Three Mysterious Numbers in Mathematics

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If I have seen further, it is by standing on the shoulders of giants – Sir Isaac Newton.

INSTRUCTIONAL BACKGROUND: WHY THIS TOPIC?

I have incorporated reading exercises in my math classes for four years because I believe that they will enhance students' ability in reading comprehension of math texts, and create a more active interaction and dialogue between my students and myself. The reading exercise is assigned at the end of each class, and students are to read and take a half to one-page notes on the section that will be taught in the next class period. They will include their own interpretations of definitions, formulas, and so on in their notes, as well as any questions that they may have.

During this period, however, I have observed that all, except a few of my students, merely do it for the sake of taking notes without any serious attempt to read with comprehension as I expected them to do. This exercise fails to produce the result I expected. The lively discussion and dialogue that I anticipate does not occur and ends up as shallow ones where I, the teacher, ask them a few questions pertaining to the assigned section. One reason that could help explain this unsuccessful activity is the time constraint that both teacher and students experience. My school uses the A/B block schedule in which students have four 90-minute periods a day. That makes a total of 8 classes they have to take on both A and B days, which amount to an average of at least 120 minutes for homework everyday. For the teacher, the materials that I have to cover in AP calculus or pre-AP precalculus classes are very demanding and do not permit me to squander time in activities that I deem unproductive. I continue this exercise, however, because some of the students still come up with great questions that stimulate discussion, which is beneficial to the whole class. I believe though, the biggest reason for this failure is that our students seriously lack the ability of reading comprehension. Ross Lence, in his general introduction to our seminar, *Reflections on a Few Good Books*, captures this problem as follows: "But reading great books is no easy task, for before one can know great books from not-so-greats, *one must know how to read*…the whole enterprise of reading great books is a complex web linking reading, thinking, and writing into one integrative whole" (emphasis added). Knowing how to read is essential to the reading of not only great books, but also to any other kind of books. The National Council of Teachers of Mathematics (NCTM) has been advocating reading in mathematics, for some time now, contending that mathematics students should be exposed to the cultural, historical, and scientific evolution of mathematics, and not merely the skills of manipulation of symbols. In this paper I will explain why I chose this particular topic, and what and how the unit will be taught.

Before I discuss more about the curriculum topic, it should be recognized also that mathematics is not all about numbers–higher math such as abstract algebra, topology, etc. is hardly constructed upon numbers, but on concept and definition. Nevertheless, the number is the main component of mathematics since the dawn of its history. Archeological relics show that the ancient people have employed numbers at the very first time they tried to measure their plots of land, or to count the number of their livestock. In other words, numbers have been developed out of necessity of mankind, and become an integral part of human activities. In a nutshell, numbers evolve from the set of natural numbers (or counting numbers), to the set of whole numbers that includes zero, to the set of integers that includes the negative numbers, to the set of rational numbers (fraction), to the set of real numbers that includes irrational numbers, and finally to the set of complex numbers that includes all numbers and the imaginary ones. In the development of the systems of numbers, there are three special numbers that marked off three significant periods in the history of mathematics. They are Pi (π), zero (0) and e. Tracing along the history of these numbers will not only illuminate the history of mathematics itself, but also of the development of mankind. In other words, in studying this topic students will have an opportunity to read math in a different context that provides linkage to other subjects such as history, language arts, English, geography, philosophy, ethics, and so on. For example, π is itself an enigma that could be traced back into ancient Egyptian time (around 1650 B.C.). Who discovered the number, how, and for what purpose are a few questions that could be used for discussion. The history of π could be taught as an introduction in a geometry course. For higher math courses, as in precalculus and calculus, π could be made to mystically appear in unusual places as in Euler's famous formula: $e^{i\pi} + 1 = 0$, or in some infinite series.

Furthermore, while tracing the development of these numbers, students will definitely sharpen their academic skills in analysis, and critical thinking. For instance e was first discovered by Euler, hence his initial was used to name this number, but e was not used as the base of the natural logarithm until much later time. In calculus, natural logarithm is defined in terms of an integral that enables one to solve and patch a hole in the power rule of integration. Nevertheless, to follow the development of e is to follow the development of logarithm with Napier's brilliant idea when he developed logarithm to handle calculation involving large numbers in astronomy. In order to follow this development, students are to recall and apply previous knowledge in algebra and geometry in a new context. In other words, students will be able, at the very least, to analyze the differences and similarities between Napier's log and natural log, and to ask themselves provocative questions such as "what is *natural* in natural logarithm?" that one has taken for granted.

WHAT DO I TEACH IN THIS UNIT?

After discussing the reasons why I chose this topic, I will discuss the objectives that I expect my students and me to achieve in this unit. My goals might seem a bit ambitious, but due to the particular environment of my school and my students, I believe that we could work it out.

DeBakey High School for Health Professions is a magnet school that is a joint venture of Houston ISD and Baylor College of Medicine. The school was initially established as a vocational school, but over the years has become a secondary school with a comprehensive and challenging pre-college academic and health-oriented educational program that enables students to pursue post-secondary health careers. Our students are selected from around the district through a competitive entrance exam and standardized test scores, TAAS and Stanford 9. The curriculum at DeBakey is rigorous and demanding. Besides health science and other traditional courses, every student at DeBakey must take five years of math and science resulting in calculus in the senior year. The state of Texas has mandated in the past three years that if calculus were offered in high school, then it must be Advanced Placement (AP) calculus. This law has created a tremendous stress for both teacher and students, since AP calculus is a college level course, and students are expected to sit for the AP exam. There are two levels of AP calculus: AP calculus AB and BC. Calculus AB is equivalent to Calculus I and BC to Calculus II in college, respectively. DeBakey offers both AP calculus AB and BC.

I have been teaching pre-AP precalculus and AP calculus AB and BC courses for the past four years. With the exception of the AP course, which follows well-designed topics by the College Board, one of the questions in teaching these classes is how to differentiate the regular and the pre-AP classes. Both regular and pre-AP must cover the same materials, but the pre-AP will have to cover them in depth and at higher level of thinking. According to Bloom's taxonomy–knowledge, comprehension, application, analysis, synthesis, and evaluation – students in a pre-AP course are expected to perform at the level of application and above. However, the notion of depth is vague. How deep in a topic should a teacher teach his students? Should the teacher teach how trigonometry was discovered and in what circumstances, for instance? Does the depth of a topic mean that students are exposed to a wider context in which the topic is applied?

The Curriculum Council of the National/State Leadership Training Institute on the Gifted and Talented (GT) developed the Principles of a Differentiated Curriculum to help teachers design lessons meeting the needs of GT students as well as those of pre-AP and AP students. The lessons should be designed to foster the skills clustered in three groups:

(i) Development of critical and creative thinking and problem-solving skills: to identify characteristics and attributes, categorize, classify, identify pattern, determine cause/effect, summarize, formulate questions, etc;

(ii) Development of research, learning-to-learn, and study skills: to use parts of a book, multiple and varied references, to conduct a research study–historical, experimental, correlational, descriptive, to organize information; and

(iii) Development of skills of production and presentation: to create a work plan and basic design to follow to develop the product, to recognize materials needed to execute the product (Brandt, 2000).

In other words, the principles mentioned here will serve as guidelines for my curriculum unit.

As earlier mentioned, reading is essential in every academic endeavor, but somehow it has not been emphasized in mathematics, even though the skills required in reading a math text are not different from other analytical reading. My first objective in teaching this unit is to cultivate in students some interest in reading mathematics by introducing to them the history, cultural backgrounds, mathematicians, and evolution of mathematics in reading materials. Students will be exposed to the development of the "mysterious" numbers in readings about each number. The discussion that follows the readings is an integral part of teaching this unit. It is during discussion that students will demonstrate their comprehension and analysis of the reading materials. Not only the mathematical portion of the reading is discussed and analyzed, but as Shmuel Avital, a mathematics professor at Israel Institute of Technology, asserts, "exposing students to some of this development has the potential to enliven the subject and to humanize it for them." Moreover, when students read about the history of the numbers, they will have to solve some of the problems that lead to the development of the numbers. They bring the readers back to the time when the problems were posed and illustrate the concerns of the period. The final objective of this unit is for the students to recognize the linkage between mathematics and other fields as in philosophy, ethics, physical science, and even religion.

In particular each of the mysterious numbers will be introduced in this unit. Each number has its own place in the history of mathematics and has tremendous effect to all aspect of mankind. For example, when study Newton's Law of Cooling (a topic covered in Algebra II, precalculus, and calculus), which states that the temperature of an object will cool down in proportion to the difference in its temperature and that of the surrounding medium, students will be able to apply this law in various ways to find the initial temperature, or the temperature of an object at time t, or the temperature of the surrounding area (in mathematical equation, the Newton's law of cooling is $y(t) = Ce^{kt} + T_s$ where *y*(t) is the temperature at time *t*, and T_s is the temperature of the surrounding area.)

However, an innocent question such as "when will the object cool down to the temperature of its surrounding area?" could spur a great debate in the notion of zero. If the object cools down to its surrounding, then there is no difference in its temperature and that of the surrounding medium, and therefore, the law of cooling breaks down, because it is impossible to solve for *t* in the equation $Ce^{kt} = 0$. What could be interpreted as *no* difference? How small is the difference for it to be considered no difference? The discussion will lead to another important concept in mathematics, infinity. The notion of

“Cool” in pop culture “cool guy” could be also differentiated in the physical and social behavioral sense (people don’t get cool, only dead people get cool).

The History of π

1. What is the number Pi (π)?

Almost all of us know what Pi (π) is and that it has something to do with circle. π is the sixteenth letter of the Greek alphabet, used to denote the ratio of the circumference of a circle and its diameter. The oldest record of π is found in the Egyptian Rhind Papyrus, dated in 1650 B.C., in which the value of π is about 3.16. More interestingly is the fact that π has been mentioned in the Bible. “And he made the Sea of cast bronze, ten cubits from one brim to the other; it was completely round all about. Its height was five cubits, and a line of thirty cubits measured its circumference.” (I Kings 7: 23). Numerically speaking π is about 3.1416, but actually π is an *irrational* number with unending billions of digits after the decimal point. In 1999, Dr. Kanada of University of Tokyo calculated 206,158,430,000 decimal digits for π (Blatner, 1997).

The key concept here is the *irrationality* of π that will lead to the discussion about the evolution of the number systems: the set of the natural numbers, the integers, the rational, the irrational, and the set of real numbers. A number is a rational if it can be expressed as a ratio of two integers, or as terminating or non-terminating and repeating decimals, such as 3/5 = 0.6, or 3/11 = 0.2727…The irrational number does not fall into this definition; it is out of place with the known numbers, and indeed it has baffled the ancient Greek mathematicians. Actually the ancient Greeks have known the existence of the irrational numbers when they tried to find the length of the diagonal of a square with side 1. Using the famous Pythagorean theorem, one can easily find the length of the diagonal to be $\sqrt{2}$, the square root of 2. However, they refused to accept irrational numbers and have swept them under the rug for a long, long time. Why did the Greeks refused to accept the irrational numbers? What was the Greek philosophy π about the natural world? What schools of thoughts dominated the ancient Greek philosophy? Those are the questions that will lead students to the realm of philosophy and history of the ancient Greece.

Pi appears across cultures, continents and times. The Babylonians and Egyptians used Pi around 2000 B.C.; the numerical values are $\pi = 3\frac{1}{8}$ and $\pi = \frac{256}{81} = 3.1605$, respectively. Around 1100 B.C., the Chinese discovered π = 3 in around 1100 B.C. In the third century B.C. Archimedes inscribed a 96-sided polygon in a circle to establish that π is between $3\frac{10}{71}$ and $3\frac{1}{7}$. This method is also called the *exhaustion* method that is the forerunner of calculus. In the second century Ptolemy used Pi = 3.14166…, and the Indian discovered Pi in around 500 C.E. In 1220 Leonardo de Pisa, a mathematician who is better known as Fibonacci, after his discovery of the Fibonacci sequence, calculated π =3.141818… Since then π returned to Europe and spread all over the world.

2. Computation of π

Mathematicians all over the world have tried to compute π since its discovery. The exhaustion method of Archimedes is the first systematic method using the perimeters of the inscribed and circumscribed polygons of a circle to approximate the circumference of the circle. Archimedes found π to be about 3.1419 (Blatner, 19). The exhaustion method used to calculate π is indeed "exhaustive." Around 1600s mathematicians began to lose interest in the laborious task of spending endless hours to add, subtract, multiply and divide polygons for the calculation of π. Mathematicians must have thought that there must be a better way to compute π, and in their search for a better way, new disciplines of mathematics were born, one of which is calculus, as we know today. With the advent of technology, computers join in the quest for the mysterious π, in 1997, Kanada and Takahashi calculated 51.5 billion digits for π using the computer Hitachi SR2201 in just over 29 hours. (Blatner, 59)

3. What are its uses?

Of course the first appearance of pi is in the area of a circle: $A = \pi r^2$, where r is the radius of the circle, and in other geometric figures that involved circles such as the sphere, the cone, etc. The most amazing place for π to appear is in the most famous formula, which was discovered by Euler: $e^{i\pi} + 1 = 0$, where it has nothing to do with geometric figures and combines all mysterious numbers together, namely the number *e*, 0, π itself and *i*. *e* is also a transcendental number that will be discussed next, as well as the number 0 which has been taken for granted nowadays. π, 0 and *e* are *real* numbers, whereas *i* (defined as $i^2 = -1$) is the *imaginary* unit of the complex numbers, which will be discussed, hopefully, in another curriculum unit.

What about the numerical value of pi? In calculation, for all practical purposes and even for the most precision machine there always exists some error. Usually if the error is about or less than one thousandth, then it is considered "pretty" good. Blatner asserts that no calculation that involved π would realistically requires more than 7 digits, even a physicist would not require more than 15 or 20. So why are mathematicians so driven in the quest for the mystery of π? The answer partly is to find a better way to determine a more accurate value for π, but in this quest, mathematics evolved into different and higher level. Newton and Leibniz both discovered infinite series that compute π while developing calculus. The power of computer is also determined by its capacity to compute the value of π. In so doing engineers were able to identify the hidden flaws in their hardware or software that could not be uncovered by any other ways. (Blatner, 3)

With more than 51 billion of digits, π is so close but still so far for human beings. It is still an enigma and an indicator of imperfection inherent in human beings.

The History of Zero (0)

1. Why Zero?

Zero holds a very special position in the set of all numbers. However, zero has not been accepted; it was even banned from Western society for more than two thousand years. This section will discuss the creation of zero, and its position in mathematics, physics, philosophy and its relation to the infinity.

Zero is different from π in a sense that it is a creation of man, whereas π appears in nature. The history of zero dated back farther than π, when human beings began to count. Archaeologist Karl Absolom in the late 1930s unearthed a 30,000-year-old wolf bone with a series of notches carved into it. (Seife, 2000) This evidence shows that ancient people were counting something. The Egyptian was the first to use symbols to represent numbers five thousand years ago, even before the time of the pyramids. Nevertheless, there was no need for the symbol and number zero. Early people did not need zero simply because they did not have to keep track of zero something, be it sheep or bananas. The creation of zero took place in around 300 B.C. when the Babylonian had started using two slanted wedges, // to represent an empty space, or a placeholder on the abacus. Similar counting machines were invented in other cultures: the suan-pan in China, the soroban in Japan, the s'choty in Russia, the coulba in Turkey (Seife, 14).

The Egyptians were the most civilized people of the ancient world. They invented geometry and became masters of mathematics. They developed methods to calculate the area of various shape of land, the volume of objects like the pyramid, etc. However, Egyptian mathematics did not evolve into an abstract system of logic. They never progressed beyond measurement or counting. Nonetheless, Egyptian mathematics influenced other countries, including Greece, whereas mathematics flourished into a higher level of abstraction.

2. Zero, Void, and Nothingness

The Greeks embraced and enhanced Egyptian mathematics. Pythagoras who is probably one of the greatest mathematicians with his famous theorem, "the square of a hypotenuse of a right triangle equals the sum of the squares of the other two sides," founded a school of thought that professes the relation between number and philosophy as inseparable. In Greek mathematics there was no significant distinction between shapes and numbers. Multiplying two numbers, for example, is the same as finding an area of a rectangle. Therefore, it makes no sense to multiply something by zero, since there is no such rectangle with zero width, or zero length.

Furthermore, to the Pythagoreans every number-shape has a hidden meaning, and the most beautiful one is considered sacred. Thus the ultimate symbol of the Pythagorean view of the universe is the *golden ratio* — dividing a line segment so that the ratio of the small part to the large part is the same as the ratio of the large part to the whole. If 1 is the length of a small line segment, and x is the length of the large part, then the golden ratio is obtained by solving the quadratic equation $\frac{1}{x} = \frac{x}{1+x}$. The golden ratio is about 1.618 (The golden ratio is beautiful in the eyes of the Pythagoreans, but it is an irrational number and irrational number endangers the Pythagorean system as much as zero). To the Pythagoreans, ratios and proportions controlled the universe, and the whole Greek universe rested upon the tenet that there is no void, because the earth is at the center of the universe, and the sun, moon, planets and stars revolved around the earth. Zero is a troublesome number because when zero is divided by something (other than zero itself) it yields zero. It is even more troublesome when something is divided by zero. The ratio does not make any sense. Therefore, zero must be rejected. But while the West rejected the void, other cultures in the East embraced it. The notion of void, and nothingness was central in Islamic thought, which dominated the Arabic world, the Middle East, and Central Asia – where God created the universe out of the void.

3. Zero, Infinity and God

Zero and infinity go hand in hand. On the one hand, dividing a number by zero does not make sense, but what if that number is divided by a very small number? The result would be obviously very large. What if that very small number is as small as zero but is not equal to zero, the result would be gigantically large, the infinity. However, Pythagoras' view and later Aristotelian doctrine became dominant philosophy of the Western world. According to this doctrine the planets move in heavenly spheres and there are but a *finite* number of spheres corresponding to the number of planets in the system. There cannot be an infinite number of nested spheres, and so the West rejected both infinity and the infinite (Seife, 40). This tenet was endangered by the paradox of Zeno, a philosopher of Elea. The paradox was put into a story about a race between Achilles and a turtle, and in this race, Achilles never could catch up with the turtle that had a head start. Suppose that a turtle is 1 foot ahead of Achilles and it runs at half a foot per second, whereas Achilles runs at one foot per second. In a mere second, Achilles has caught up with the turtle, but in that second, the turtle has moved half a foot ahead. And a half of a half of a foot ahead, and so on, ad infinitum. Achilles never could catch up the turtle.

The History of *e*

1. Logarithm and Natural Logarithm:

e is the base of the natural logarithm as we know today. But logarithm was developed by Napier without the concept of a base. Napier developed logarithm to ease the labor astronomers had to undertake in the extensive plane and spherical, trigonometrical calculations necessary for astronomy. The mathematician Pierre Laplace, who is well known by his Laplace Transform in calculus, and also known as the "Newton of France," remarked: "By shortening the labors, the invention of logarithms doubled the life of the astronomer" (Maor, 22). Napier began his logarithm by applying the rule of exponent—the product of b^m and b^n equals b raised to the exponent (or power) of m+n: $b^m \cdot b^n = b^{m+n}$—to the extent that "if we could write any positive number as a power of some given, fixed number (later to be called base), then multiplication and division of numbers would be equivalent to addition and subtraction of their exponents" (Maor, 6). Napier began a painstaking task of constructing the table of logarithms in twenty years, even though his initial table contains only 101 entries. Napier named his creation after twenty of patient and hard work *logarithm*, meaning "ratio number." Even though Napier did not think of a base for his logarithms (he chose $1-10^{-7} = 0.9999999$ for his "base"), Napier logarithm is not far from having the base *e*, which was discovered later by Euler. (Students will be asked to demonstrate how Napier log works and how was it linked to the natural log).

2. e and the Limit process

Euler established that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2.71828...$This expression contains two important concepts in mathematics: the limit and the infinity. Literally the expression reads as the limit of $(1+1/n)^n$ as n approaches infinity is *e*. But what does it really mean by infinity? A number that is greater than any number that anyone can think of? What does it mean by the limit? Can anyone ever reach that number *e*? Those are the questions that push mathematics to a higher level of abstraction, and open up new frontiers for mathematics, science, art, and philosophy. Mathematically speaking, as *n* approaches infinity, 1/n approaches 0, and $(1+1/n)^n$ approaches 1^∞, which is not 1, but indeterminate.

3. e in the physical world

Not only is *e* the base of the natural logarithm, *e* appears mysteriously at many places both in the mathematical and physical worlds. Since *e* is the base of natural log, denoted by the function $f(x) = \ln x$, the inverse function is $f(x) = e^x$ models many physical phenomena: the growth and decay model, the law of cooling, the interest in finance, the curve of a hanging chain (the catenary), or in the Gateway Arch at St. Louis, Missouri. But the most mysterious place for *e* to appear is in the famous Euler formula: $e^{i\pi} + 1 = 0$, which combines all three mysterious numbers appear together with the imaginary unit $i^2 = -1$, and in this combination of mathematical operations produces a real number.

TEACHING STRATEGY

This unit is intended for pre-AP precalculus, as well as AP calculus AB and BC (eleventh and twelfth grades), although the regular classes could be benefited if the reading materials are introduced to students as historical anecdotes to humanize the subject and as introduction to new idea and concept.

I plan to use this curriculum unit as a project that lasts a whole grading cycle (6 weeks), in which the teacher will serve only as coach or mentor, some kind of "a guide on the side, not a sage on the stage." Students will have to report their progress in a predetermined timeline. Since my school follows A/B day schedule with each period 90 minutes long, this unit could be developed as a major project consisting of 6 phases and the time spent for each phase in class will not be more than 45 minutes.

1. The class will be divided into three groups of about five to six students. Each group will be assigned a topic, namely, $\pi, 0,$ or e for research. Each group will select a group leader who will be responsible for coordinating the group's activities.

2. Each group will present its progress report in writing in two weeks after receiving the assignment, including their findings in books, articles, etc. This progress report is 10 percent of the total grade.

3. First presentation is in the third week during the first 45 minutes of the class. In this briefing, the whole group will sit as a panel facing the whole class, and each member of the group will present his/her findings on the background of the topic using the basic journalistic format 5 W and 1 H (what, where, when, why, who and how). After the briefing about 20 to 30 minutes, the panel will answer questions from their peers and from the teacher. This presentation is 20 percent of the grade.

4. Second presentation in the fifth week is also in the first 45 minutes of the class. In this presentation the group will present the mathematics of the topics including the techniques, the reasoning, etc. And again, the panel will be quizzed by its peers and by the teacher. This presentation is 20 percent of the grade.

5. Third presentation in the sixth week is also 45 minutes in length. Each group will present the linkage, or influence of the topic to other fields and answer questions from their peers and from the teacher. This presentation is 20 percent of the grade.

6. Final paper. Each group will submit its final paper on the topic that captures the essence of the previous three presentations and incorporates the discussions emanated during the presentation. This paper is 30 percent of the grade. Another possibility for the final report is in electronic format, either in power-point presentation, or in a web site. Using an electronic format is not only a cross-curriculum activity between math and computer classes, but also an activity incorporating technology into teaching and learning.

In order to monitor each group's progress, I will use the Group Evaluation form, which was developed by Ellen Kamischke in her book "A Watched Cup Never Cools," published by Key Curriculum Press (1999), with some modification of my own to fit my students' project (see Appendix A). Kamischke remarks in her book about the Group Evaluation form:

> I use this as a confidential report from them to me. Students should feel free to vent frustrations with other group members or with the activity itself in this document. They should be encouraged to complete the evaluation thoughtfully. To provide this encouragement, I award points for turning in a thoughtful evaluation for each lab. Reading the evaluation from various group members gives a picture of what went on as they pursued the activity…You may also want to have that group check in more frequently and report on each individual is doing.

With three groups per class, the topics of this project will be covered sequentially in a whole semester. The second group that is assigned the topic of the number *e* will also present its findings in the second 6 weeks to correspond with the topics in the course.

The next very important question is the assessment of the project. How do I know that the project is a success? The grade awarded to each phase does not truly reflect the success of the project, however, because students could simply do what is required of them to earn grade. The assessment that comes at the end of the project is not only about the grade students earned, but whether the intended purposes are achieved. I believe what emanates from discussion is far more important than the grade students earned at each phase. The teacher's role as a guide or facilitator, therefore, is very important in keeping the discussion alive, not only for the panel, but also for the whole class.

One disadvantage of this format is that only one group really has to do its assigned project while other groups may not do anything during the whole 6 weeks, or the whole twelve weeks. To compensate for this, during each group presentation, the whole class is encouraged to take notes on the topic as well as on the discussion during the presentation. The ideas, concepts, reasoning from the topic could be incorporated as essay questions in regular quizzes or tests during the semester or in final exam.

Suggested discussion questions

Presentation of Pi

In the first presentation, the group will present its findings on the history of Pi based on the 5W and 1H format. The following questions will help to stimulate thinking for the class as a whole:

1. Give an explanation on why Pi appears across cultures and continents. (This is an open-ended question and has no definite answer. Students could give some plausible explanation).

2. Discuss the evolution of the number system: natural, whole, rational, irrational, real and complex numbers. This discussion will lead to the concept of cardinality of sets of numbers: which set of numbers is countable; which is uncountable?

In the second presentation, the group will present the computation of Pi. The possible questions for discussion in this presentation are:

1. How did ancient people derive the formula for area of a circle?

2. Compare and contrast the methods used by the exhaustion method in calculating the area and the circumference of a circle. Which method is easier, why? What concept is implied in using the exhaustion method?

In the final presentation of Pi, the following questions and exercises will stimulate thinking on the practical use of computation of Pi

1. Use a graphing calculator to calculate the value of Pi. (This exercise will require the use of series, a topic in the second semester of both precalculus and AP calculus BC; however, students could use the built-in function of the TI graphing calculator to compute the value of Pi. The infinite series will be revisited and studied in depth later on in the second semester of calculus BC).

2. Why have mathematicians tried to compute the value of Pi? What are the practical uses of calculating Pi to billions of digits?

Presentation of Zero

In the first presentation of zero, the group will present findings on the history of zero. The following questions will help stimulate thinking for the class as a whole:

1. In what circumstances did zero first appeared? (Students will try to give plausible answers to this question, and in doing so will look into some aspects of the ancient society: agricultural, nomadic, mercantile, etc.)

2. Explain why a number divided by zero does not make sense.

3. How did the Greeks explain the existence of God, and why does zero threaten the Greek universe?

In the second presentation, mathematics that involve zero will be presented. Here are some suggested questions that involve limit operation:

1. Resolve the Zeno paradox. (Zeno paradox could be resolved either by using the sum of infinite geometric series [precalculus], or by the limit process [calculus]).

2. Solve the equation $C e^{k t} = 0$ for *t*. What difficulty (difficulties) arises in solving this equation? What is considered *nothing*?

3. What is the empty set? Why is the empty set a subset of any set?

In the final presentation, students will present the linkage between the number zero to other fields such as arts, philosophy, religion, etc. The following questions could be used to guide the discussion:

1. How does zero fit in perspective drawing/painting?

2. Explain the relation between zero and infinity. (Students could use the exhaustion method of Archimedes, mentioned in the presentation of Pi, or the limit process in calculus).

3. What religions employ the notion of void and nothingness? Assuming that Greek philosophy dominated the West, and thus hampered the mathematical and scientific progress for more than 2 thousand years, discuss the danger of close-mindedness and dogmatism in a society.

Presentation of *e*

In the first presentation of *e*, students will present their findings on the background of *e*. The following questions could be used to guide the discussion:

1. For what ends and how did Napier invent logarithm?

2. What is natural logarithm and how is it different from Napier logarithm or from common logarithm? How did *e* appear to be the base of natural log?

3. Why is natural logarithm “natural”?

In the second presentation, students will present the mathematics involved in the development of *e*. Here are some possible questions for discussion:

1. Use log table to calculate $\left(\dfrac{(493.8)(23.67)^2}{5.104} \right)^{1/3}$

2. Derive the formula for compound interest and calculate the monthly payment of a mortgage.

3. Evaluate $\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$ by numerical process and by calculus.

In the final presentation, students will present the linkage between *e* and other fields as in arts, science, philosophy, etc. The following questions could be used to guide discussion:

1. Explain the equiangular spiral (logarithmic spiral) or spira mirabilis.

2. What are transcendental numbers?

CONCLUSION

This curriculum unit is about readings in mathematics using the three mysterious numbers in the history of mathematics, and is developed as a project that encompasses the principles promulgated by the Council on Gifted and Talented. The intended audience is students in pre-AP precalculus or AP calculus. Students are expected to do their own readings and research about the topics. It is further assumed that students are self-motivated, and the teacher will serve as a coach or facilitator who guides them towards the intended goals.

For students who are in regular math classes, the readings about the historical background of these numbers could also be assigned and discussed as an introduction to new concepts in the course.

APPENDIX A

Group Evaluation Form

Project Title:______________________________ Group #______________

Group Members' Contribution: (In the parentheses following each name, rate the contribution of each member on a scale from 1 to 5, with 1 indicating a limited contribution and 5 indicating a strong contribution)

- Leader:____ ( )
- Your name:____( )
- Other group members:_____________________________________ ( )

  _____________________________________ ( )

  _____________________________________ ( )

1. **Describe your contribution to the project** (Be specific, especially what you did to assist in the completion of the project. Use the back of this page if you need more space.)

2. **How is the workload divided among group members?** (Fairly? On research, presentation, etc?)

3. **What type of workload were you assigned?**

4. **Were there difficulties in working as a group?** (Be specific.)

ANNOTATED BIBLIOGRAPHY

The following is the list of books that pertains to the unit. This list will serve as a starting point for further researching into the topic.

Blatner, David. *The Joy of Pi.* New York: Walker and Co., 1997.
Perhaps this is one of most completed books about Pi. The author traces the history of Pi from ancient Egypt to other developments of the number in other continents. The history of Pi provides other historical developments in other related fields of mathematics.

Dunham, William. *The Mathematical Universe.* New York: John Wiley & Sons, 1994.
The author presents a selection of great proof, notorious disputes, and intriguing unsolved mysteries with subjects ranging from Greek geometry to infinite series.

Hoffman, Paul. *Archimedes' Revenge.* New York: W.W.Norton, 1988.
This book covers a wide range of topics from number system to application in physics, chemistry, computer and political science.

Maor, Eli. *To Infinity and Beyond.* Princeton, N.J.: Princeton University Press, 1991.
The author "gives a thorough survey of infinity as seen by mathematicians, by the artists and by those who have been fascinated by it." The concept of infinity is viewed as mathematical infinity, geometric infinity to aesthetic infinity and cosmological infinity. This book covers the "cultural history of the infinite."

———. *e: The Story of A Number.* Princeton, N.J.: Princeton University Press, 1994.
The development of this number was traced back from Archimedes of ancient Greek to David Hilbert of modern time. Closely related to the number e is the development of logarithm and complex numbers.

Nahin, Paul J. *An Imaginary Tale–The Story of* $\sqrt{-1}$. Princeton, N.J.: Princeton University Press, 1998.
Imaginary number is demystified in this book. The author tells the story of i where $i^2 = -1$ from a historic as well as from human perspective. The book encompasses historical facts, math discussions, and application of the complex number.

Paulos, John Allen. *Beyond Numeracy–Ruminations of a Numbers Man*. New York: Random House, 1991.
This book covers a wide range of topics in mathematics, from the basic to the advanced. In writing this book, the author demystifies the belief that mathematics is hierarchical. "Often very 'advanced' mathematical ideas are more intuitive and comprehensible than are certain areas of elementary algebra."

Seife, Charles. *Zero–The Biography of a Dangerous Idea.* New York: Penguin Books, 2000.
The author traces the history of zero from its birth as a philosophical concept in the East to its struggle for acceptance in the West. This book encompasses the ideas and theory from Aristotle, Pythagoras, Descartes, to Einstein.

Teacher Resources

Bell, Eric Temple. *Mathematics: Queen & Servant of Science*. Mathematical Association of America, 1987.
As the title indicates, this book covers a wide range of topics in mathematics and science from elementary to advanced pure and applied mathematics. In this book, Bell also tries to answer one of the most profound questions: "How is it that abstract patterns, created in the minds of mathematicians, so beautifully mesh with the physical structure of the universe? Could it be there is no reality other than our shifting mental impressions?"

Field, J.V. *The Invention of Infinity–Mathematics and Arts in the Renaissance*. New York: Oxford University Press, 1997.
It is not surprising for the application of mathematics in arts, for the wide use of math in arts is the symmetry of figure. However, Field traces the history of arts in the Renaissance and discovers that one of the basic concepts of modern mathematics, namely infinity, has been applied in the creation of arts in the renaissance, thus presenting a clear linkage of mathematics and arts.

Gleick, James. *Chaos: Making a New Science*. New York: Penguin Books, 1987.
This book explores the phenomena of chaos in relation to patterns in mathematics with extensive use of computer model.

Klein, Jacob. *Greek Mathematical Thought and the Origin of Algebra*. MIT Press, 1968.
Klein investigates the revival and assimilation of Greek mathematics in the sixteenth century from the symbolic concept of number to modern science. The author traces the development of formal mathematical language from the *Arithmetic* of Diophantus to modern formalism of Vieta and the transformation of the arithmos concept.

Lakoff, George and Rafael Nunez. *Where Mathematics Comes From*. New York: Basic Books, 2000.
The authors link cognitive science to mathematics –the motion that abstract ideas arrive via conceptual metaphor within the cognitive unconscious—from arithmetic and algebra to set and logic to infinity. In short, this is a study of mathematical idea analysis, which provides a Theory of Embodied Mathematics.

Smith, Sanderson. *Agnesi to Zeno*. Key Curriculum Press, 1996.
This book provides linkage of mathematics to the development of mankind encompassing the history of mathematics from counting systems to modern development in chaos theory.

Swetz, Frank, ed. *Learn From The Masters*. MAA, 1995.
This book presents a series of essays addressing a wide range of topics in mathematics from the history in school mathematics to history in higher mathematics. This book is considered to be a manual for teaching the history of mathematics.

White, Alvin M. ed. *Humanistic Mathematics.* MAA, 1993.
As the title suggests, this book is a collection of essays that presents mathematics as a discipline within human perspective. Thomas Tymoczko, one of the contributors of the collection, asserts that humanistic mathematics is mathematics with a human face. It provides for an understanding of not only what is investigated, but also on how and why it is investigated. Humanistic mathematics has become a major part of mathematical culture.

McLeish, John, *Number*. Ballantine Books, 1991.
In a nutshell, this book is about number, or rather the history of numbers. But it is more than just the history of numbers? The author traces the history of numbers and mathematics through ancient time to modern mathematics and electronic age, and their application.