How Do I Choose?
A Study of the Tools Used to Calculate Real Life Combinations

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INTRODUCTION

The mathematics curriculum presented to middle school students includes the study of probability to inform and explain to youngsters how the likelihood of "uncertain" events can be estimated. "In probability theory, an event is a set of outcomes (a subset of the sample space) to which a probability is assigned. Typically, any subset of the sample space is an event" ("Event"). For example, when playing games with dice, throwing a single die offers six possible results from 1 to 6; we call this group of possibilities the sample space in the experiment of throwing a die; getting any of those results constitutes an event. In weather forecast, rain, storm, tornado, and hail are possible events predicted after analyzing and interpreting the readings from devices to measure variables in the environment, such as humidity, pressure, or speed of the wind. Informally, an event is the result we get from any experience to which we are exposed, understanding the experiences in our entire life as a collection of experiments.

"Uncertain" events are those for which we do not know for sure what is going to happen. We face "uncertain" events everywhere in real life: drawing a card from a deck of playing cards, guessing the winner in a race, winning or losing money for changes in the stock market, or being diagnosed with any kind of cancer. We observe that some events have more chances to occur than others; we might say they are more "probable" to occur than others. The probability of an event is a positive decimal number between 0 and 1, which measures the relative frequency of occurrences for that particular event in several repetitions within an experiment (Carlson 86).

The study of probability has become increasingly important to analyze trends in the behavior of a variety of processes and populations. My purpose is to convince students about how relevant probability is to their life. Students will be able to estimate all the possible combinations in experiments like selecting the correct six numbers from a pool of 54 numbers in a lottery game and estimating the probability of that event occurring (Paulson 202). Convincing students about the fact that they can use probabilities and statistics to be in more control of their life is not an easy task, but the use of technology and concrete materials might ease this quest for knowledge.

Throughout my life, I have studied and admired mathematics and science, and I have strived to explain and to be in command of everything that happens to me, besides improving my performance when decision making is involved, but it is not possible to get exactly what I wish all the time. That is why I do not feel disappointed when I do not get a door prize when somebody is drawing names from a hat in a room with 400 people. In probabilities my little Jacob can find the explanation why he does not receive his paper back faster when I return the graded test. Probabilities go far beyond schools and libraries. In the world of casinos and gambling, for example, probability is widely used to attempt to beat the "odds." Most of the time, people cannot beat the "odds" and sadly see how the house usually wins because devices in casinos or the lottery system are "not fair."

To understand how the house wins, it is necessary to identify the possible results that can be observed when playing a poker game, Russian roulette, or any other trick used for amusement inside a casino. Gamblers need to know how many different ways a smaller number of items can be taken from a larger; the items taken from the larger group can be cards, numbers in a spinner, funds in the stock market, or horses in a race, where the number of four-horse accumulators can be taken from a total of eight selections. These practical problems are solved calculating combinations and permutations (Webb). A combination is the collection of "one or more elements selected from a set without regard to the order of selection" ("Combination"). If the number of possible combinations is small, we can calculate the number of events using patterns as valuable visual tools. All events present in any combination of r elements selected from a set of n elements ($n>r$) are given by the coefficients contained in a famous mathematical pattern known as Pascal's Triangle. Pascal's Triangle's properties are taught in 8th grade mathematics, but its applications to probabilities are not explored in depth. If provided relevant examples and concrete experiences, students can understand the importance of such a tool in probability.

When the number of possible combinations is large, the use of technology becomes very important to improve computations. Technology is available in schools. Many schools count with scientific calculators, computers, and software programs with special functions capable of processing the data from probability experiments.

The thematic unit from this seminar will have as its main purpose the study of problems solved by establishing combinations using Pascal's Triangle as a tool to get the sample space of the problem in the quickest way possible. Pascal's Triangle is a treasure chest full of mathematical properties, and my task is to discover it with my students. I want to provide my students tools and sources of information, such as books, software programs, and websites where they can observe and interact with the Pascal's Triangle's properties, along with the principles to calculate the probabilities of events whose frequencies can be observed in the triangle while being able to make connections with real life situations involving probabilities. They will build the set of all possible events in a sample space from several experiments. This activity will help them immensely to manipulate a concept, usually perceived as abstract. Even though teaching Pascal's Triangle belongs to the 8th grade mathematics curriculum, the way I am planning to present it may serve as enrichment for students participating in Pre-Advanced Placement programs or even Algebra I. The unit can be adapted to be taught to English as Second Language students by incorporating more pictures and graphical models along with the concrete objects.

ACADEMIC SETTING AND CURRICULUM OBJECTIVES

The lessons contained in this thematic unit are designed to be taught for two weeks following a block schedule, each block 90 minutes long.

I will open the unit with exploratory activities where the students are able to observe the context where probabilities can occur and how they occur. In each situation students have to identify elements—sample space, event, tree diagram, combination, pattern, Pascal's Triangle, probability, factorial, probability distribution, and bar graph—after they have been exposed to physical experiences and definitions. When performing an experiment, students should be able to construct a sample space by observing the outcomes in a tree diagram (Project CLEAR: Grade 6th Mathematics). The frequency of outcomes will be calculated and compared with the coefficients present in the Pascal's Triangle; each row will contain the number of events corresponding to the sample space of a particular experiment with certain number of elements. Once the number of combinations is determined, the probability will be calculated using a variety of tools. Coins, dice, paper bags with numbers and cards will be used to perform experiments; the experiments will provide raw data to compare with the theoretical probabilities.
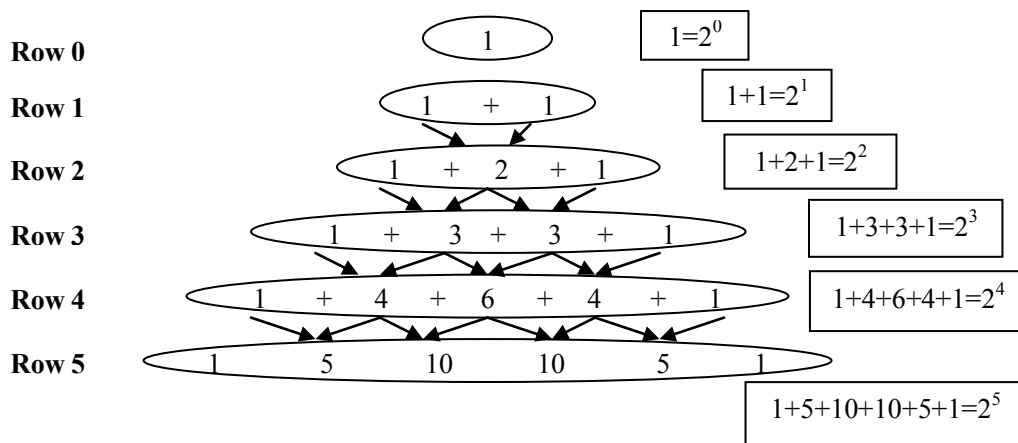
SAMPLE SPACES AND PROBABILITY

Exploring Combinations inside the Pascal's Triangle.

Note: Every graph was developed by the author of this unit.

To build the Pascal's triangle is necessary to arrange whole numbers in a 2- dimensional arrangement such as:

Row 0   1
Row 1   1   1

Row 0 has 1 as unique coefficient. Row 1 has 2 coefficients with a value of 1 each. To generate the next line, we add together each pair of adjacent numbers in the line above:

Row 0   1 $1=2^0$

Row 1   1 + 1 $1+1=2^1$

Row 2   1 + 2 + 1 $1+2+1=2^2$

Row 3   1 + 3 + 3 + 1 $1+3+3+1=2^3$

Row 4   1 + 4 + 6 + 4 + 1 $1+4+6+4+1=2^4$

Row 5   1   5   10   10   5   1

$1+5+10+10+5+1=2^5$

Figure 1: Building a Pascal's Triangle and observing the patterns being formed.

Each Row in the triangle will have 1 as first and last coefficient, for example in Row 2:   1 2   1, Row 3: 1   3   3   1. When adding together all the coefficients present in a row, the result is a power of 2. A pattern with all the powers of 2 can be seen then starting from Row 0: $1 = 2^0$, Row 1: $1+ = 2^1$, Row 2: $1+ !+ = 2^2$, Row 3: $1+ + + = !^3$.

In the experiment of tossing a coin a certain number of times, getting a head will be considered a success and getting a tail will be considered a failure. If we toss the coin 5 times, the number of combinations of outcomes can be observed in Row 5 in the Pascal's Triangle. When we add together all the coefficients present in the row: 1+5+10+10+5+1=32, the result will be the total number of possible events when tossing a coin five times, which is known as sample space. Our sample space will include:1 possible event getting 0 heads and 5 tails, 5 possible events getting 1 head and 4 tails, 10 possible events getting 2 heads and 3 tails, 10 possible events getting 3 heads and 2 tails, 5 possible events getting 4 heads and 1 tail and 1 possible event getting 5 heads and 0 tails. The easiest way to identify the possible outcomes is observing a tree diagram. The tree diagram corresponding to the experiment of tossing 5 coins is provided in figure 2.
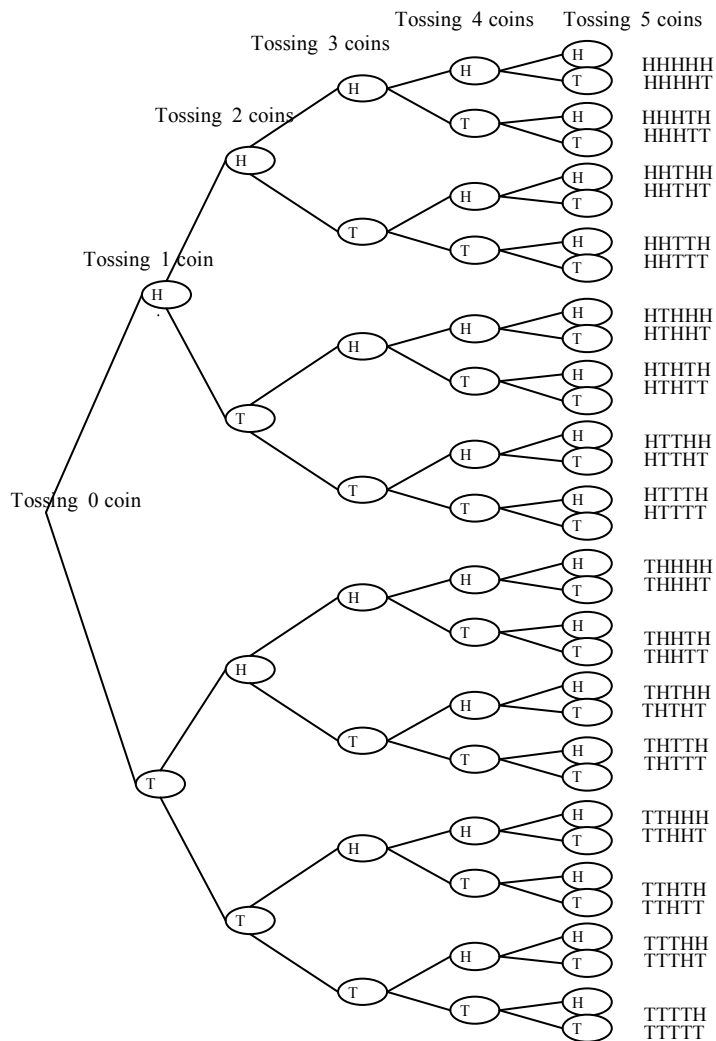
Figure 2: Tree diagram used for the experiment of tossing 5 coins.

The combinations shown to students usually have a small number of elements, and for that reason the Pascal's Triangle eases calculations and brings interest to the study of the relationships between number theory, patterns, and probabilities.

Calculating Probabilities within the Pascal Triangle

To calculate the probability of any event in any experiment, we have to set up the ratio between the number of events being studied and the total number of events known as sample space. The result will be gotten dividing # of successful events by total number of events.

Going back to Pascal's Triangle, if we divide each coefficient on each row, taking for now the nth row, by the total number of events present in that row calculated by 2^n (Hubbard and Roby), we will get a triangle showing the probabilities corresponding to the possible events when tossing certain number of coins (only for fair coins, with probability of getting a head 0.5 or tail 0.5), considering getting a head (H) as a success and getting a tail (T) as a failure:
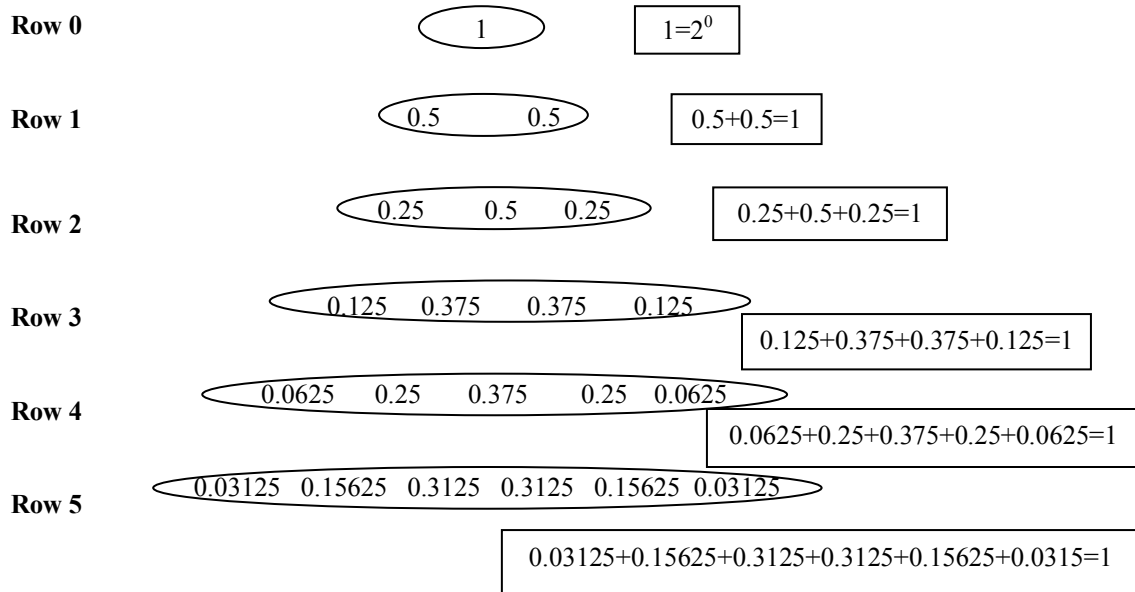
Figure 3: Triangle of Probabilities for the experiments of tossing 0, 1, 2, 3, 4 and 5 fair coins (p=0.5)

Each row on this triangle shows the probability for each type of event on its sample space. Each row corresponds to the sample space for one experiment. The sum of all events yields 1. It is important to mention, when the coins are not fair, the probability can not be calculated dividing the number of combinations by the total number of events.

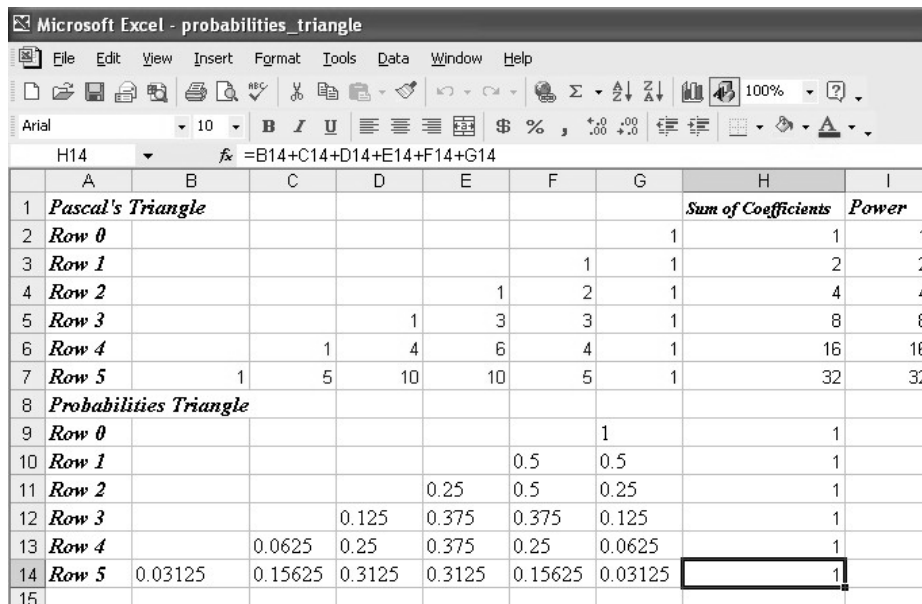Using the Microsoft Excel worksheet to observe the property just mentioned:

Figure 4: Using a Microsoft Excel Worksheet to reproduce the triangles.

If we graph each row of the probabilities triangle in a bar graph, in our Microsoft Excel worksheet using the Chart Wizard, we will represent a probability distribution:
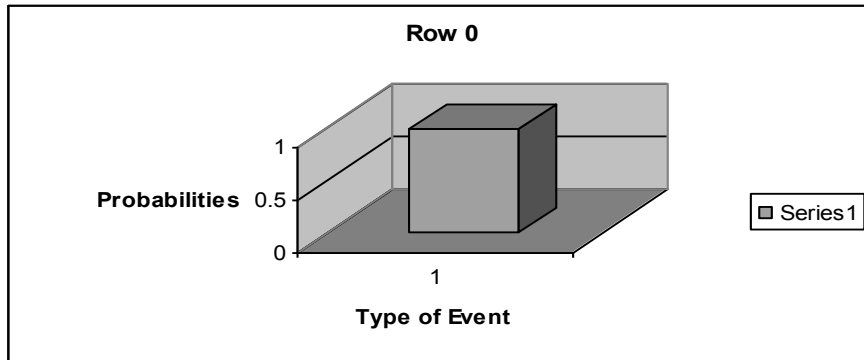

Figure 5: Probability Distribution observed when tossing 0 coins.

When tossing 0 coins, I have a coin face up in my hand showing a head (H), success. The probability of getting 1 head is 1.
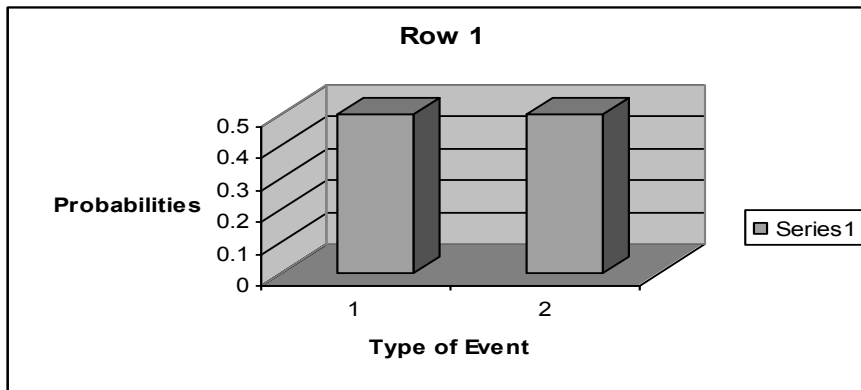

Figure 6: Probability Distribution observed when tossing 1 coin.

Tossing 1 fair coin, I can get a head (H), success or tail (T) failure. The probability of getting a head is $\frac{1}{2} = 0.5$ and the probability of getting a tail is $\frac{1}{2} = 0.5$.
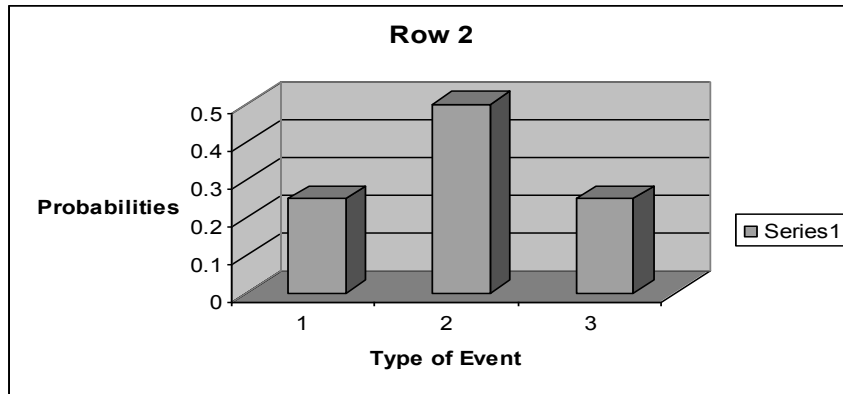
Figure 7: Probability Distribution observed when tossing 2 coins.

Tossing 2 fair coins, I can get 1 event with 0 heads, 2 tails (TT), 2 events with 1 head 1 tail (HT, TH) or 1 event with 2 heads 0 tails (HH). The probability of getting 0 heads, 2 tails is $\frac{1}{4} = 0.25$, the probability of getting 1 head 1 tail is $\frac{2}{4} = 0.5$ and the probability of getting 2 heads 0 tails.

Figure 8: Probability Distribution observed when tossing 3 coins.

Tossing 3 fair coins, I can get 1 event with 0 heads, 3 tails (TTT), 3 events with 1 head 2 tails (HTT, THT, TTH), 3 events with 2 heads 1 tail (HHT, HTH, THH) or 1 event with 3 heads 0 tails (HHH). The probability of getting 0 heads, 3 tails is $\frac{1}{8} = 0.125$, the probability of getting 1 head 2 tails is $\frac{3}{8} = 0.375$, the probability of getting 2 heads 1 tail is $\frac{3}{8} = 0.375$ and the probability of getting 3 heads 0 tails is $\frac{1}{8} = 0.125$.
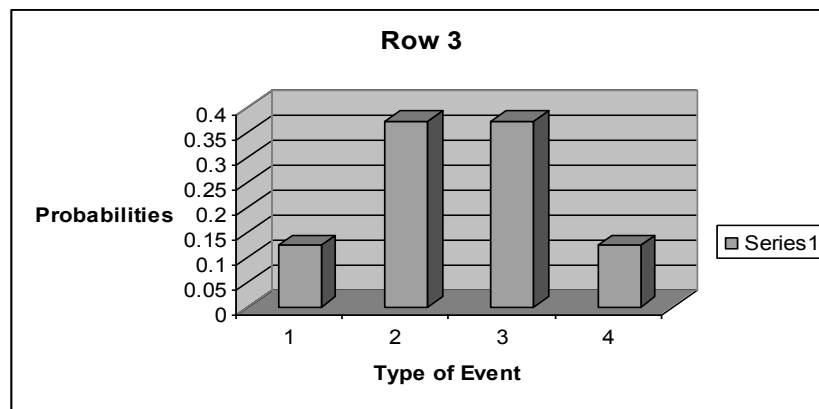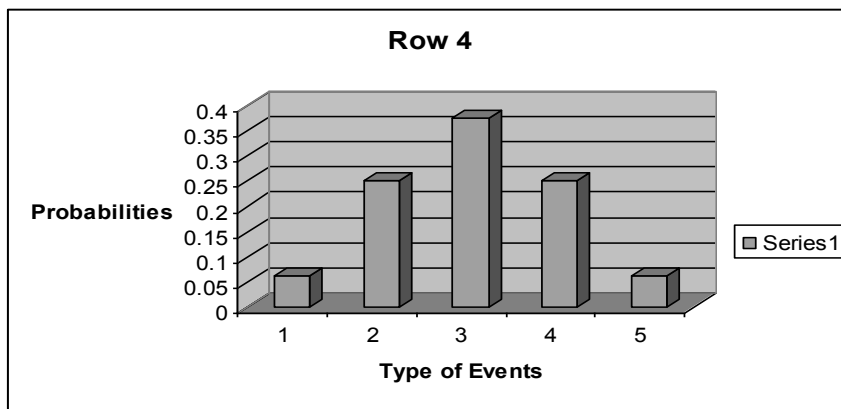
Figure 9: Probability Distribution observed when tossing 4 coins.

Tossing 4 fair coins, I can get an event with 0 heads, 4 tails (TTTT), 4 events with 1 head 3 tails (HTTT, THTT, TTHT, TTTH), 6 events with 2 heads 2 tails (HHTT, HTHT, THTH, HTTH, THHT, TTHH), 4 events with 3 heads 1 tail (HHHT, HHTH, HTHH, THHH) or 1 event with 4 heads 0 tails (HHHH). The probability of getting 0 heads, 4 tails is $\frac{1}{16} = 0.0625$, the probability of getting 1 head 3 tails is $\frac{4}{16} = 0.25$, the probability of getting 2 heads 2 tails is $\frac{6}{16} = 0.375$, the probability of getting 3 heads 1 tail is $\frac{4}{16} = 0.25$ and the probability of getting 4 heads 0 tails is $\frac{1}{16} = 0.0625$.
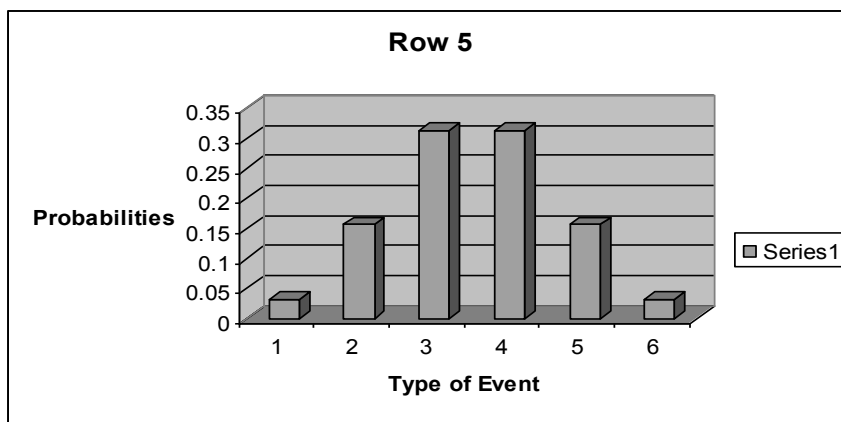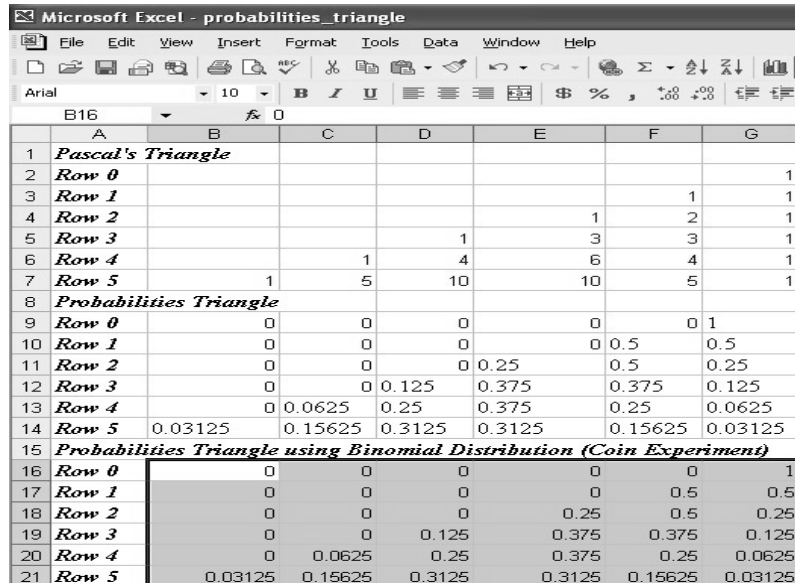
Figure 10: Probability Distribution observed when tossing 5 coins.

Tossing 5 fair coins, I can get 1 event with 0 heads, 5 tails (TTTTT), 5 events with 1 head 4 tails (HTTTT, THTTT, TTHTT, TTTHT, TTTTH), 10 events with 2 heads 3 tail (HHTTT, HTHTT, HTTHT, HTTTH, THTTH, TTHTH, TTTHH, TTHHT, THHTT, THTHT), 10 events with 3 heads 2 tails (HHHTT, HHTHT, HTHHT, THHHT, THHTH, THTHH, TTHHH, HTTHH, HHTTH, HTHTH), 5 events with 4 heads 1 tail (HHHHT, HHHTH, HHTHH, HTHHH, THHHH) or 1 event with 5 heads 0 tails (HHHHH).

The distribution we have formed receives a special name: Binomial Distribution. The Binomial Distribution can be generated once a binomial experiment is performed. Tossing coins constitutes a binomial experiment. A binomial experiment has the following conditions:

1. Each trial can have only two outcomes or outcomes that can be reduced to two outcomes. These outcomes can be considered as either success or failure.
2. There must be a fixed number of trials.

	A	B	C	D	E	F	G
1	Pascal's Triangle						
2	Row 0						1
3	Row 1					1	1
4	Row 2				1	2	1
5	Row 3			1	3	3	1
6	Row 4		1	4	6	4	1
7	Row 5	1	5	10	10	5	1
8	Probabilities Triangle						
9	Row 0	0	0	0	0	0	1
10	Row 1	0	0	0	0	0.5	0.5
11	Row 2	0	0	0	0.25	0.5	0.25
12	Row 3	0	0	0.125	0.375	0.375	0.125
13	Row 4	0	0.0625	0.25	0.375	0.25	0.0625
14	Row 5	0.03125	0.15625	0.3125	0.3125	0.15625	0.03125
15	Probabilities Triangle using Binomial Distribution (Coin Experiment)						
16	Row 0	0	0	0	0	0	1
17	Row 1	0	0	0	0	0.5	0.5
18	Row 2	0	0	0	0.25	0.5	0.25
19	Row 3	0	0	0.125	0.375	0.375	0.125
20	Row 4	0	0.0625	0.25	0.375	0.25	0.0625
21	Row 5	0.03125	0.15625	0.3125	0.3125	0.15625	0.03125

Figure 11: Data processed in a Microsoft Excel Worksheet used to generate 3-dimensional representation of probability distributions.

3. The outcomes of each trial must be independent of each other.
4. The probability of a success must remain the same for each trial. (Clarke)

A Binomial Distribution generates the discrete probability distribution of obtaining exactly *r* successes out of n trials (where the result of each trial is true with probability *p* and false with probability *q=1-p*). The binomial distribution is therefore given by

$$P_p(n|N) = \binom{N}{n} p^n q^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

"A probability distribution, more properly called a probability density, assigns to every interval of the real numbers a probability, so that the probability axioms are satisfied" (Weisstein).

The best way to understand how the distribution works is by doing an experiment with many trials. The probabilities obtained from the physical experiment of tossing a coin zero, one, two, ... , n times is a good example of a Binomial Distribution. This example in particular can be simulated with interactive applets (Siegrist).

To condense the data entered in Microsoft Excel in one graph, select the data, use Chart Wizard, in Step 1 of 4 specify Chart Type: Column, within Standard Types, then select 3-D column, click next, in Step 2 of 4 Series must be in Rows and click on series to modify the series' names, click next, in Step 3 of 4 provide a chart title, X axis: leave blank, Y axis: coins for every experiment, Z axis: Probabilities, click next, in Step 4 of 4 select As Object in: Sheet 1. Make sure the array of data is completed with 0 in the worksheet, as we can see in figure 11.

Figure 12 represents a 3-dimensional graph with the information corresponding to 5 experiments whose coefficients are present in the Pascal's Triangle from Row 0 to Row 5.
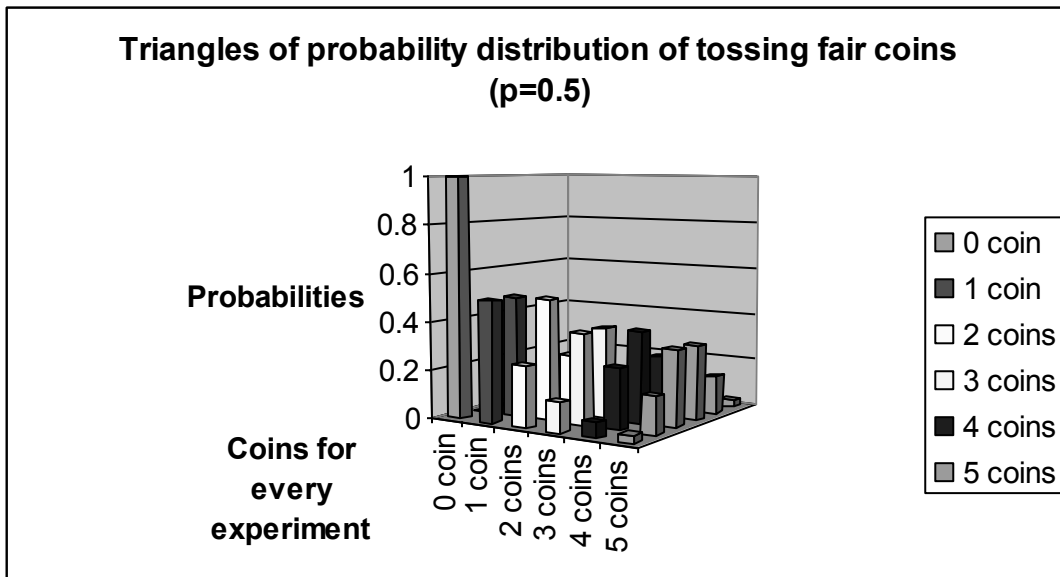

Figure 12: Probability Distributions observed when tossing 0, 1, 2, 3, 4 and 5 fair coins.

TOOLS TO CALCULATE PROBABILITIES

In a Binomial Distribution the probability for each event is calculated using the formula:

$$\binom{n}{r} p^r \bullet (1-p)^{n-r}$$, as we established before.

Where $\binom{n}{r}$ is the combination of r elements taken from n elements in total.

The full formula for the combination function is $C_r^n = \dfrac{n!}{(n-r)!*r!}$

The symbol "!" in the formula means factorial and it can be calculated by multiplying n by all natural numbers previous to it until 1: n!=n*(n-1)*(n-2)*(…)…*1, for example, 5!=5*4*3*2*1.

When the number of elements we are dealing with starts to grow, the use of spreadsheets and modern calculators gains big importance. Microsoft Excel spreadsheets allow using a function to calculate the combination of r elements selected from a group of n elements, written as **combin(n,r)** (Webb).

In excel a factorial is entered **=fact(n)** where n is the number of the factorial needed to be worked out. Big factorials are estimated from the Stirling's approximation.

To use the new tools and put our skills in practice let us look at an everyday life problem.

I know many people who might buy a lottery ticket when they feel broke or just feel lucky, without any money. Tom, my neighbor, bought a ticket today, and he wants to know what the probability is of winning tonight because he is almost certain he selected the correct six numbers from a pool of 54 numbers. There are many ways of selecting the six numbers from the 54. Every single way is equally likely; it has the same probability.

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To find the total number of ways to select the numbers, we can use the formula for combinations:

$$C_6^{54} = \frac{54!}{(54 - 6)! \, 6!} = 25827165 \, , \; C_6^{54} \text{ the probability of having the lucky ticket if}$$

$$P(W) = \frac{1}{25827165} = 3.87189E\text{-}08$$

The probability is very low so I guess Tom will not be winning the lottery any sooner.

Stirling's approximation for factorials

Factorials can be approximated for large values of n using the Stirling approximation which is given by:

$$n! = \sqrt{2\pi} \, * n^n \exp\left[-n + \frac{1}{12n} + 0\left(\frac{1}{n^2}\right) \right]$$

This equation yields $n!$ with a relative error smaller than 1/4 % for all integer values of n and with a relative error smaller than 3.5 x 10^{-7} for $n > 20$" (Van Zeghbroeck).

Binomial Distribution Function

Microsoft Excel spreadsheets provide a function to calculate probabilities found on a Binomial Distribution once its parameters has been specified. Microsoft Excel Help specifies the necessary attributes to declare the function Binomial Distribution.

The function is written by BINOMDIST(number_s,trials,probability_s,cumulative) where:
number_s = number of successes
trials = number of trials.
probability_s = probability of success.

Cumulative is a logical value, 0 or 1. If cumulative is TRUE=1, BINOMDIST returns the cumulative distribution function; if FALSE=0, it returns the probability mass function. The function Binomial Distribution in a Microsoft Excel spreadsheet will return the individual term binomial distribution probability in a single cell. The function BINOMDIST can be used in problems with a fixed number of trials, when the outcomes of any trial are only success or failure, when trials are independent, and when the probability of success remains constant during the experiment. ("Help File").

When performing the experiment of tossing a coin 5 times, the probability of getting 2 heads is calculated using the formula $P(2) = \frac{5!}{(5-2)! \, 2!} \, 0.5^2 \bullet (1 - 0.5)^3 = 0.3125$

$$C_2^5 = \frac{5!}{(5-2)! \, 2!} = 0 \, , \; C_2^5 \text{ is the combination of 2 elements taken from 5 elements in total.}$$

In this case:

n=5, r=2 (successes) and p=0.5, p is the probability of getting a head from a fair coin (say what is each letter). Using the function Binomial Distribution in Excel, on any cell in a blank worksheet type =BINOMDIST(2,5,0.5,FALSE) and the cell will show 0.3125 which corresponds to the probability of getting 2 heads when tossing 5 fair coins. When the coins are unfair (biased), the probabilities will change. In real life the coins are not necessarily "balanced"; therefore, the experiment will follow a distribution with different characteristics. Let us take, for instance, the

case when the probability of getting a head (success) corresponds to p=1/3 (unbalanced coin). The distributions of probability can be seen in figure 13.
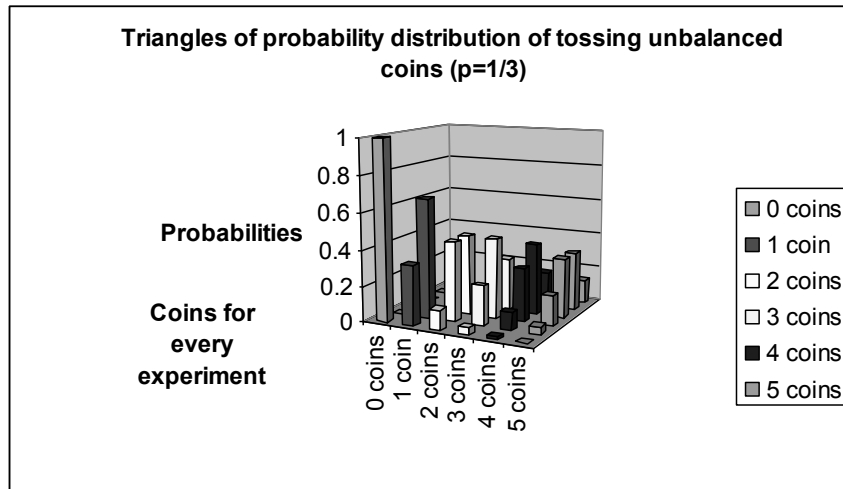

Figure 13: Probabilities Distributions observed when tossing 0, 1, 2, 3, 4 and 5 "unfair" or "unbalanced coins".

Now if we use dice instead of coins. The probabilities shown on each row of our probabilities triangle must be calculated using the formula corresponding to the Binomial Distribution function.

When we roll a die 5 times, the probability of getting a 3 (3 dots in the upper face) 2 times, can be calculated using the formula $P\ell = \dfrac{5!}{(5-2)!\cdot 2!}\left(\dfrac{1}{5}\right)^2 \cdot \left(1-\dfrac{1}{6}\right) = 0.160751$

$C_2^5 = \dfrac{5!}{(5-2)!\cdot 2!} = 0$, C_2^5 is the combination of 2 elements taken from 5 total elements.

In this case:

n=5, r=2 and p=$\dfrac{1}{6}$, p is the probability of getting a 3 (3 dots in the upper face) from a fair die.

Considering:
Getting a 3 = success
Getting any other number = failure

A very interesting problem for students is the number of questions they can get correct in a test. If we take, for example, a TAKS (Texas Assessment of Knowledge and Skills) test with 48 questions, our "lucky" student needs to get 70% or more of the answers correct to pass; to have an idea of how likely this condition is, our student needs to calculate the probability of guessing 34 or more correct answers out of 48 questions. Therefore:

P(passing) = P(34)+P(35)+P(36)+P(37)+P(38)+P(39)+P(40)+P(41)+P(42)+P(43)+P(44)
        +P(45)+ P(46)+P(47)+P(48)

The TAKS test is a multiple choice test and every question has 4 answer choices, therefore p(correct)=0.25, using the formula for the Binomial Distribution to calculate the probability of getting 34 correct answers out of 48 questions:

$$P(4) = \frac{48!}{(48-4)! \cdot 4!}\left(\frac{1}{4}\right)^{4} \cdot \left(1-\frac{1}{4}\right)^{8-4}$$ =2.91175E-11, the rest of the values of

probability will be calculated in the same way.

P(passing) = 2.91175E-11+ 3.88233E-12+ 4.67318E-13+ 5.05208E-14+ 4.87482E-15+ 4.16651E-16+ 3.12488E-17+ 2.03244E-18+ 1.12914E-19+ 5.25179E-21+ 1.98932E-22+ 5.89427E-24+ 1.28136E-25+ 1.81754E-27+ 1.26218E-29=3.3523E-11

The result is a very small number, so I would suggest my "lucky" student to study because he will not have a good chance to pass if he decides to guess.

In our popular culture TV shows have made impact in the way of thinking. A very popular TV game, *Let's Make a Deal*, whose debut was on NBC on December 30, 1963 ("Let's Make a Deal") and has been on air for forty-two years, posed a very interesting problem where a contestant has to choose among three doors to find a car:
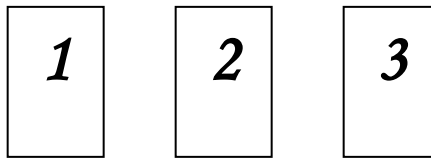
Figure 14: Monty Hall Problem

The problem has been named after the TV game host Monty Hall. The contestant chooses a door, for example, door 1, the content is not revealed; then the host opens a door, showing a goat. Of course, the game host knows where the car is. The host will offer the contestant the opportunity to switch if he wants to. The probability of winning the car will change depending on switching or not switching the door previously chosen.

The probability of choosing the door with the car behind it is $\frac{1}{3}$ and it is equally likely. The car can be behind door 1, or behind door 2, or behind door 3.
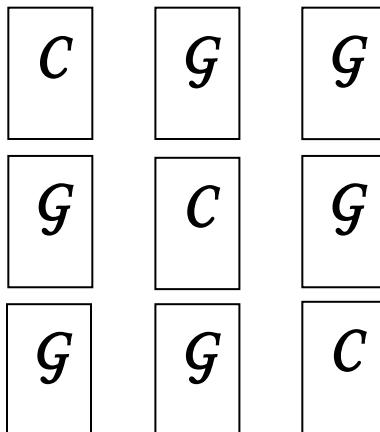
Figure 15: Possible Combinations in the Monty Hall Problem

In this analysis of the problem the contestant will choose door 1 and will always switch. In the first combination CGG, the contestant chooses door 1, and the car is behind it. Since the content is not revealed, the contestant does not know he is winning; then the game host opens either one of the two doors with a goat behind it and offers the opportunity to switch. The contestant switches the car for the goat and loses this time. In the second combination GCG, the contestant chooses door 1, and a goat is behind it; the content is not revealed. Then the game host

opens the door with the other goat behind it and offers the opportunity to switch. The contestant switches a goat for the car and wins this time. In the third combination GGC, the contestant chooses door 1, and a goat is behind it; the content is not revealed. Then the game host opens the door with the other goat behind it and offers the opportunity to switch. The contestant switches a goat for the car and wins this time. The Monty Hall problem represents a conditional probability. For the first combination, the probability of winning the car P(C) is equal to the sum of the probability of choosing the car given that we have chosen the first combination considering the probability of choosing the first combination plus the probability of choosing the car given that we have chosen the second combination considering the probability of choosing the second combination plus the probability of choosing the car given that we have chosen the third combination considering the probability of choosing the third combination. The solution to the problem can be written as follows:

$$P(C) = P(C|C1) \cdot P(C1) + P(C|C2) \cdot P(C2) + P(C|C3) \cdot P(C3)$$

$$P(C) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{2}{3}$$

In the second case of the problem, the contestant will choose door 1 and will not switch. In the first combination CGG, the contestant chooses door 1, and the car is behind it. Since the content is not revealed, the contestant does not know he is winning. Then the game host opens either one of the two doors with a goat behind it and offers the opportunity to switch. The contestant does not switch, and he keeps the car; he wins this time. In the second combination GCG, the contestant chooses door 1, and a goat is behind it. The content is not revealed; then the game host opens the door with the other goat behind it and offers the opportunity to switch. The contestant does not switch, and he does not get the car. He takes the goat home and loses this time. In the third combination GGC, the contestant chooses door 1, and a goat is behind it. The content is not revealed. Then the game host opens the door with the other goat behind it and offers the opportunity to switch. The contestant does not switch and takes the goat. He loses this time. The solution to this case of the problem can be written as follows:

$$P(C) = 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{1}{3}$$

By switching doors, the contestant keeps his chances to win as high as $\frac{2}{3}$.

Considering that the TV show *Let's Make a Deal* has been transmitted for long time every night, a trial of the probability experiment is being performed. Let us say the show is broadcast 5 nights a week, 20 nights during a month, every night a different contestant. The probability of 5 contestants winning the car in a month, considering that they all switch, is:

$$P(5) = \frac{20!}{(20-5)! \cdot 5!}\left(\frac{2}{3}\right)^5 \cdot \left(1-\frac{2}{3}\right)^{20-5} = 0.000142$$

If we continue to calculate the probabilities for every event, we can find that the largest probabilities correspond to 12, 13 and 14 people winning in a month, P(12)=0.147979646

P(13)=P(14)=0.182128795. The Probability of 12, or 13, or 14 people winning in a month is P(12)+P(13)+P(14)= 0.147979646+0.182128795+0.182128795=0.512237.

The result suggests that the TV game's producer has to plan a budget to buy many cars.

Problems like this one give some insight about how possible is to improve the chances of getting a favorable outcome in completely random situations.

IMPLEMENTING THE UNIT

Following the objectives on the TEKS (Texas Essential of Knowledge and Skills) test, students will be able to perform experiments and identify different events from a sample space. Once the sample space is given, students should be able to calculate the probability from the number of events declared as favorable and the total number of events in the sample space. Results can be generated using the formulas already known when the number of elements and trials are small and using the technology available in Microsoft Excel spreadsheets when the number of elements and combinations increase.

To make the experience relevant and engaging to students, it is necessary to use concrete objects to perform the experiments in the most realistic way possible: coins, dice, spinners, Russian Roulettes, and tickets in a bag with different numbers.

When real objects are not available, the technology available on computers, by using Microsoft Excel or interactive websites (applets) on the Internet which display the mentioned devices used in probability experiments, becomes very useful. Technology awareness is important for the development of students' skills.

LESSONS WHERE THE ODDS ATTACK

Lesson 1

The lesson starts with an exploratory activity where the students will identify situations where results may be "unpredictable," such as discussing the weather section of the newspaper and posing a question as to whether it was going to rain the next day. When discussing the schedule of basketball, football, soccer or baseball games, try to establish with the students who the winner might be and ask questions, such as: Are you sure? Why? What if Roger Clemens gets sick in the next Astros game? Students will collect a series of newspaper articles and selections where random events occur to be presented in a flip-book to the whole class and finally posted outside the classroom.

Objectives

The problems previously discussed in the article will help prepare teacher and students to understand the relationship between the theoretical results and the experimental data gathered in the experiments of drawing numbers from a pool, tossing coins and tossing dice. Students will draw a tree diagram for each experiment to identify the events in the sample space. Students will compare theoretical results and experimental data after calculating probabilities.

Materials

- Newspapers
- Scissors
- Glue
- 1 Paper bag for every pair of students
- 54 little squares with the numbers 1 to 54 printed in one side to be put inside the paper bag for each pair of students
- 1 Table # 1 of Lottery Game from the Appendix to record the numbers drawn from the paper bag
- 1 Plastic coin is to be tossed four times or 4 plastic coins to be tossed once each
- 1 Table #2 to record results from 16 trials tossing coins (one trial is tossing 1 coin four times or tossing 4 coins)

- 1 die is to be rolled four times or 4 dice will be roll once.
- 1 Table #3 to record results from 16 trials rolling dice (one trial is rolling 1 die four times or rolling four dice)
- Pencil
- Construction Paper
- Paper

Activity 1

Students will work in pairs and will alternate in drawing 6 numbers from a paper bag with 54 numbers inside until they complete 30 drawings. Every time a student draws the 6 numbers, the student has to record each number in table #1 in the exact order they appear and put the numbers back in the bag. The teacher has to choose a combination of numbers. Calculate the probability through dividing number of successes (desired combination) by the number of trials (30). Explain how it is possible to have 0 as a probability, which constitutes not to get the "magic combination." Introduce the formula to calculate combinations, along with the concept of factorial. Open a Microsoft Excel Worksheet and use the **combin(n,r)** function to calculate the number of possible ways 6 numbers can be selected from 54. Students should prepare a report of the experience, reflecting their ideas, suggestions and frustrations. Students should answer the question: Would you buy a lottery ticket in the future?

Activity 2

The teacher will allow students to work with a partner; one student will perform the experiment; the other student will record the outcomes. The teacher will distribute 1 plastic coin to be tossed four times or 4 plastic coins to be tossed once to each pair of students. Students will record in table #2 the outcomes of the experiment. Students should be encouraged to determine the number of combinations for a particular event, for example, how many combinations of 2 heads and 2 tails were obtained. Students will explore how to generate the Pascal's Triangle. Allow students to manipulate a virtual manipulative to generate Pascal's Triangle ("Interactive Pascal's Triangle"). Students must find how the pattern is generated. The teacher should discuss properties present in the triangle and strategies to find them. The students will draw the tree diagram to identify the combinations of outcomes. Students work with the "Probability Tree" (Holmes) to generate the tree diagram of the experiment. Teacher discusses with students how to draw a tree diagram from any experiment. Teacher facilitates the process of drawing the tree diagram for the experiment of tossing 4 coins. Students calculate experimental probabilities using ratios. Students are introduced to finding probabilities within the Pascal's triangle using the model of fair coins. Students are introduced to formulas to calculate the probabilities within a Microsoft Excel Worksheet. Students generate a poster using construction paper and a report, journal format to present the information to class and outside the classroom.

Activity 3

Students will follow the same plan of action found in this lesson in activity 2, except for the fact that they will be rolling 1 die 4 times or 4 dice once. Students record the experimental results in table #3. Students have to generate a report to collect results and write their ideas about the experiment.

Assessment

Every pair of students will present posters and reports with reflections about each experiment. Participation is encouraged. A classroom presentation will serve to enforce individual accountability. Refer to the Examples of Rubrics to Assess a Report, Poster and Presentation in the Appendix.

Lesson 2

The lesson will serve as evidence to students of how hard is to pass a test by guessing. The probabilities will be calculated using the Binomial Distribution function in Microsoft Excel once the students are presented with the formula and are able to explain how to manipulate the variables to generate the result.

Objectives

Observe the sample space obtained in the experiment of guessing to answer a test. Show how probability can influence a major event in a student's life. Calculate probabilities using technology.

Materials

- 10 Paper bag labeled Q1, Q2, Q3,…, Q10 for every pair of students
- 4 little squares one with the letter A, one with the letter B, one with the letter C and one with the letter D, for each Bag/Question, that yields a total of forty little squares with letters printed on one side
- 1 Table #4 of Test results to record the answers chosen for each of the 10 questions
- Pencil
- Construction Paper
- Paper

Activity 1

Students will work in groups of 4 and will alternate in drawing 1 answer from each Bag/Question, record the answers in table #4, put the little squares back in all the Bags/Questions. Every student must perform the process just described until 20 trials are complete, which corresponds to 5 trials per student. Students are encouraged to discuss the experimental results and calculate the experimental probabilities dividing #of test passed by total number of trials (20 trials). Manipulate the binomial distribution setting up the formula as follows:

$$P() = \frac{10!}{(10-7)! \cdot 7!}\left(\frac{1}{4}\right)^7 \cdot \left(1 - \frac{1}{4}\right)^{10-7} + \frac{10!}{(10-8)! \cdot 8!}\left(\frac{1}{4}\right)^8 \cdot \left(1 - \frac{1}{4}\right)^{10-8} +$$

$$\frac{10!}{(10-\ )! \cdot \ !}\left(\frac{1}{4}\right)^{ } \cdot \left(-\frac{1}{4}\right)^{10-\ } + \frac{10!}{(10-10)! \cdot 10!}\left(\frac{1}{4}\right)^{10} \cdot \left(-\frac{1}{4}\right)^{10-10}$$

Calculate probabilities with the formula. Students open a Microsoft Excel Worksheet to calculate the probabilities with the Binomial Distribution function; then they should compare the values of probability and conclude they are the same.

Assessment

Students should reflect their data in a report, whose main idea is to answer the question: Would you guess in a test? Would you guess in the TAKS test? Refer to the Example of Rubric to Assess a Report in the Appendix.

Lesson 3

The lesson will reproduce the Monty Hall Problem using three cards; the car is represented by the ace of hearts and the goat is represented by any two cards with spades. Students will repeat the experience of a contestant every night in the show 20 times corresponding to 20 nights in a month. Students will calculate the experimental probabilities from the data gathered in a table.

Objectives

Students will identify the sample space in the Monty Hall Problem. Students will compare theoretical probabilities with the experimental probabilities. Students will reflect about random situations in real life contexts.

Materials

- 3 cards from a 52 card deck, ace of hearts and any two cards with spades
- 1 Table #5 of Monty Hall Problem to record the combinations of doors, successes and failures, over a month, 5 nights per week, 4 weeks in a month yield 20 nights
- Pencil
- Construction Paper
- Paper

Activity 1

Students will work in pairs and will alternate roles, game host and contestant, until they complete 20 trials. The results are recorded in table 5 Monty Hall Problem. Students count the number of wins and calculate the experimental probability dividing wins by total number of trials (20). Students will create a tree diagram with all possible results. The teacher encourages students to use the Binomial Distribution formula to calculate theoretical probabilities. Students compare experimental data with the theoretical results. Students will explore the problem using an interactive webpage showing the 3 doors on a stage simulating the Monty Hall Problem (Garcia).

Assessment

Students will create a big stage with construction paper and will represent part of their results in front of the class. Report of results in journal format answering the question: Would you like to participate in the show *Let's Make a Deal*? Would you win the car? Refer to the Example of Rubric to Assess a Role Play in the Appendix.

CONCLUSION

The purpose of this thematic unit is to provide relevant examples to middle school students to motivate them in the study of mathematics, specifically probability. Students and teachers alike might find challenging to identify any other valid examples of binomial experiments, but in similar situations to the ones posed in my previous analysis, beating the "odds" can be achieved after careful consideration of the available options which we formally call sample space. The use of technology offers new possibilities of growth to different kinds of learners and increases the efficiency of calculations and the presentation of data. Gradually, more and more people are becoming involved with technology to study mathematical and physical processes. It is important to offer to students all the available tools to succeed. In the study of mathematics, there is no room for superstition or guessing. The study of probability is opening the door towards the understanding of how to defeat the "unpredictable."

APPENDIX

Table 1: Lottery Game

Events	1st Number	2nd Number	3rd Number	4th Number	5th Number	6th Number
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						
14						
15						
16						
17						
18						
19						
20						
21						
22						
23						
24						
25						
26						
27						
28						
29						
30						

Table 2: Tossing 4 coins

Events	1st Coin	2nd Coin	3rd Coin	4th Coin	# of heads	# of tails	Outcome
1							
2							
3							
4							
5							
6							
7							
8							
9							
10							
11							
12							
13							
14							
15							
16							

Table 3: Tossing 4 dice

Events	1st Die	2nd Die	3rd Die	4th Die	# of 3, success (3 dots in upper face)	# of failures, other than 3 (in upper face	Outcome
1							
2							
3							
4							
5							
6							
7							
8							
9							
10							
11							
12							
13							
14							
15							
16							

Table 4: Test Results

Events	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	# of correct answers	Score (%)
1												
2												
3												
4												
5												
6												
7												
8												
9												
10												
11												
12												
13												
14												
15												
16												
17												
18												
19												
20												

Table #5: Monty Hall Problem

Events	D1	D2	D3	Results (W=wins or L=loses)
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				

Example of Rubric to Assess a Report

Category Number	Category to be assessed	Available Points			Totals
1	Title of Report: (Experiment)	0	1	2	
2	Table with Experimental Data	0	1	2	
3	Clear Tree Diagram	0	1	2	
4	Possible Combinations	0	1	2	
5	Declare Sample Space	0	1	2	
6	Probability from raw data	0	1	2	
7	Theoretical Probability	0	1	2	
8	Analysis of results, reflection	0	1	2	
9	Originality	0	1	2	
10	Overall Appearance	0	1	2	
	Total				

Example of Rubric to Assess a Poster

Category Number	Category to be assessed	Available Points			Totals
1	Title of Poster: (Experiment)	0	1	2	
2	Table with Experimental Data	0	1	2	
3	Clear Tree Diagram	0	1	2	
4	Possible Combinations	0	1	2	
5	Declare Sample Space	0	1	2	
6	Probability from raw data	0	1	2	
7	Theoretical Probability	0	1	2	
8	Analysis of results, reflection	0	1	2	
9	Picture to represent experiment	0	1	2	
10	Overall Appearance	0	1	2	
	Total				

Example of Rubric to Assess a Presentation

Category Number	Category to be assessed	Available Points			Totals
1	Title of Presentation stated: (Experiment)	0	1	2	
2	Description of experiment, directions	0	1	2	
3	Explanation of experimental data	0	1	2	
4	Clear ideas presented	0	1	2	
5	Use of Visual Aids	0	1	2	
6	Declare Sample Space	0	1	2	
7	Probability from raw data	0	1	2	
8	Theoretical Probability	0	1	2	
9	Analysis of results, reflection	0	1	2	
10	Originality	0	1	2	
	Total				

Example of Rubric to Assess a Role Play

Category Number	Category to be assessed	Available Points			Totals
1	Title of Play stated by narrator: (Experiment)	0	1	2	
2	Story	0	1	2	
3	Characters	0	1	2	
4	Elements of the Stage	0	1	2	
5	Clear plot	0	1	2	
6	Present a problem	0	1	2	
7	Model solution	0	1	2	
8	Analysis of results, reflection	0	1	2	
9	Originality	0	1	2	
10	Overall Appearance	0	1	2	
	Total				

ANNOTATED BIBLIOGRAPHY

Works Cited

Carlson, Bruce. *Communication Systems*. New York: McGraw-Hill.1975.
    Applications of probabilities to communications systems.

Clarke, Aaron. "The Binomial Distribution." 29 June 2006.
    <http://elderlab.yorku.ca/~aaron/Stats2022/BinomialDistribution.htm>.
    Basic Definition of a Binomial Experiment.

"Combination". *Answers.com* (2006). 19 March 2006. <http://www.answers.com/topic/combination>.
    Definition of combination in a mathematical context.

"Event (probability theory)." *Wikipedia* (2006). 19 March 2006.
    <http://en.wikipedia.org/wiki/Event_%28probability_theory%29>.
    Basic Definition of Event in a mathematical context.

Garcia, Adriano. "The Monty Hall page" (2006). 30 June 2006. <http://math.ucsd.edu/~crypto/Monty/monty.html>.
    Virtual Manipulative to simulate the Monty Hall Problem.

HISD. Project CLEAR. Model Lessons. 6th and 8th Grade Mathematics.
    Curriculum framework for the Houston Independent School District.

Holmes, Susan. "Probability Tree." 19 March 2006. <http://www-
    stat.stanford.edu/~susan/surprise/ProbabilityTree.html>.
    Virtual Manipulative to draw tree diagrams.

Hubbard, Matthew and Tom Roby. "Approximating the Bell Curve by Pascal's Triangle." *Pascal's Triangle: From
    Top to Bo*ttom. 19 March 2006. <http://binomial.csuhayward.edu/applets/appletNormApprox.html>.
    Pascal's Triangle properties from top to bottom.

"Interactive Pascal's Triangle." The Mathforum @ Drexel. 30 June 2006.
    <http://mathforum.org/dr.cgi/pascal.cgi?rows=1>.
    Virtual Manipulative to generate a Pascal's Triangle.

Let's Make a Deal. The Official Let's Make a Deal website (1999-2006). 30 June 2006.
    <http://www.letsmakeadeal.com/>.
    Informative website about the TV game.

"Help File." *Microsoft® Excel 2002*. Microsoft Corporation (1985-2001). 19 March 2006.

Paulson, Richard A. "Using Lottery Games to Illustrate Statistical Concepts and Abuses." *The American Statistician*.
    August 1992, Vol. 46, No. 3.

Siegrist, Kyle. "Binomial Coin Experiment." *Virtual Laboratories in Probability and Statistics*. Math. UAH. (1997-
    2006). 19 March 2006. <http://www.math.uah.edu/stat/applets/BinomialCoinExperiment.xhtml>.

Van Zeghbroeck, Bart J. "Stirling's Approximation for Factorials." (1997) 12 February 2006. 02/12/06. <http://ece-www.colorado.edu/~bart/book/stirling.htm>.
    Definition of the Stirling's Approximation to calculate big factorials.

Webb, Peter. "Understanding Combinations and Permutations." (1996-2005). 29 January 2006.
    <http://www.probabilitytheory.info/topics/pascal_combinations_permutations.htm>.
    This site talks about origins and applications of the Pascal's Triangle.

Weisstein, Eric W. "Binomial Distribution." (2006) 19 March 2006.
    <http://mathworld.wolfram.com/BinomialDistribution.html>.
    Mathematical demonstrations and definition of a Binomial Distribution.

Supplemental References

Easton, Valerie J. and John H. McColl. *Statistics Glossary*. (September 1997). 12 February 2006.
    <http://www.stats.gla.ac.uk/steps/glossary/probability.html#probability>.
    Definitions frequently used in probabilities and statistics.

"Probability Axioms". *Wikipedia*. (2006). 19 March 2006. <http://en.wikipedia.org/wiki/Probability_axioms>.

"Probability Distribution" *Wikipedia*. (2006). 19 March 2006. <http://en.wikipedia.org/wiki/Probability_distribution>.

Shipman, Jerry R. *The Algebra & Trigonometry Problem Solver.* (2004).
    Solution Guide to textbooks.