_{h,0}$, the backward Euler scheme for (34a)-(34d) reads:
Find $(\hat{c}^m, \hat{w}^m)$ such that for all $v \in Z$ it holds

$$\sigma\ (\hat{c}^m, v)_{0,\Omega} + \sigma\tau\Delta t\ (\hat{c}^m, v)_{0,\Omega} + \Delta t\ (\nabla \hat{w}^m, \nabla v)_{0,\Omega} - \sigma\ (\hat{c}^{m-1}, v)_{0,\Omega} = 0, \tag{37a}$$

$$a_h^{DG}(\hat{c}^m, v) + (\hat{g}(\hat{c}^m, v)_{0,\Omega} - (\hat{w}^m, v)_{0,\Omega} = 0. \tag{37b}$$

The unique solvability of (37a),(37b) follows in the same way as that of (9a)-(9c).
Likewise, the backward Euler scheme for (36a)-(36c) is given by:
Find $(\hat{c}_h^m, \hat{w}_h^m)$ such that for all $v_h \in V_h^{(p)}$ it holds

$$\sigma\ (\hat{c}_h^m - \hat{c}_h^{m-1}, v_h)_{0,\Omega} + \sigma\tau\Delta t\ (\hat{c}_h^m, v_h)_{0,\Omega} + \Delta t\ (\nabla \hat{w}_h^m, \nabla v_h)_{0,\Omega} = 0, \tag{38a}$$

$$a_h^{DG}(\hat{c}_h^m, v_h) + \sum_{K\in\mathscr{T}_h(\Omega)} (\hat{g}(\hat{c}_h^m, v_h)_{0,K} - (\hat{w}_h^m, v_h)_{0,\Omega} = 0. \tag{38b}$$

Again, the unique solvability of (38a),(38b) follows in the same way as that of (26a)-(26c).

Remark 7 *(i) The C0IPDG approximation* (38a),(38b) *is consistent with* (37a),(37b) *in the sense that for all* $v_h \in V_h^{(p)}$ *it holds*

$$\sigma\ (\hat{c}^m-\hat{c}^{m-1},v_h)_{0,\Omega}+\sigma\tau\Delta t\ (\hat{c}^m,v_h)_{0,\Omega}+\Delta t\ (\nabla\hat{w}^m,\nabla v_h)_{0,\Omega}=0, \tag{39}$$
$$\tilde{a}_h^{DG}(\hat{c}^m,v_h)+\sum_{K\in\mathscr{T}_h(\Omega)}(\hat{g}(\hat{c}^m,v_h)_{0,K}-(\hat{w}^m,v_h)_{0,\Omega}=0.$$

(ii) Using similar arguments as in [26, 29] it can be shown that $\hat{c}_h^m$ is bounded in the C0IPDG norm uniformly in h, i.e., there exists a constant $C_B^{(1)}>0$, independent of h, such that

$$\|\hat{c}_h^m\|_{2,h,\Omega}\le C_B^{(1)},\quad 0\le m\le M. \tag{40}$$

Since $V_h^{(p)}$ is continuously embedded in $C(\bar{\Omega}$, there exists another constant $C_B^{(2)}>0$, independent of h, such that

$$\max_{x\in\bar{\Omega}}|\hat{c}_h^m(x)|\le C_B^{(2)},\quad 0\le m\le M. \tag{41}$$

Lemma 1. *Let $\hat{g}$ be given by* (34e). *Then there exists a constant C_1, independent of h, such that for $\hat{c}^m\in H^r(\Omega), r\ge 5, 0\le m\le M$, and $\hat{c}_h^m, v_h\in V_h^{(p)}, p\ge 2$, it holds*

$$|(\hat{g}(\hat{c}^m)-\hat{g}(\hat{c}_h^m),v_h)_{0,\Omega}|\le C_1\ \|\hat{c}^m-\hat{c}_h^m\|_{2,h,\Omega}\ \|v_h\|_{0,\Omega}. \tag{42}$$

Proof. Observing (34e) we have

$$\sum_{K\in\mathscr{T}_h(\Omega)}(\hat{g}(\hat{c}^m)-\hat{g}(\hat{c}_h^m),v_h)_{0,K}=-\sum_{K\in\mathscr{T}_h(\Omega)}(\hat{a}(\hat{c}^m)\Delta\hat{c}^m-\hat{a}(\hat{c}_h^m)\Delta\hat{c}_h^m,v_h)_{0,K} \tag{43}$$
$$-\sum_{K\in\mathscr{T}_h(\Omega)}a_2\exp(2\tau t)(\hat{c}^m|\nabla\hat{c}^m|^2-\hat{c}_h^m|\nabla\hat{c}_h^m|^2,v_h)_{0,K}+(\hat{f}_0(\hat{c}^m)-\hat{f}_0(\hat{c}_h^m),v_h)_{0,\Omega}.$$

In view of (3)

$$\hat{a}(\hat{c}_h^m)\Delta\hat{c}_h^m-\hat{a}(\hat{c}^m)\Delta\hat{c}^m=(\hat{a}(\hat{c}^m)-\hat{a}(\hat{c}_h^m))\Delta\hat{c}^m+\hat{a}(\hat{c}_h^m)(\Delta\hat{c}^m-\Delta\hat{c}_h^m)\ =$$
$$a_2\exp(2\tau t)(\hat{c}^m+\hat{c}_h^m)(\hat{c}^m-\hat{c}_h^m)\Delta\hat{c}^m+\hat{a}(\hat{c}_h^m)(\Delta\hat{c}^m-\Delta\hat{c}_h^m).$$

Then the first term on the right-hand side of (43) can be estimated according to

$$|\sum_{K\in\mathscr{T}_h(\Omega)}(\hat{a}(\hat{c}^m)\Delta\hat{c}^m-\hat{a}(\hat{c}_h^m)\Delta\hat{c}_h^m,v_h)_{0,K}|\ \le \tag{44}$$
$$D_1\sum_{K\in\mathscr{T}_h(\Omega)}\|\hat{c}^m-\hat{c}_h^m\|_{0,K}\ \|v_h\|_{0,K}+D_2\sum_{K\in\mathscr{T}_h(\Omega)}\|\Delta\hat{c}^m-\Delta\hat{c}_h^m\|_{0,K}\ \|v_h\|_{0,K},$$

where the constants $D_i, 1\le i\le 2$, are given by

$$D_1:=\max_{x\in\bar{\Omega}}|(\hat{c}^m+\hat{c}_h^m)(x)\Delta\hat{c}^m(x)|,\quad D_2:=\max_{x\in\bar{\Omega}}|a_0+a_2\exp(2\tau T)(\hat{c}_h^m(x))^2|.$$

We note that $\hat{c}^m, \Delta\hat{c}^m \in C(\bar{\Omega})$, since for $r \geq 5$ the spaces $Z \cap H^r(\Omega)$ and $H^{r-2}(\Omega)$ are continuously embedded in $C(\bar{\Omega})$. Moreover, due to (41) $\hat{c}_h^m$ is bounded in $C(\bar{\Omega})$ uniformly in h. Hence, the constants $D_i, 1 \leq i \leq 2$, are well defined and bounded from above independent of h.
For the second term on the right-hand side of (43) we split

$$a_2 \exp(2\tau t)\ (\hat{c}^m\ |\nabla\hat{c}^m|^2 - \hat{c}_h^m\ |\nabla\hat{c}_h^m|^2, v_h)_{0,K}, K \in \mathscr{T}_h(\Omega),$$

by means of

$$\begin{aligned}
&(a_2 \exp(2\tau t)\ (\hat{c}^m\ |\nabla\hat{c}^m|^2 - \hat{c}_h^m\ |\nabla\hat{c}_h^m|^2, v_h)_{0,K}\ = \\
&a_2 \exp(2\tau t)\ ((\hat{c}^m - \hat{c}_h^m)\ |\nabla\hat{c}^m|^2, v_h)_{0,K} + a_2 \exp(2\tau t)\ (\hat{c}_h^m\ \nabla\hat{c}^m \cdot (\nabla\hat{c}^m - \nabla\hat{c}_h^m), v_h)_{0,K} \\
&+ a_2 \exp(2\tau t)\ (\hat{c}_h^m\ \nabla\hat{c}_h^m \cdot (\nabla\hat{c}^m - \nabla\hat{c}_h^m), v_h)_{0,K}.
\end{aligned} \tag{45}$$

For the first term on the right-hand side of (45) we obtain

$$\begin{aligned}
&|\sum_{K \in \mathscr{T}_h(\Omega)} a_2 \exp(2\tau t)\ ((\hat{c}^m - \hat{c}_h^m)\ |\nabla\hat{c}^m|^2, v_h)_{0,K}|\ \leq \\
&D_3 \sum_{K \in \mathscr{T}_h(\Omega)} \|\hat{c}^m - \hat{c}_h^m\|_{0,K}\ \|v_h\|_{0,K},
\end{aligned} \tag{46}$$

where

$$D_3 := a_2 \exp(2\tau T)\ \max_{x \in \bar{\Omega}} |\nabla\hat{c}^m(x)|^2$$

which is well defined, since $\nabla\hat{c}^m \in C(\bar{\Omega})^2$.
Likewise, observing (23), the second term on the right-hand side of (45) can be estimated from above as follows

$$\begin{aligned}
&|\sum_{K \in \mathscr{T}_h(\Omega)} a_2 \exp(2\tau t)\ (\hat{c}_h^m\ \nabla\hat{c}^m \cdot (\nabla\hat{c}^m - \nabla\hat{c}_h^m), v_h)_{0,K}|\ \leq \\
&D_4 \sum_{K \in \mathscr{T}_h(\Omega)} \|\nabla\hat{c}^m - \nabla\hat{c}_h^m\|_{0,K}\ \|v_h\|_{0,K} \leq D_4 \|\nabla(\hat{c}^m - \hat{c}_h^m)\|_{0,\Omega}\ \|v_h\|_{0,\Omega}\ \leq \\
&C_{PF} D_4\ |\hat{c}^m - \hat{c}_h^m|_{2,h,\Omega}\ \|v_h\|_{0,\Omega},
\end{aligned} \tag{47}$$

where due to (41)

$$D_4 := a_2 \exp(2\tau T)\ \max_{x \in \bar{\Omega}} |\hat{c}_h^m(x)\ \nabla\hat{c}^m(x)| \leq a_2 \exp(2\tau T) C_B^{(2)}\ \max_{x \in \bar{\Omega}} |\nabla\hat{c}^m(x)|.$$

Since $\nabla\hat{c}^m \in C(\bar{\Omega})^2$, we note that D_4 is well defined and independent of h.
For the third term on the right-hand side of (45) we use the generalized Hölder inequality (1) with $v_1 = \nabla\hat{c}_h^m, v_2 = \nabla\hat{c}^m - \nabla\hat{c}_h^m, v_3 = v_h$, and $p_1 = 4/(1+2\varepsilon), p_2 = 4/(1-2\varepsilon), 0 < \varepsilon \ll 1$, and $p_3 = 2$.

$$
\begin{aligned}
&|\sum_{K\in\mathscr{T}_h(\Omega)} a_2\exp(2\tau t)\ (\hat{c}_h^m\ \nabla\hat{c}_h^m\cdot(\nabla\hat{c}^m-\nabla\hat{c}_h^m),v_h)_{0,K}|\ \le \\
&D_5 \sum_{K\in\mathscr{T}_h(\Omega)} \int_K |\nabla\hat{c}_h^m|\ |\nabla\hat{c}^m-\nabla\hat{c}_h^m|\ |v_h|\ dx\ \le \\
&D_5 \sum_{K\in\mathscr{T}_h(\Omega)} \|\hat{c}_h^m\|_{1,4/(1+2\varepsilon),K}\ \|\hat{c}^m-\hat{c}_h^m\|_{1,4/(1-2\varepsilon),K}\ \|v_h\|_{0,K}\ \le \\
&D_5\ \|\hat{c}_h^m\|_{1,4/(1+2\varepsilon),\Omega} \sum_{K\in\mathscr{T}_h(\Omega)} \|\hat{c}^m-\hat{c}_h^m\|_{1,4/(1-2\varepsilon),K}\ \|v_h\|_{0,K},
\end{aligned}
\tag{48}
$$

where

$$D_5 := a_2\exp(2\tau T)\ \max_{x\in\Omega}|\hat{c}_h^m(x)| \le a_2\exp(2\tau T)C_B^{(2)}.$$

Since $H^{3/2-\varepsilon}(\Omega)$ is continuously embedded in $W^{1,4/(1+2\varepsilon)}(\Omega)$ and $V_h^{(p)}$ is continuously embedded in $H^{3/2-\varepsilon}(\Omega)$ (cf., e.g., [5]), there exists a constant $D_6 > 0$ such that

$$\|\hat{c}_h^m\|_{1,4/(1+2\varepsilon),\Omega} \le D_6\ \|\hat{c}_h^m\|_{2,h,\Omega}. \tag{49}$$

Moreover, $H^2(K)$ is continuously embedded in $W^{1,4/(1-2\varepsilon)}(K)$ and hence, there exists a constant $D_7 > 0$, which can be chosen independent of h, such that for all $K \in \mathscr{T}_h(\Omega)$ it holds

$$\|\hat{c}^m-\hat{c}_h^m\|_{1,4/(1-2\varepsilon),K} \le D_7\ \|\hat{c}^m-\hat{c}_h^m\|_{2,K}. \tag{50}$$

Using (49) and (50) in (48), it follows that

$$
\begin{aligned}
&|\sum_{K\in\mathscr{T}_h(\Omega)} a_2\exp(2\tau t)\ (\hat{c}_h^m\ \nabla\hat{c}_h^m\cdot(\nabla\hat{c}^m-\nabla\hat{c}_h^m),v_h)_{0,K}|\ \le \\
&D_8 \sum_{K\in\mathscr{T}_h(\Omega)} \|\hat{c}^m-\hat{c}_h^m\|_{2,K}\ \|v_h\|_{0,K},
\end{aligned}
\tag{51}
$$

where due to (40)

$$D_8 := D_5\ D_6\ D_7\ \|\hat{c}_h^m\|_{2,h,\Omega} \le D_5\ D_6\ D_7\ C_B^{(1)}.$$

Finally, for the third term on the right-hand side of (43) we use that

$$\hat{f}_0(\hat{c}^m) - \hat{f}_0(\hat{c}_h^m) = \int_0^1 \hat{f}_0'(\hat{c}^m + s\ (\hat{c}_h^m - \hat{c}^m))\ ds\ (\hat{c}^m - \hat{c}_h^m)$$

to obtain

$$|(\hat{f}_0(\hat{c}^m) - \hat{f}_0(\hat{c}_h^m), v_h)_{0,\Omega}| \le D_9 \sum_{K \in \mathscr{T}_h(\Omega)} \|\hat{c}^m - \hat{c}_h^m\|_{0,K} \, \|v_h\|_{0,K}^2, \tag{52}$$

where

$$D_9 := \max_{x \in \Omega} \int_0^1 |\hat{f}_0'(\hat{c}^m + s\,(\hat{c}_h^m - \hat{c}^m))| \, ds.$$

Now, (42) is a direct consequence of (44),(46),(47),(51), and (52).

Corollary 1. *Under the assumptions of* Lemma 1 *there exists a constant* $C_2 > 0$*, independent of* h*, such that for* $0 \le m \le M$ *it holds*

$$\|I_h\hat{w}^m - \hat{w}_h^m\|_{0,\Omega} \le C_2 \, h^{-2} \, \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega} + \|\hat{w}^m - I_h\hat{w}^m\|_{0,\Omega}. \tag{53}$$

Proof. Obviously, we have

$$\|I_h\hat{w}^m - \hat{w}_h^m\|_{0,\Omega} = \sup_{v_h \in V_h^{(p)}} \frac{|(I_h\hat{w}^m - \hat{w}_h^m, v_h)_{0,\Omega}|}{\|v_h\|_{0,\Omega}}. \tag{54}$$

Using (6b) and (26b) we find

$$\begin{aligned} &(I_h\hat{w}^m - \hat{w}_h^m, v_h)_{0,\Omega} = (I_h\hat{w}^m - \hat{w}^m, v_h)_{0,\Omega} + (\hat{w}^m - \hat{w}_h^m, v_h)_{0,\Omega} \ = \\ &(I_h\hat{w}^m - \hat{w}^m, v_h)_{0,\Omega} + a_h^{DG}(\hat{c}^m - \hat{c}_h^m, v_h) + (\hat{g}(\hat{c}^m) - \hat{g}(\hat{c}_h^m), v_h)_{0,\Omega}. \end{aligned} \tag{55}$$

In view of (25), for the second term on the right-hand side of (55) we obtain

$$|a_h^{DG}(\hat{c}^m - \hat{c}_h^m, v_h)| \le \Gamma \ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega} \ \|v_h\|_{2,h,\Omega}. \tag{56}$$

On the other hand, using (42) from Lemma 1 we find

$$|a_h^{DG}(\hat{c}^m - \hat{c}_h^m, v_h) + (\hat{g}(\hat{c}^m) - \hat{g}(\hat{c}_h^m), v_h)_{0,\Omega}| \le C_1 \ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega} \ \|v_h\|_{2,h,\Omega}. \tag{57}$$

The inverse inequalities (15a)(15b) and the trace inequality (15c) imply the existence of a constant $D_{10} > 0$, independent of h, such that

$$\|v_h\|_{2,h,\Omega} \le D_{11} \ h^{-2} \ \|v_h\|_{0,\Omega}. \tag{58}$$

Using (57) and (58) in (55) gives the assertion.

We introduce the interpolation errors:

$$
\begin{aligned}
&e_{int}^{(1)}(\hat{c}^\ell) := \Delta t\ \|\hat{c}^\ell - I_h\hat{c}^\ell\|_{0,\Omega}^2, \quad e_{int}^{(2)}(\hat{c}^\ell) := \ \Delta t\ \|\nabla(\hat{c}^\ell - I_h\hat{c}^\ell)\|_{0,\Omega}^2, \quad 0 \le \ell \le m,\\
&e_{int}^{(3)}(\hat{c}^\ell) := \Delta t\ \|\frac{\hat{c}^\ell - \hat{c}^{\ell-1}}{\Delta t} - I_h(\frac{\hat{c}^\ell - \hat{c}^{\ell-1}}{\Delta t})\|_{0,\Omega}^2, \quad 1 \le \ell \le m,\\
&e_{int}^{(4)}(\hat{c}^\ell) := \Delta t\ \|\hat{c}^\ell - I_h\hat{c}^\ell\|_{2,h,\Omega}^2, \quad 0 \le \ell \le m, \qquad (59)\\
&e_{int}^{(5)}(\hat{c}^\ell) := \Delta t\ \|\frac{\hat{c}^\ell - \hat{c}^{\ell-1}}{\Delta t} - I_h(\frac{\hat{c}^\ell - \hat{c}^{\ell-1}}{\Delta t})\|_{2,h,\Omega}^2, \quad 1 \le \ell \le m,\\
&e_{int}^{(1)}(\hat{w}^\ell) := \Delta t\ \|\hat{w}^\ell - I_h\hat{w}^\ell\|_{0,\Omega}^2, \quad e_{int}^{(2)}(\hat{w}^\ell) := \Delta t\ \|\nabla(\hat{w}^\ell - I_h\hat{w}^\ell)\|_{0,\Omega}^2, \quad 0 \le \ell \le m,\\
&e_{int}^{(3)}(\hat{w}^\ell) := \Delta t\ \|\frac{\hat{w}^\ell - \hat{w}^{\ell-1}}{\Delta t} - I_h(\frac{\hat{w}^\ell - \hat{w}^{\ell-1}}{\Delta t})\|_{0,\Omega}^2, \quad 1 \le \ell \le m.
\end{aligned}
$$

Lemma 2. *Under the assumptions of Theorem 5, for $\eta, \xi > 0$ there exists a constant $C_3 > 0$, independent of h, such that it holds*

$$
\begin{aligned}
&\frac{1}{2}\eta\sigma\ \|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}^2 + \frac{1}{2}\tau\eta\sigma\Delta t\ \|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}^2\ \le \qquad (60)\\
&\frac{3}{2}\eta\xi^{-1}MC_{PF}\Delta t\ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}^2 + \frac{1}{3}\eta\xi M\Delta t\ \|\nabla(\hat{w}^m - \hat{w}_h^m)\|_{0,\Omega}^2\ +\\
&C_3\Big((1+\Delta t)\ \|\hat{c}^{m-1} - \hat{c}_h^{m-1}\|_{0,\Omega}^2 + \|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}^2 + \sum_{i=1}^{3} e_{int}^{(i)}(\hat{c}^m)\Big).
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&\eta\sigma\ \|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}^2 + \tau\eta\sigma\Delta t\ \|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}^2\ = \qquad (61)\\
&\eta\sigma\ (\hat{c}^m - \hat{c}_h^m, \hat{c}^m - I_h\hat{c}^m)_{0,\Omega} + \tau\eta\sigma\Delta t\ (\hat{c}^m - \hat{c}_h^m, \hat{c}^m - I_h\hat{c}^m)_{0,\Omega}\ +\\
&\eta\sigma\ (\hat{c}^m - \hat{c}_h^m, I_h\hat{c}^m - \hat{c}_h^m)_{0,\Omega} + \tau\eta\sigma\Delta t\ (\hat{c}^m - \hat{c}_h^m, I_h\hat{c}^m - \hat{c}_h^m)_{0,\Omega}.
\end{aligned}
$$

By Young's inequality with $\varepsilon = 1/4$ the first two terms on the right-hand side of (61) can be estimated from above according to

$$
\begin{aligned}
&\eta\sigma\ |(\hat{c}^m - \hat{c}_h^m, \hat{c}^m - I_h\hat{c}^m)_{0,\Omega}| \le \eta\sigma\ \|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}\ \|\hat{c}^m - I_h\hat{c}^m\|_{0,\Omega}\ \le \qquad (62a)\\
&\eta\sigma\ \|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}\ \Big(\Delta t\ \|\frac{\hat{c}^m - \hat{c}^{m-1}}{\Delta t} - I_h(\frac{\hat{c}^m - \hat{c}^{m-1}}{\Delta t})\|_{0,\Omega} + \|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}\Big)\\
&\le \frac{1}{4}\eta\sigma(1+\tau\Delta t)\ \|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}^2 + \eta\sigma\ \|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}^2 + \eta\sigma\tau^{-1}\ e_{int}^{(3)}(\hat{c}^m),\\
&\tau\eta\sigma\Delta t\ |(\hat{c}^m - \hat{c}_h^m, \hat{c}^m - I_h\hat{c}^m)_{0,\Omega}| \le \frac{1}{4}\tau\eta\sigma\Delta t\ \|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}^2 + \tau\eta\sigma\ e_{int}^{(1)}(\hat{c}^m). \qquad (62b)
\end{aligned}
$$

In view of (37a) and (38a), for the last two terms on the right-hand side of (61) we find

$$\eta\sigma\ (\hat{c}^m - \hat{c}_h^m, I_h\hat{c}^m - \hat{c}_h^m)_{0,\Omega} + \tau\eta\sigma\Delta t\ (\hat{c}^m - \hat{c}_h^m, I_h\hat{c}^m - \hat{c}_h^m)_{0,\Omega} = \tag{63}$$
$$\eta\sigma\ (\hat{c}^{m-1} - \hat{c}_h^{m-1}, I_h\hat{c}^m - \hat{c}_h^m)_{0,\Omega} - \eta M\Delta t\ (\nabla(\hat{w}^m - \hat{w}_h^m), \nabla(I_h\hat{c}^m - \hat{c}_h^m))_{0,\Omega}.$$

The first term on the right-hand side of (63) can be estimated from above as follows:

$$\eta\sigma\ |(\hat{c}^{m-1} - \hat{c}_h^{m-1}, I_h\hat{c}^m - \hat{c}_h^m)_{0,\Omega}| \le \tag{64}$$
$$\eta\sigma\ |(\hat{c}^{m-1} - \hat{c}_h^{m-1}, I_h\hat{c}^m - \hat{c}^m)_{0,\Omega}| + \eta\sigma\ |(\hat{c}^{m-1} - \hat{c}_h^{m-1}, \hat{c}^m - \hat{c}_h^m)_{0,\Omega}|.$$

As in (62a), for the first term on the right-hand side of (64) Young's inequality with $\varepsilon = 1/4$ yields

$$\eta\sigma\ |(\hat{c}^{m-1} - \hat{c}_h^{m-1}, \hat{c}^m - I_h\hat{c}^m)_{0,\Omega}| \le \tag{65}$$
$$\frac{1}{4}\eta\sigma(1+\tau\Delta t)\ \|\hat{c}^{m-1} - \hat{c}_h^{m-1}\|_{0,\Omega}^2 + \eta\sigma\|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}^2 + \tau^{-1}\eta\sigma\ e_{int}^{(3)}(\hat{c}^m).$$

For the second term on the right-hand side of (64) we obtain

$$\eta\sigma\ |(\hat{c}^{m-1} - \hat{c}_h^{m-1}, \hat{c}^m - \hat{c}_h^m)_{0,\Omega}| \le \eta\sigma\Big(\frac{1}{4}\|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}^2 + \|\hat{c}^{m-1} - \hat{c}_h^{m-1}\|_{0,\Omega}^2\Big). \tag{66}$$

For the second term on the right-hand side of (63) Young's inequality with $\varepsilon = 1/6$ and the Poincaré-Friedrichs inequality (23) yield

$$\eta M\Delta t\ |(\nabla(\hat{w}^m - \hat{w}_h^m), \nabla(I_h\hat{c}^m - \hat{c}_h^m))_{0,\Omega}| \le \tag{67}$$
$$\eta M\Delta t\ \Big(|(\nabla(\hat{w}^m - \hat{w}_h^m), \nabla(I_h\hat{c}^m - \hat{c}^m))_{0,\Omega}| + |(\nabla(\hat{w}^m - \hat{w}_h^m), \nabla(\hat{c}^m - \hat{c}_h^m))_{0,\Omega}|\Big) \le$$
$$\eta M\Delta t\Big(\frac{1}{3}\xi\ \|\nabla(\hat{w}^m - \hat{w}_h^m)\|_{0,\Omega}^2 + \frac{3}{2}\xi^{-1}C_{PF}\ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}^2 + \frac{3}{2}\eta\xi^{-1}M\ e_{int}^{(2)}(\hat{c}^m).$$

The assertion follows from (61)-(67).

Lemma 3. *Under the assumptions of Theorem 5, for* $\lambda > 0$ *there exist constants* $C_i > 0, 4 \le i \le 6$, *independent of h, such that it holds*

$$\frac{5}{6}\lambda M\Delta t\ \|\nabla(\hat{w}^m - \hat{w}_h^m)\|_{0,\Omega}^2 + \frac{1}{2}\lambda\sigma\gamma(1+\Delta t)\ \ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}^2 \le \tag{68}$$
$$\lambda\sigma(C_4 + \tau C_5\Delta t)\ \|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}^2 + C_6\Big((1+\Delta t)\ \|\hat{c}^{m-1} - \hat{c}_h^{m-1}\|_{0,\Omega}^2 +$$
$$\|\hat{c}^{m-1} - \hat{c}_h^{m-1}\|_{2,h,\Omega}^2 + (1+h^{-4})(\|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}^2 + \|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{2,h,\Omega}^2) +$$
$$\|\hat{w}^{m-1} - I_h\hat{w}^{m-1}\|_{0,\Omega}^2 + (1+h^{-4})(e_{int}^{(1)}(\hat{c}^m) + e_{int}^{(3)}(\hat{c}^m)) + e_{int}^{(4)}(\hat{c}^m) + e_{int}^{(5)}(\hat{c}^m) +$$
$$e_{int}^{(1)}(\hat{w}^m) + e_{int}^{(2)}(\hat{w}^m) + e_{int}^{(1)}(\hat{c}^{m-1}) + e_{int}^{(1)}(\hat{w}^{m-1})\Big).$$

Proof. We have

$$\lambda M\Delta t\ \|\nabla(\hat{w}^m - \hat{w}_h^m)\|_{0,\Omega}^2 = \lambda M\Delta t\ (\nabla(\hat{w}^m - \hat{w}_h^m), \nabla(\hat{w}^m - I_h\hat{w}^m)_{0,\Omega} + \\ \lambda M\Delta t\ (\nabla(\hat{w}^m - \hat{w}_h^m), \nabla(I_h\hat{w}^m - \hat{w}_h^m))_{0,\Omega}. \tag{69}$$

For the first term on the right-hand side of (69) Young's inequality with $\varepsilon = 1/6$ yields

$$\lambda M\Delta t\ |(\nabla(\hat{w}^m - \hat{w}_h^m), \nabla(\hat{w}^m - I_h\hat{w}^m))_{0,\Omega}| \ \leq \\ \frac{1}{6}\lambda M\Delta t\ \|\nabla(\hat{w}^m - \hat{w}_h^m)\|_{0,\Omega}^2 + \frac{3}{2}\lambda M\ e_{int}^{(2)}(\hat{w}^m). \tag{70}$$

Taking advantage of (37a) and (38a), for the second term on the right-hand side of (69) it follows that

$$\lambda M\Delta t\ (\nabla(\hat{w}^m - \hat{w}_h^m), \nabla(I_h\hat{w}^m - \hat{w}_h^m))_{0,\Omega} = \lambda\sigma\ (\hat{c}^{m-1} - \hat{c}_h^{m-1}, I_h\hat{w}^m - \hat{w}_h^m)_{0,\Omega} \\ -\tau\lambda\sigma\Delta t\ (\hat{c}^m - \hat{c}_h^m, I_h\hat{w}^m - \hat{w}_h^m)_{0,\Omega} - \lambda\sigma\ (\hat{c}^m - \hat{c}_h^m, I_h\hat{w}^m - \hat{w}_h^m)_{0,\Omega}\ = \\ \lambda\sigma\ (\hat{c}^{m-1} - \hat{c}_h^{m-1}, I_h\hat{w}^m - \hat{w}^m)_{0,\Omega} - \tau\lambda\sigma\Delta t\ (\hat{c}^m - \hat{c}_h^m, I_h\hat{w}^m - \hat{w}^m)_{0,\Omega} \\ -\lambda\sigma\ (\hat{c}^m - \hat{c}_h^m, I_h\hat{w}^m - \hat{w}^m)_{0,\Omega} + \lambda\sigma\ (\hat{c}^{m-1} - \hat{c}_h^{m-1}, \hat{w}^m - \hat{w}_h^m)_{0,\Omega} \\ -\tau\lambda\sigma\Delta t\ (\hat{c}^m - \hat{c}_h^m, \hat{w}^m - \hat{w}_h^m)_{0,\Omega} - \lambda\sigma\ (\hat{c}^m - \hat{c}_h^m, \hat{w}^m - \hat{w}_h^m)_{0,\Omega}. \tag{71}$$

The first and the third term on the right-hand side of (71) can be estimated from above as the corresponding terms in Lemma 2 using Young's inequality with $\varepsilon = 1$ and $\varepsilon = 1/6$:

$$\lambda\sigma\ |(\hat{c}^{m-1} - \hat{c}_h^{m-1}, I_h\hat{w}^m - \hat{w}^m)_{0,\Omega}| \leq \lambda\sigma(1+\Delta t)\ \|\hat{c}^{m-1} - \hat{c}_h^{m-1}\|_{0,\Omega}^2 + \\ \frac{1}{4}\lambda\sigma\|\hat{w}^{m-1} - I_h\hat{w}^{m-1}\|_{0,\Omega}^2 + \frac{1}{4}\lambda\sigma\ e_{int}^{(3)}(\hat{w}^m), \tag{72a}$$

$$\lambda\sigma\ |(\hat{c}^m - \hat{c}_h^m, I_h\hat{w}^m - \hat{w}^m)_{0,\Omega}| \leq \lambda\sigma(1+\frac{1}{6}\Delta t)\ \|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}^2 + \\ \frac{1}{4}\lambda\sigma\ \|\hat{w}^{m-1} - I_h\hat{w}^{m-1}\|_{0,\Omega}^2 + \frac{3}{2}\lambda\sigma\ e_{int}^{(3)}(\hat{w}^m). \tag{72b}$$

For the second term on the right-hand side of (71) Young's inequality with $\varepsilon = 1$ implies

$$\tau\lambda\sigma\Delta t\ |(\hat{c}^m - \hat{c}_h^m, I_h\hat{w}^m - \hat{w}^m)_{0,\Omega}| \leq \tau\lambda\sigma\Big(\Delta t\ \|\hat{c}^m - \hat{c}_h^m\|_{0,\Omega}^2 + \frac{1}{4}\ e_{int}^{(1)}(\hat{w}^m)\Big). \tag{73}$$

For the last three terms on the right-hand side of (71) we obtain

$$\lambda\sigma\ (\hat{c}^{m-1} - \hat{c}_h^{m-1}, \hat{w}^m - \hat{w}_h^m)_{0,\Omega} - \tau\lambda\sigma\Delta t\ (\hat{c}^m - \hat{c}_h^m, \hat{w}^m - \hat{w}_h^m)_{0,\Omega} \\ -\lambda\sigma\ (\hat{c}^m - \hat{c}_h^m, \hat{w}^m - \hat{w}_h^m)_{0,\Omega} = \lambda\sigma\ (\hat{c}^{m-1} - I_h\hat{c}^{m-1}, \hat{w}^m - \hat{w}_h^m)_{0,\Omega} \\ -\tau\lambda\sigma\Delta t\ (\hat{c}^m - I_h\hat{c}^m, \hat{w}^m - \hat{w}_h^m)_{0,\Omega} - \lambda\sigma\ (\hat{c}^m - I_h\hat{c}^m, \hat{w}^m - \hat{w}_h^m)_{0,\Omega} \\ +\lambda\sigma\ (I_h\hat{c}^{m-1} - \hat{c}_h^{m-1}, \hat{w}^m - \hat{w}_h^m)_{0,\Omega} - \tau\lambda\sigma\Delta t\ (I_h\hat{c}^m - \hat{c}_h^m, \hat{w}^m - \hat{w}_h^m)_{0,\Omega} \\ -\lambda\sigma\ (I_h\hat{c}^m - \hat{c}_h^m, \hat{w}^m - \hat{w}_h^m)_{0,\Omega}. \tag{74}$$

Using Corollary (1) and Young's inequality with $\varepsilon = 1/18$ and $\varepsilon = 1/2$, the first term on the right-hand side of (74) can be estimated from above as follows:

$$
\begin{aligned}
&\lambda\sigma\ |(\hat{c}^{m-1} - I_h\hat{c}^{m-1}, \hat{w}^m - \hat{w}_h^m)_{0,\Omega}|\ \le \qquad (75)\\
&\lambda\sigma\ \|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}\ \Big(\|\hat{w}^m - I_h\hat{w}^m\|_{0,\Omega} + \|I_h\hat{w}^m - \hat{w}_h^m\|_{0,\Omega}\Big)\ \le\\
&\lambda\sigma\ \|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}\ \Big(2\ \|\hat{w}^m - I_h\hat{w}^m\|_{0,\Omega} + C_2 h^{-2}\ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}\Big)\ \le\\
&\lambda\sigma\ \|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}\ \Big(2\Delta t\ \|\frac{\hat{w}^m - \hat{w}^{m-1}}{\Delta t} - I_h(\frac{\hat{w}^m - \hat{w}^{m-1}}{\Delta t})\|_{0,\Omega} +\\
&2\ \|\hat{w}^{m-1} - I_h\hat{w}^{m-1}\|_{0,\Omega} + C_2 h^{-2}\ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}\Big)\ \le\\
&\frac{1}{18}\lambda\sigma\ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}^2 + \frac{9}{2}\lambda\sigma C_2^2 h^{-4}\ \|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}^2 +\\
&\lambda\sigma\Big(\|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}^2 + \|\hat{w}^{m-1} - I_h\hat{w}^{m-1}\|_{0,\Omega}^2 + e_{int}^{(1)}(\hat{c}^{m-1}) + e_{int}^{(3)}(\hat{w}^m)\Big).
\end{aligned}
$$

Likewise, by Young's inequality with $\varepsilon = 1/14, \varepsilon = 1/18$, and $\varepsilon = 1/2$ and observing $\Delta t \le T$, for the third term on the right-hand side of (74) we get

$$
\begin{aligned}
&\lambda\sigma\ |(\hat{c}^m - I_h\hat{c}^m, \hat{w}^m - \hat{w}_h^m)_{0,\Omega}|\ \le \qquad (76)\\
&\lambda\sigma\ \Big(\Delta t\ \|\frac{\hat{c}^m - \hat{c}^{m-1}}{\Delta t} - I_h(\frac{\hat{c}^m - \hat{c}^{m-1}}{\Delta t})\|_{0,\Omega} + \|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}\Big)\ \cdot\\
&\Big(2\Delta t\ \|\frac{\hat{w}^m - \hat{w}^{m-1}}{\Delta t} - I_h(\frac{\hat{w}^m - \hat{w}^{m-1}}{\Delta t})\|_{0,\Omega} + 2\ \|\hat{w}^{m-1} - I_h\hat{w}^{m-1}\|_{0,\Omega}\\
&+ C_2 h^{-2}\ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}\Big)\ \le\\
&\frac{1}{14}\lambda\sigma\Delta t\ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}^2 + \lambda\sigma\Big((\frac{7}{2}C_2^2 h^{-4} + T)\ e_{int}^{(3)}(\hat{c}^m) + T\ e_{int}^{(3)}(\hat{w}^m)\Big) +\\
&\lambda\sigma\Big(e_{int}^{(3)}(\hat{c}^m) + e_{int}^{(1)}(\hat{w}^m) + e_{int}^{(3)}(\hat{w}^m) + e_{int}^{(1)}(\hat{c}^{m-1})\Big) +\\
&\frac{1}{18}\lambda\sigma C_3\ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}^2 + \frac{9}{2}\lambda\sigma C_2^2 h^{-4}\ \|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}^2 +\\
&\lambda\sigma\ \|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega}^2 + \lambda\sigma\|\hat{w}^{m-1} - I_h\hat{w}^{m-1}\|_{0,\Omega}^2.
\end{aligned}
$$

Finally, for the second term on the right-hand side of (74) Young's inequality with $\varepsilon = 1/14$ and $\varepsilon = 1/2$ gives

$$
\begin{aligned}
&\tau\lambda\sigma\Delta t\ |(\hat{c}^m - I_h\hat{c}^m, \hat{w}^m - \hat{w}_h^m)_{0,\Omega}|\ \le \qquad (77)\\
&\tau\lambda\sigma\Delta t\ \|\hat{c}^m - I_h\hat{c}^m\|_{0,\Omega}\Big(\|\hat{w}^m - I_h\hat{w}^m\|_{0,\Omega} + \|I_h\hat{w}^m - \hat{w}_h^m\|_{0,\Omega}\Big)\ \le\\
&\tau\lambda\sigma\Delta t\ \|\hat{c}^m - I_h\hat{c}^m\|_{0,\Omega}\Big(2\ \|\hat{w}^m - I_h\hat{w}^m\|_{0,\Omega} + C_2\ h^{-2}\ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}\Big)\ \le\\
&\frac{1}{14}\tau\lambda\sigma\gamma\Delta t\ \|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}^2 + \frac{7}{2}\tau\lambda\sigma\Big(1 + \gamma^{-1}C_2^2 h^{-4}\Big)\ e_{int}^{(1)}(\hat{c}^m) + \tau\lambda\sigma\ e_{int}^{(1)}(\hat{w}^m).
\end{aligned}
$$

Using (37b) and (38b), for the first of the last three terms on the right-hand side of (74) we obtain

$$\begin{aligned}&\lambda\sigma\ (I_h\hat{c}^{m-1}-\hat{c}_h^{m-1},\hat{w}^m-\hat{w}_h^m)_{0,\Omega}\ = \\ &\lambda\sigma\ \Big(a_h^{DG}(\hat{c}^m-\hat{c}_h^m,I_h\hat{c}^{m-1}-\hat{c}_h^{m-1})+(\hat{g}(\hat{c}^m)-\hat{g}(\hat{c}_h^m),I_h\hat{c}^{m-1}-\hat{c}_h^{m-1})_{0,\Omega}\Big)\ = \\ &\lambda\sigma\ \Big(a_h^{DG}(\hat{c}^m-\hat{c}_h^m,I_h\hat{c}^{m-1}-\hat{c}^{m-1})+a_h^{DG}(\hat{c}^m-\hat{c}_h^m,\hat{c}^{m-1}-\hat{c}_h^{m-1})\Big)\ + \\ &\lambda\sigma\ \Big((\hat{g}(\hat{c}^m)-\hat{g}(\hat{c}_h^m),I_h\hat{c}^{m-1}-\hat{c}^{m-1})+(\hat{g}(\hat{c}^m)-\hat{g}(\hat{c}_h^m),\hat{c}^{m-1}-\hat{c}_h^{m-1})_{0,\Omega}\Big).\end{aligned} \tag{78a}$$

Similarly, for the second term we get

$$\begin{aligned}&\tau\lambda\sigma\Delta t\ (I_h\hat{c}^m-\hat{c}_h^m,\hat{w}^m-\hat{w}_h^m)_{0,\Omega}\ = \\ &\tau\lambda\sigma\Delta t\ \Big(a_h^{DG}(\hat{c}^m-\hat{c}_h^m,I_h\hat{c}^m-\hat{c}^m)+a_h^{DG}(\hat{c}^m-\hat{c}_h^m,\hat{c}^m-\hat{c}_h^m)\Big)\ + \\ &\tau\lambda\sigma\Delta t\ \Big((\hat{g}(\hat{c}^m)-\hat{g}(\hat{c}_h^m),I_h\hat{c}^m-\hat{c}^m)+(\hat{g}(\hat{c}^m)-\hat{g}(\hat{c}_h^m),\hat{c}^m-\hat{c}_h^m)_{0,\Omega}\Big),\end{aligned} \tag{78b}$$

whereas for the third term we obtain

$$\begin{aligned}&-\lambda\sigma\ (I_h\hat{c}^m-\hat{c}_h^m,\hat{w}^m-\hat{w}_h^m)_{0,\Omega}\ = \\ &-\lambda\sigma\ \Big(\tilde{a}_h^{DG}(\hat{c}^m-\hat{c}_h^m,I_h\hat{c}^m-\hat{c}^m)+\tilde{a}_h^{DG}(\hat{c}^m-\hat{c}_h^m,\hat{c}^m-\hat{c}_h^m)\Big) \\ &-\lambda\sigma\ \Big((\hat{g}(\hat{c}^m)-\hat{g}(\hat{c}_h^m),I_h\hat{c}^m-\hat{c}^m)_{0,\Omega}+(\hat{g}(\hat{c}^m)-\hat{g}(\hat{c}_h^m),\hat{c}^m-\hat{c}_h^m)_{0,\Omega}\Big).\end{aligned} \tag{78c}$$

Taking advantage of (25) and (42) from Lemma 1 and using Young's inequality with $\varepsilon = 1/18$, for (78a) we can establish the upper bound

$$\begin{aligned}&\lambda\sigma\ |(I_h\hat{c}^{m-1}-\hat{c}_h^{m-1},\hat{w}^m-\hat{w}_h^m)_{0,\Omega}|\le\frac{2}{9}\lambda\sigma\ \|\hat{c}^m-\hat{c}_h^m\|_{2,h,\Omega}^2\ + \\ &\frac{9}{2}\lambda\sigma(\Gamma^2+C_1^2)\ \|\hat{c}^{m-1}-\hat{c}_h^{m-1}\|_{2,h,\Omega}^2+\frac{9}{2}\lambda\sigma(\Gamma^2+C_1^2)\ \|\hat{c}^{m-1}-I_h\hat{c}_h^{m-1}\|_{2,h,\Omega}^2.\end{aligned} \tag{79}$$

Similarly, for (78b) Gårding's inequality (24) and Young's inequality with $\varepsilon = 1/14$ yield

$$\begin{aligned}&-\tau\lambda\sigma\Delta t\ (I_h\hat{c}^m-\hat{c}_h^m,\hat{w}^m-\hat{w}_h^m)_{0,\Omega}\le-\tau\lambda\sigma\gamma\Delta t\ \|\hat{c}^m-\hat{c}_h^m\|_{2,h,\Omega}^2\ + \\ &\tau\lambda\sigma\beta\Delta t\ \|\hat{c}^m-\hat{c}_h^m\|_{0,\Omega}^2+\frac{3}{14}\tau\lambda\sigma\Delta t\ \|\hat{c}^m-\hat{c}_h^m\|_{2,h,\Omega}^2\ + \\ &\frac{7}{2}\tau\lambda\sigma\gamma^{-1}\Big(C_1^2\Delta t\ \|\hat{c}^m-\hat{c}_h^m\|_{0,\Omega}^2+\Big(C_1^2\ e_{int}^{(1)}(\hat{c}^m)+\Gamma^2\ e_{int}^{(4)}(\hat{c}^m)\Big)\Big).\end{aligned} \tag{80}$$

Finally, for (78c) another application of Gårding's inequality (24) and Young's inequality with $\varepsilon = 1/14$ and $\varepsilon = 1/18$ we obtain

$$
\begin{aligned}
&-\lambda\sigma\ (I_h\hat{c}^m-\hat{c}_h^m,\hat{w}^m-\hat{w}_h^m)_{0,\Omega}\le\lambda\sigma\Big(-\gamma\,\|\hat{c}^m-\hat{c}_h^m\|_{2,h,\Omega}^2+\beta\ \|\hat{c}^m-\hat{c}_h^m\|_{0,\Omega}^2\Big)+\\
&\frac{3}{18}\lambda\sigma\ \|\hat{c}^m-\hat{c}_h^m\|_{2,h,\Omega}^2+\frac{1}{7}\lambda\sigma\Delta t\ \|\hat{c}^m-\hat{c}_h^m\|_{2,h,\Omega}^2+\frac{9}{2}\lambda\sigma C_1^2\ \|\hat{c}^m-\hat{c}_h^m\|_{0,\Omega}^2\ +\\
&\frac{9}{2}\lambda\sigma\Big(C_1^2\ \||\|\hat{c}^{m-1}-I_h\hat{c}^{m-1}\|_{0,\Omega}^2+\Gamma^2\ \|\hat{c}^{m-1}-I_h\hat{c}^{m-1}\|_{2,h,\Omega}^2\Big)+\\
&\frac{7}{2}\lambda\sigma\Big(C_1^2\ e_{int}^{(3)}(\hat{c}^m)+\Gamma^2\ e_{int}^{(5)}(\hat{c}^m)\Big). \qquad (81)
\end{aligned}
$$

The assertion follows from (69)-(81).

Proposition 8 *Under the assumptions of Theorem 5 there exists a constant $C_7>0$, independent of h, such that it holds*

$$
\begin{aligned}
&\|\hat{c}^m-\hat{c}_h^m\|_{0,\Omega}^2+\tau\Delta t\ \|\hat{c}^m-\hat{c}_h^m\|_{0,\Omega}^2+\frac{1}{2}\lambda\sigma\gamma\ \Big(\|\hat{c}^m-\hat{c}_h^m\|_{2,h,\Omega}^2 \qquad (82)\\
&+\Delta t\ \|\hat{c}^m-\hat{c}_h^m\|_{2,h,\Omega}^2\Big)+M\Delta t\ \|\nabla(\hat{w}^m-\hat{w}_h^m)\|_{0,\Omega}^2\ \le\\
&C_7\Big(\|\hat{c}^{m-1}-\hat{c}_h^{m-1}\|_{0,\Omega}^2+\|\hat{c}^{m-1}-\hat{c}_h^{m-1}\|_{2,h,\Omega}^2+h^{-4}\ \|\hat{c}^{m-1}-I_h\hat{c}^{m-1}\|_{0,\Omega}^2\ +\\
&\|\hat{c}^{m-1}-I_h\hat{c}^{m-1}\|_{2,h,\Omega}^2+\|\hat{w}^{m-1}-I_h\hat{w}^{m-1}\|_{0,\Omega}^2+h^{-4}(e_{int}^{(1)}(\hat{c}^m)+e_{int}^{(3)}(\hat{c}^m))\ +\\
&\sum_{i=4}^{5}e_{int}^{(i)}(\hat{c}^m)+\sum_{i=1}^{3}e_{int}^{(i)}(\hat{w}^m)+e_{int}^{(1)}(\hat{c}^{m-1})+e_{int}^{(1)}(\hat{w}^{m-1})\Big).
\end{aligned}
$$

Proof. The estimates (60) from Lemma 2 and (68) from Lemma 3 imply the existence of a constant $D_{10}>0$, independent of h, such that

$$
\begin{aligned}
&\sigma(\eta-\frac{3}{2}\lambda C_6)\ \|\hat{c}^m-\hat{c}_h^m\|_{0,\Omega}^2+\tau\sigma(\eta-\frac{1}{2}\lambda C_7)\Delta t\ \|\hat{c}^m-\hat{c}_h^m\|_{0,\Omega}^2\ + \qquad (83)\\
&\frac{1}{2}\lambda\sigma\gamma\,\|\hat{c}^m-\hat{c}_h^m\|_{2,h,\Omega}^2+\frac{1}{2}\lambda\sigma(\tau\gamma-\frac{3}{2}\eta\xi^{-1}C_{PF})\Delta t\|\hat{c}^m-\hat{c}_h^m\|_{2,h,\Omega}^2\ +\\
&M(\frac{5}{6}\lambda-\frac{1}{3}\eta\xi)\Delta t\ \|\nabla(\hat{w}^m-\hat{w}_h^m)\|_{0,\Omega}^2\le D_{10}\Big((1+\Delta t)\ \|\hat{c}^{m-1}-\hat{c}_h^{m-1}\|_{0,\Omega}^2\ +\\
&+\|\hat{c}^{m-1}-\hat{c}_h^{m-1}\|_{2,h,\Omega}^2+(1+h^{-4})(\|\hat{c}^{m-1}-I_h\hat{c}^{m-1}\|_{0,\Omega}^2+\|\hat{c}^{m-1}-I_h\hat{c}^{m-1}\|_{2,h,\Omega}^2)\\
&+\|\hat{w}^{m-1}-I_h\hat{w}^{m-1}\|_{0,\Omega}^2+(1+h^{-4})(e_{int}^{(1)}(\hat{c}^m)+e_{int}^{(3)}(\hat{c}^m))+e_{int}^{(2)}(\hat{c}^m)\ +\\
&\sum_{i=4}^{5}e_{int}^{(i)}(\hat{c}^m)+\sum_{i=1}^{3}e_{int}^{(i)}(\hat{w}^m)+e_{int}^{(1)}(\hat{c}^{m-1})+e_{int}^{(1)}(\hat{w}^{m-1})\Big).
\end{aligned}
$$

We choose $\frac{6}{5}<\lambda<2$ and $\eta>0$ such that

$$
\eta-\max(\frac{3}{2}C_6,\frac{1}{2}C_7)\lambda\ge\sigma^{-1}.
$$

Then, we choose $\xi>0$ by means of

$$\frac{5}{6}\lambda - \frac{1}{3}\eta\xi \geq 1 \quad \Longleftrightarrow \quad \xi \leq \frac{5\lambda - 6}{2\eta}.$$

Finally, we choose $\tau > 0$ according to

$$\gamma\tau - \frac{3}{2}\eta\xi^{-1}C_{PF} \geq \gamma \quad \Longleftrightarrow \quad \tau \geq \frac{2\xi\gamma + 3\eta C_{PF}}{2\xi\gamma}.$$

For this choice of λ, η, ξ, and τ, the assertion follows from (83) observing that $e_{int}^{(1)}(\hat{c}^m) \leq e_{int}^{(4)}(\hat{c}^m), e_{int}^{(2)}(\hat{c}^m) \leq C_{PF}^2 e_{int}^{(4)}(\hat{c}^m), e_{int}^{(3)}(\hat{c}^m) \leq e_{int}^{(5)}(\hat{c}^m)$, and $\|v\|_{0,\Omega} \leq \|v\|_{2,h,\Omega}, v \in V_h^{(p)} + Z$.

Proposition 9 *Under the assumptions of Theorem 5 there exists a constant $C_8 > 0$, independent of h, such that it holds*

$$\begin{aligned}
&\|\hat{c}^m - \hat{c}_h^m\|_{2,h,\Omega}^2 + \Delta t \sum_{\ell=1}^{m} \|\hat{c}^\ell - \hat{c}_h^\ell\|_{2,h,\Omega}^2 + \Delta t \sum_{\ell=1}^{m} \|\nabla(\hat{w}^\ell - \hat{w}_h^\ell)\|_{0,\Omega}^2 \ \leq \\
&C_8 \Big(h^{-4} \sum_{\ell=1}^{m} (e_{int}^{(1)}(\hat{c}^\ell) + e_{int}^{(3)}(\hat{c}^\ell)) + \sum_{i=4}^{5}\sum_{\ell=1}^{m} e_{int}^{(i)}(\hat{c}^\ell) + \sum_{i=1}^{3}\sum_{\ell=1}^{m} e_{int}^{(i)}(\hat{w}^\ell) + \\
&h^{-4}\ \|c_0 - I_h c_0\|_{0,\Omega}^2 + (1 + h^{-4})\ \|c_0 - I_h c_0\|_{2,h,\Omega}^2 + \|w_0 - I_h w_0\|_{0,\Omega}^2 \Big),
\end{aligned} \tag{84}$$

where w_0 by (9b) *with $c = c_0$ and $w = w_0$.*

Proof. The proof is by induction on m. For $m = 1$ the assertion follows from (82) taking into account that $\hat{c}^0 = c_0$ and $\hat{w}^0 = w_0$. Let us assume that (84) holds true for $m - 1$. Observing

$$\|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{0,\Omega} \leq \Delta t \sum_{\ell=1}^{m-1} \|\frac{\hat{c}^\ell - \hat{c}^{\ell-1}}{\Delta t} - I_h \frac{\hat{c}^\ell - \hat{c}^{\ell-1}}{\Delta t}\|_{0,\Omega} + \|c_0 - I_h c_0\|_{0,\Omega}$$

and the same for $\|\hat{c}^{m-1} - I_h\hat{c}^{m-1}\|_{2,h,\Omega}$ and $\|\hat{w}^{m-1} - I_h\hat{w}^{m-1}\|_{0,\Omega}$, it follows from (82) that (84) is satisfied for m as well.

Proof of Theorem 5: We have $t_m \to t$ as $\Delta t \to 0$. Due to the regularity assumptions (28a),(28b) for $\Delta t \to 0$ the left-hand side of (84) converges to

$$\|(\hat{c} - \hat{c}_h)(\cdot, t)\|_{2,h,\Omega}^2 + \int_0^t \|\hat{c} - \hat{c}_h\|_{2,h,\Omega}^2\ ds + \int_0^t \|\nabla(\hat{w} - \hat{w}_h)\|_{0,\Omega}^2\ ds.$$

On the other hand, for the sum of the interpolation errors (59) it holds

$$\sum_{\ell=1}^{m} e_{int}^{(1)}(\hat{z}^{\ell}) \to \int_0^t \|\hat{z} - I_h\hat{z}\|_{0,\Omega}^2 \, ds, \quad \hat{z} = \hat{c} \text{ and } \hat{z} = \hat{w},$$

$$\sum_{\ell=1}^{m} e_{int}^{(3)}(\hat{z}^{\ell}) \to \int_0^t \|\frac{\partial \hat{z}}{\partial s} - I_h \frac{\partial \hat{z}}{\partial s}\|_{0,\Omega}^2 \, ds, \quad \hat{z} = \hat{c} \text{ and } \hat{z} = \hat{w},$$

$$\sum_{\ell=1}^{m} e_{int}^{(4)}(\hat{c}^{\ell}) \to \int_0^t \|\hat{c} - I_h\hat{c}\|_{2,h,\Omega}^2 \, ds,$$

$$\sum_{\ell=1}^{m} e_{int}^{(5)}(\hat{c}^{\ell}) \to \int_0^t \|\frac{\partial \hat{c}}{\partial s} - I_h \frac{\partial \hat{c}}{\partial s}\|_{2,h,\Omega}^2 \, ds,$$

$$\sum_{\ell=1}^{m} e_{int}^{(2)}(\hat{w}^{\ell}) \to \int_0^t \|\nabla(\hat{w} - I_h\hat{w})\|_{0,\Omega}^2 \, ds.$$

Hence, taking (30),(31) into account, (32) holds true for $c = \hat{c}$. Finally, backtransformation according to (33) allows to conclude. $\square$

6 Discretization in time by singly diagonally implicit Runge-Kutta methods

For the discretization in time of the C0IPDG approximation (26a)-(26c) we use (s,q) Singly Diagonally Implicit Runge-Kutta (SDIRK) methods of stage s and order q with respect to a partitioning of the time interval $[0,T]$ into subintervals $[t_{m-1},t_m]$ of length $\tau_m := t_m - t_{m-1}, 1 \le m \le M$ (cf., e.g., [1, 7, 19]). In particular, for polynomial order $p = 2$ of the C0IPDG approximation we use a $(2,2)$ SDIRK method with coefficients given by the Butcher scheme in Table 6.1

Table 1 Butcher scheme of a 2-stage SDIRK method of order $q = 2$

$$\begin{array}{c|cc} \kappa & \kappa & 0 \\ 1 & 1-\kappa & \kappa \\ \hline & 1-\kappa & \kappa \end{array} \qquad \kappa = 1 \pm \tfrac{1}{2}\sqrt{2}$$

If the polynomial degree is $p = 3$, we use a $(3,3)$ SDIRK method with Butcher scheme given by Table 6.2, and for $p = 4$ we use a $(3,4)$ SDIRK method with Butcher scheme given by Table 6.3.

The fully discrete approximation represents a parameter dependent nonlinear algebraic system with the time-step size as a parameter which is solved by a predictor-corrector continuation strategy with constant continuation as a predictor and Newton's method as a corrector [9, 20]. The predictor-corrector continuation strategy

Table 2 Butcher scheme of a 3-stage SDIRK method of order $q=3$

$$\begin{array}{c|ccc} \alpha & \alpha & 0 & 0 \\ \frac{1+\alpha}{2} & \frac{1-\alpha}{2} & \alpha & 0 \\ 1 & b_0 & b_1 & \alpha \\ \hline & b_0 & b_1 & \alpha \end{array}$$

where $\alpha \approx 0.44$ is the root of $p(x) = x^3 - 3x^2 + \frac{3}{2}x - \frac{1}{6}$, $b_0 = -\frac{6\alpha^2-16\alpha+1}{4}$, and $b_1 = \frac{6\alpha^2-20\alpha+5}{4}$ (cf. [1]).

Table 3 Butcher scheme of a 3-stage SDIRK method of order $q=4$

$$\begin{array}{c|ccc} (1+\kappa)/2 & (1+\kappa)/2 & 0 & 0 \\ \frac{1}{2} & -\kappa/2 & (1+\kappa)/2 & 0 \\ (1-\kappa)/2 & 1+\kappa & -(1+2\kappa) & (1+\kappa)/2 \\ \hline & 1/(6\kappa^2) & 1-1/(3\kappa^2) & 1/(6\kappa^2) \end{array} \qquad \kappa = 2\cos(\pi/18)/\sqrt{3}$$

features an adaptive choice of the continuation parameter. For details we refer to [2].

7 Numerical results

We consider the initial-boundary value problem (2a)-(2c) in $Q := \Omega \times (0,T]$ with $\Omega := (0,L)^2, L := 1.0 \cdot 10^{-4} m$, and $T := 1.0 \cdot 10^{-1} s$. The physical parameters β, κ, σ, and a_0, a_2, h_0, M are given in Table 4 in their physical units. We use the reference quantities

$$L_{ref} := 1.0 \cdot 10^{-5} m, \quad T_{ref} := 1.0 \cdot 10^{-2} s, \quad \sigma_{ref} := 1.0 Jm^{-2} \tag{85}$$

and scale all independent variables and parameters to dimensionless form. Hence, the scaled domain and the scaled time interval become $\Omega = (0,10)^2$ and $[0,10]$. The values of the parameters in dimensionless form are also listed in Table 4. The initial concentration c_0 has been chosen as a smooth function $c_0 \in C^\infty(\Omega)$ satisfying the compatibility conditions (5).

Table 4 Physical parameters in the sixth order Cahn Hilliard equation

Symbol	Value	Unit	Dimensionless Value
σ	1.0	Jm^{-2}	1.0
β	5.0	Jm^{-2}	5.0
h_0	$5.0 \cdot 10^{-1}$	1	$5.0 \cdot 10^{-1}$
M	$1.0 \cdot 10^{-13}$	$m^2 s^{-1}$	$1.0 \cdot 10^{-3}$
κ	$1.0 \cdot 10^{-25}$	Jm^2	$1.0 \cdot 10^{-1}$
a_0	$-4.0 \cdot 10^{-12}$	J	-4.0
a_2	$1.0 \cdot 10^{-12}$	J	1.0

Figure 1 shows a visualization of the microemulsification process obtained by the numerical solution of the sixth order Cahn-Hilliard equation using a C^0IPDG approximation with $p = 2$ and penalization parameter $\alpha = 25.0$ and a 2-stage SDIRK with $q = 2$ at time instants $t = 0.60$ (left) and $t = 3.86$ (right). The pure water phase ($c = 1$) is depicted in dark blue, the pure oil phase ($c = -1$) in dark red, and the microemulsion phase ($c = 0$) in light green. In Figure 1 (right), the formation of oil-in-water and water-in-oil droplets is clearly visible.

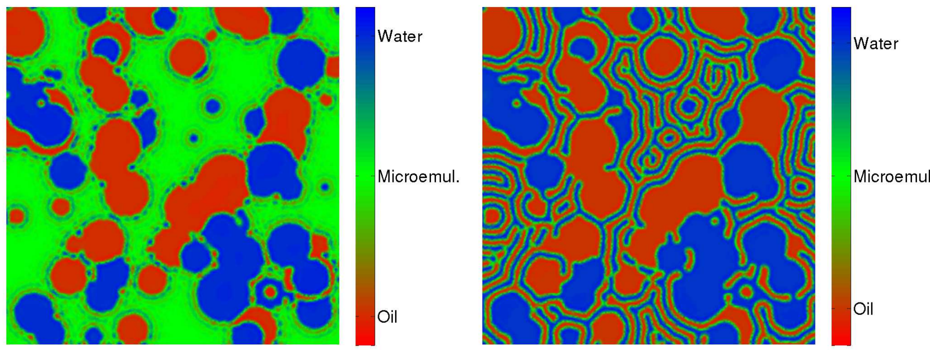

Fig. 1 Formation of oil-in-water and water-in-oil droplets at time instants $t = 0.60$ (left) and $t = 3.86$ (right). C^0IPDG approximation with $p = 2$ on a 128×128 grid and 2-stage SDIRK with $q = 2$ (from [2]).

The underlying finite element mesh is a geometrically conforming, simplicial triangulation $\mathscr{T}_h(\Omega)$ of mesh size h. For $h = 1/24, 1/48$ and at $t = 2.5$ we have computed the convergence rates in the mesh dependent C0IPDG-norm. Obviously, the domain Ω does not have a boundary Γ of class $C^{r+1}, r \geq 5$, and hence, we cannot expect quasi-optimal convergence rates. Therefore, we also computed the convergence rates for a patch Ω of elements around the midpoint m_Ω of Ω given by

$$\omega := \bigcup \{K \in \mathscr{T}_{2h}(\Omega) \mid m_\Omega \in \mathscr{N}_{2h}(K)\},$$

where $\mathscr{N}_{2h}(K)$ is the set of nodal points in K. The convergence rates are as follows

$$\begin{aligned} \mathrm{err}_\omega(t) &:= \log_2 \frac{\|u_h(\cdot,t) - u_{2h}(\cdot,t)\|_{2,2h,\omega}}{\|u_{h/2}(\cdot,t) - u_h(\cdot,t)\|_{2,2h,\omega}}, \\ \mathrm{err}_\Omega(t) &:= \log_2 \frac{\|u_h(\cdot,t) - u_{2h}(\cdot,t)\|_{2,2h,\Omega}}{\|u_{h/2}(\cdot,t) - u_h(\cdot,t)\|_{2,2h,\Omega}}. \end{aligned}$$

In each case the time-step size has been chosen sufficiently small so that the error due to discretization in time do not affect the error due to spatial discretization. The convergence rates are shown in Table 5, Table 6, and Table 7.

Table 5 Patchwise and global convergence rates for the semidiscrete C0IPDG approximation with $p = 2$

	$err_\omega(2.5)$	$err_\Omega(2.5)$
$h = 1/24$	1.06	0.66
$h = 1/48$	1.02	0.91

Table 6 Patchwise and global convergence rates for the semidiscrete C0IPDG approximation with $p = 3$

	$err_\omega(2.5)$	$err_\Omega(2.5)$
$h = 1/24$	1.83	1.68
$h = 1/48$	1.91	1.79

Table 7 Patchwise and global convergence rates for the semidiscrete C0IPDG approximation with $p = 4$

	$err_\omega(2.5)$	$err_\Omega(2.5)$
$h = 1/24$	2.83	2.56
$h = 1/48$	2.90	2.67

For domains Ω with boundary Γ of class $C^{r+1}, r \geq 5$, the quasi-optimal convergence rates are 1.0 for $p = 2$, 2.0 for $p = 3$, and 3.0 for $p = 4$ (cf. Theorem 5). We see that we get almost quasi-optimal convergence rates on the patch ω in the $\|\cdot\|_{2,2h,\omega}$-norm, but as expected not quite as good convergence rates on the entire domain Ω in the $\|\cdot\|_{2,2h,\Omega}$-norm.

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