What a Drag…

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INTRODUCTION

This unit is designed to introduce the concept of abstract modeling to high school students. The mathematics involved could be taught from tenth grade to twelfth grade. The ideas are fairly simple. Together they combine into a long process. With the use of spreadsheet programs, the tasks can become possible for any student.

The modeling will use probability and statistics to model the effects of the atmosphere on objects that move through it. This is commonly called air drag. I will not use pre-developed formulas. The formulas will be developed from basic concepts. These concepts are the laws of motion and momentum. The laws will be simplified into common sense concepts.

The models that we will create are of the atmosphere, rain falling in the atmosphere, a baseball falling from the sky, a baseball being hit by a bat, and a rocket being launched from the ground.

RATIONALE

The first question that arises is *why teach abstract modeling?* With all of the material and skills that must be taught in school, it seems to be way beyond the level of what should be taught. The common educational wisdom is first we will teach students the basics and then we will concern ourselves with enrichment. This unit is designed to make a frontal assault on common educational wisdom. I am declaring war on traditional education.

So what is wrong with traditional education? Let's apply the conventional wisdom to something dear to the hearts of teenage students-driving a car. Imagine if we said that before anyone could learn to drive, they must first learn everything about an automobile. Most people would never get the opportunity to drive! Maybe that may sound good to some people. It sure doesn't sound good to someone who wants to learn to drive.

A modern educational answer is we'll teach through a "hands-on" approach. This would be great for learning how to drive. It is not good in building a large bridge. We can't keep trying to build bridges until we get it right. There must be a better approach.

The approach used in this unit is to design abstract models from concepts students already have. The goal is to have the models be only as complex as necessary to develop basic understanding. While the models may not perfectly describe what's happening, they will serve to get results that are close enough to reality.

MY STUDENTS

I teach “gifted and talented” students who are seeking careers in professional area. The quotes around gifted and talented are there because I have no clue what anybody means by that term. I do know that my students are highly motivated and have developed a strategy to meet their aspirations. Ninety percent of this year’s graduating class has indicated a desire to study engineering or computer science in college.

This unit, however, is designed at a level that does not presuppose an above average motivation or ability. The idea is to make modeling an interesting exercise. Therefor this unit will concentrate on the concepts and use spreadsheets to make calculations. This unit will assume that teachers can set up the spreadsheets.

THE BEGINNING – ASSUME, ASSUME, ASSUME…

The very first step in modeling is to make assumptions. The answers that are based on these assumptions are not exactly correct. These assumptions will cause concerns among many teachers and students. Engineers and scientists do assume all the time. The calculations that they make are never exact. By designing models using assumptions, we are entering the real world and leaving the unreal world of education.

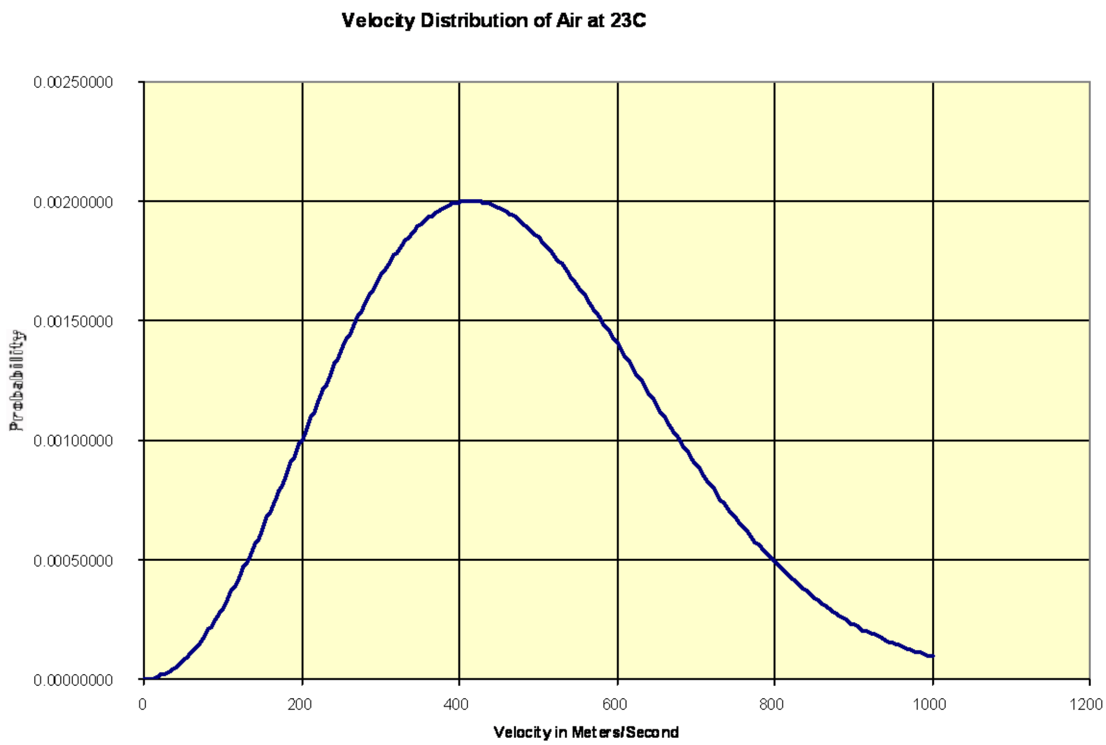
The major point in modeling is to state the assumptions clearly so that someone else can know the limitations of the model. These assumptions will be stated in terms of the simplifications that will be made to enhance understanding. Students need to know the assumptions so that they know that more complex but more accurate models can be made.

Our first assumptions are about the nature of the molecules in the atmosphere. The atmosphere contains nitrogen (78%), oxygen (21%), argon (0.6%) and small amounts of carbon dioxide, water, ozone, volatile organic compounds, particulate matter, and other gases. We will simplify the atmosphere to be one type of molecule with a mass that is the weighted average of nitrogen and oxygen. This mass is 4.79×10^{-26} Kilograms.

Molecules come in various shapes. They are also rotating in random directions. We will assume that the molecule for our models is a perfect sphere that has a radius of 1.4 angstroms (1.4×10^{-10} meters). We will assume that they are not rotating. This assumption will simplify our calculations of collisions.

No collision between any two objects is perfectly elastic. A perfectly elastic collision results in no loss of energy. We all know that when molecules collide with an object entering our atmosphere from space, the object gets very hot. The space shuttle reentry is a classic case. The tiles on the shuttle glow red hot upon reentry. Nevertheless, we will assume that all collisions with molecules are elastic collisions. This will allow us to use simple formulas to describe air drag.

Molecules at room temperature move in almost random directions with different speeds. The direction is almost at random because gravity makes the preferred direction to be towards the Earth. The speed of the molecules follows a distribution with an average of 468 meters per second. A graph of the velocity of our air molecules at room temperature is shown below:



When the molecules collide with themselves we will assume that the one molecule is moving at 468 meters per second and the others are still. When the molecules collide with other objects, we will assume that they are not moving.

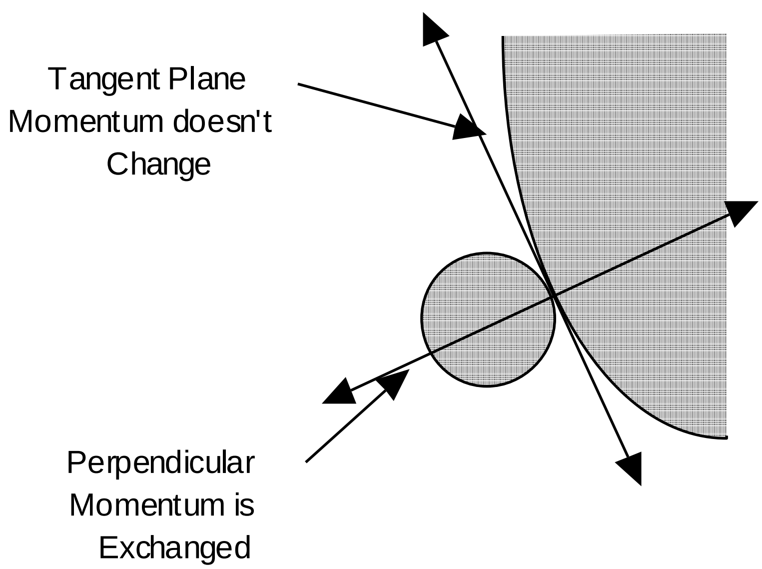
We will also make some statistical assumptions. There is an average of 2.69×10^{25} molecules in one cubic meter. This is such a large number that we can assume using averages will yield reasonable results for our models. Therefor we can assume, for example that the average velocity is zero, even though the average speed is 468 meters/second. (This assumption can help reinforce student understanding of the difference between speed and velocity). Of course, this idea will only work with no wind, which is another of our assumptions.

Clearly, we made a large number of assumptions that are not correct. However, we can do that if the behavior predicted is close to the actual behavior. We must return to the original purpose of this unit. We are trying to get a conceptual understanding of behavior through modeling. We want to be able to use our model to predict behavior. We are not trying to do advanced scientific research.

FOLLOW THE BOUNCING BALL…

I have seen physics texts that state that the results of two and three-dimensional elastic collisions cannot be found. This would mean that if an elastic collision occurs, no one could predict what would happen. If the result of the collision were indeterminate, then the result would be a random event. This clearly is not real. What the author probably meant to say is that the results of such collisions are too difficult for students to find.

I disagree with either argument. The results of two and three-dimensional elastic collisions can be found. They follow a basic premise. The objects exchange momentum in the direction of the perpendicular to the tangent plane at the point of contact. There is no momentum change parallel to the tangent plane. The diagram below illustrates the concept.



We will use m_1 and m_2 to represent the two masses, v_{1I} and v_{2I} their respective initial velocities before collision, and v_{1f} and v_{2f} to represent their velocities after collision. Then along the perpendicular, the final velocities will be given by the following equations:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{m_1 - m_2}{m_1 + m_2} v_{2i} + \frac{2m_2}{m_1 + m_2} v_{1i}$$

If the two objects have the same mass, the equations become:

$$v_{1f} = v_{2i}$$

$$v_{2f} = v_{1i}$$

If one mass is very much larger than the other mass ($m_1 >> m_2$), then the equations become

$$v_{1f} \approx \frac{m_1 - m_2}{m_1} v_{1i}$$

$$v_{2f} \approx 2v_{1i}$$

We will use this last set of equations to create the models of the raindrop, the baseball drop, the baseball flight, and the rocket launch. We will slightly modify the equation for v_{if} to find the change in velocity:

$$\Delta v = v_{1f} - v_{1i}$$

$$\Delta v \approx \frac{m_1 - m_2}{m_1} v_{1i} - v_{1i}$$

$$\Delta v \approx \frac{m_1 - m_2}{m_1} v_{1i} - \frac{m_1}{m_1} v_{1i}$$

$$\Delta v \approx \frac{m_1 v_{1i} - m_2 v_{1i} - m_1 v_{1i}}{m_1}$$

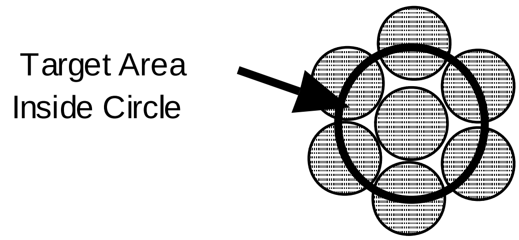
$$\Delta v \approx -\frac{m_2}{m_1} v_{1i}$$

THE GIANT BILLIARDS GAME…

In our first model, we will predict how far a molecule travels before colliding with another molecule. We need some data first. We will assume that we are at standard temperature and pressure. At standard temperature and pressure, there are 6.023×10^{24} molecules in 22.4 liters of gas. There are 1,000 liters in a cubic meter. We can divide $6.023 \times 10^{23}$by 22.4 and then multiply by 1,000 to find that there are 2.69×10^{25} molecules in a cubic meter.

The average space between molecules is found by taking the reciprocal of the cube root of the number of molecules. This turns out to be 3.34×10^{-9} meters.

We also need to find the target area for a collision between two molecules. The area would be the area as shown below:



This area has a radius of 2R ($R=1.4x10^{-10}$). Therefor, the target area is πR^2 or $9.85x10^{-19}$ square meters.

We will assume that one molecule is moving and all of the others are stationary. We will also assume that the other molecules are evenly distributed. Since the moving molecule is just as likely to move in any direction, we will use a sphere to represent its paths. This suggests a spherical distribution of the other molecules. We will do this by creating spherical shells of the other molecules.

Each sphere will be $3.34x10^{-9}$ meters larger in radius than the sphere inside. This number represents the average distance between molecules. We can find the volume of each sphere and the number of molecules in it. By subtracting the molecules in the sphere just inside it, we can find the number of molecules in the shell between the spheres. By multiplying the number of molecules in each shell by the target area for one molecule, we can find the target area of the shell. The probability of the moving molecule hitting a molecule in the shell is the target area divided by the area of the shell ($4\pi R^2$).

Finally we really need to find the probability of not hitting each shell to find the probability of hitting at any level. We find the probability of hitting by subtracting the probability of not hitting from 1. The tables below show the results of calculations made using a spreadsheet.

Dist. $X10^{-9}$	Volume $X10^{-25}$	#Mole.	#/Layer	Area $X10^{-16}$	Target area $X10^{-18}$	Phit Layer	Pmiss Layer	Pmiss Total	Phit Total
3.34	1.56	4.19	4.19	1.40	4.13	0.029	0.971	0.971	0.030
6.68	12.50	33.51	29.32	5.60	28.9	0.052	0.948	0.920	0.080
10.00	42.10	113.1	79.59	12.60	78.4	0.062	0.938	0.863	0.137
13.40	99.70	268.1	155.0	2.2.4	153	0.068	0.932	0.804	0.196
16.70	195.0	523.6	255.5	35.0	251	0.072	0.928	0.747	0.254
20.0	336.0	904.8	381.2	50.4	376	0.075	0.925	0.691	0.309
23.40	534.0	1,437	532.0	68.6	524	0.076	0.924	0.638	0.362
26.70	798.0	2,145	707.9	896	697	0.078	0.922	0.588	0.412
30.00	1140	3,054	909.0	1130	895	0.079	0.921	0.542	0.458
33.40	1560	4,189	1,136	1400	1118	0.080	0.920	0.499	0.501

Title	What it means
Distance	The distance from the center of the sphere (radius)
Volume	Volume of the sphere (Volume=$4\pi R^3/3$)
#Mole	The number of molecules in the sphere. It was found by multiplying the volume by the number of molecules per cubic meter.
#/Layer	The number of molecules between the spheres. It was found by subtracting away the number of molecules in the previous sphere.
Area	The area of each sphere (Area =$4\pi R^2$)
Target Area	The target area of the molecules. This was found by multiplying the number of molecules in a layer by the target area per molecule ($9.85x10^{-19}$)
Phit Layer	This is the probability of a hit in a layer. It was found by dividing the target area by the area.
Pmiss Layer	This is the probability of a miss in each layer. It was found by using 1-Phit. This value was used to find the cumulative probability of a miss.
Pmiss Total	This is a cumulative probability of a miss. It was found by multiplying the previous total Pmiss by the new layer Pmiss.
Phit Total	This is just 1-Pmiss Total. This method is the only way to determine the probability of one event for a series of probabilities. This way prevents probabilities from being counted twice.

The data table shows that there is approximately a 50% chance of a collision when the molecule travels $33.4x10^{-9}$ meters. Since the average molecule travels at 468 m/s, the molecule will travel for approximately $7.1x10^{-11}$seconds. If we take the reciprocal of the time we find that the molecule makes approximately $1.4x10^{10}$ collisions per second.

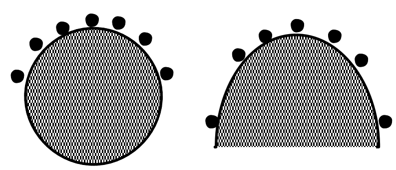
What direction will the molecule take after collision? It is impossible to tell unless we know precisely the location of each molecule and the exact paths. Slight variations will make great difference in what will happen next. The large number of collision leads to chaotic results. The large number of collisions also leads to a stable result at the end. This is a simple case of Chaos Theory.

LIKE A TRUCK HITTING A PING PONG BALL…

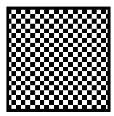
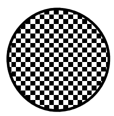
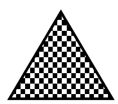
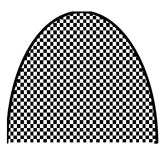
The mass of an air molecule is almost infinitesimal when compared to a raindrop, a baseball or a rocket. Therefore, the equation for the change in velocity becomes:

$$\Delta v = \frac{m_2}{m_1} v_{1i}$$

Remember that the change in velocity is along the perpendicular to the tangent between the object and molecule. On a curved surface this change varies. We can find an average of the changes along the surface of the object. We will call these a Drag Profile. A drawing if what this looks like is shown below:



The Drag Profile, D_P, is the average of the cosine of the slope angle. As the slope angle increases from 0^0 to 90^0, the change in velocity will decrease from 100% of the maximum to 0%. The average of the slopes can be found for objects with axial symmetry (circles when looked at from the top) by finding the two-dimensional average over half of the shape and using that as the average. The averages were found using a spreadsheet. The shapes used all have a 1:1 aspect ration (They are as wide as they are high.) The table below shows some Drag Profiles:

Object	Drag Profile
 Cube	1.0000
 Sphere	0.7823
 Cone	0.7071
 Paraboloid	0.7218

We are going to use a Time Step, ΔT, in our calculations. The Volume of the molecules is the product of the Area of the Object, A_O the Velocity of the Object, V, and the Time Step, ΔT. The mass of the molecules is the product of the Volume, the number of molecules per cubic meter, N, and the mass per molecule, m_m. Finally, we find multiply by the Drag Profile, D_P. We find the change in velocity as:

$$\Delta V_D = D_P \frac{A_O N m_m V \Delta T}{m_O} V$$

$$\Delta V_D = D_P \frac{A_O N m_m \Delta T}{m_O} V^2$$

The mass of the object is m_O. The equation looks complicated. Almost all of the numbers, however, are a constant. For each of our models, we will calculate all of the constants and simplify the equation. We are now ready to design our models.

RAINDROPS KEEP FALLING ON MY HEAD…

Our first model will be the raindrop model. Raindrops accelerate as they fall due to gravity. The gravitational effect is counteracted by the air drag. When the air drag effect balances the gravitational effect, the raindrop reaches terminal velocity. In our model we are not considering evaporation of the drop.

Our first step is to simplify the ΔV equation. Since we are assuming that the molecule is spherical, we can simplify the A_O divided by m_O. The mass of the raindrop is its density times its volume. The density of water is 1000 kilograms per cubic meter. We also know that the Drag Profile is 0.7823 Using the volume of the sphere and the area of the sphere, we get the following derivation:

$$\Delta V_D = D_P \frac{A_O N m_m \Delta T}{m_O} V^2$$

$$\Delta V_D = D_P \frac{A_O N m_m \Delta T}{\rho V_O} V^2$$

$$\Delta V_D = 0.7823 \frac{\pi R^2 N m_m \Delta T}{(1000)\frac{4}{3}\pi R^3} V^2$$

$$\Delta V_D = \frac{0.5867 N m_m \Delta T}{1000 R} V^2$$

We also know that there are $2.69\text{x}10^{25}$ molecules in a cubic meter (N) and the mass of a molecule is $4.79\text{x}10^{-26}$ kilograms. Therefor, the equation can be simplified to:

$$\Delta V_D = \frac{0.5867\left(2.69x10^{25}\right)\left(4.79x10^{-26}\right)\Delta T}{1000R} V^2$$

$$\Delta V_D = \frac{0.7568 \Delta T}{1000 R} V^2$$

Terminal velocity is reached when the change in velocity due to gravity is counteracted by the change in velocity due to air drag. The equation for terminal velocity is derived below:

$$
\begin{aligned}
\Delta V_D &= \Delta V_g \\
\frac{0.7568\Delta T}{1000R} V_T^2 &= g\Delta T \\
V_T^2 &= \frac{1000Rg\Delta T}{0.7568\Delta T} \\
V_T^2 &= \frac{1000R(9.81)\Delta T}{0.7568\Delta T} \\
V_T^2 &= 12,962.4R
\end{aligned}
$$

$$V_T = 113.9\sqrt{R}$$

Therefor, the larger the raindrop, the larger the terminal velocity. Let's assume that a raindrop starts at 1000 meters. From physics we can find the velocity if there were no air drag:

$$
\begin{aligned}
V &= \sqrt{2gh} \\
V &= \sqrt{19.62(1000)} \\
V &= 140m/s
\end{aligned}
$$

Converting 140 m/s to miles per hour, the raindrop would hit the ground at 313 miles per hour. Ouch! The table below indicates the terminal velocity of a raindrop for different radii (A 10 mm radius is a large marble size). The two measured values were provided by Carl Morgan, Forecaster, Meteorology, National Weather Service (www.madsci.org/posts/archives/jul2000/96226446.ph.r.html)

Radius of Raindrop	V (m/s)	V(mph)	V (measured) m/s
1 mm (.001 m)	3.60	8.38	6.49
2 mm (.002 m)	5.09	11.84	8.83
5 mm (.005)	8.05	18.73	
10 mm (.01 m)	11.39	26.50	
20 mm (.02 m)	16.10	37.46	
50 mm (.05 m)	25.47	59.27	
100 mm (.1 m)	36.02	83.82	
36.7 mm (.0367 m) Baseball Size	21.79	50.71	

How far will the raindrop fall before it reaches terminal velocity? The answer is forever! A better question is how far will the raindrop fall before it reaches 99% of terminal velocity? This problem can be solved using an iterative loop. As an example, we will start at a height of 50 meters and let a 2mm raindrop fall. We will find how far it travels before reaching 99% of 5.09 m/s. The velocity we are looking for is 5.04 m/s. We will us a time step of 0.1 seconds. Using the radius of .002 in the equation for the change in velocity would be:

$$\Delta V_D = .0378V^2$$

The iterative loop is:

Initial Conditions
  Height: H = 1000 m
  Distance Traveled =D= 0 m
  Initial Velocity: V = 0 m/s
  Time Step: ΔT = .1 s
  g= 9.81m/s^2
  ΔV_g =.98
Loop until velocity is 5.01
  $V_{OLD} = V$
  $H_{OLD} = H$
  $\Delta V_D = .0378V^2$
  $V = V_{OLD} + \Delta V_D - \Delta V_g$
  $H = H_{OLD} + .5(V + V_{OLD})\Delta T$
  $D = H_{OLD} - H$

Time	H	V	ΔVg	ΔVd	D
0	1000.000	0.000	0.981	0.000	0.000
0.1	999.510	0.981	0.981	0.036	0.490
0.2	998.056	1.926	0.981	0.140	1.944
0.3	995.710	2.766	0.981	0.289	4.290
0.4	992.598	3.458	0.981	0.452	7.402
0.5	988.875	3.987	0.981	0.601	11.125
0.6	984.698	4.367	0.981	0.721	15.302
0.7	980.201	4.627	0.981	0.809	19.799
0.8	975.488	4.799	0.981	0.871	24.512
0.9	970.634	4.909	0.981	0.911	29.366
1	965.689	4.979	0.981	0.937	34.311
1.1	960.688	5.023	0.981	0.954	39.312
1.2	955.651	5.050	0.981	0.964	44.349

The raindrop falls for about 42.5 meters before reaching 99% of terminal velocity.

IT'S A HIGH FLY BALL…

Imagine if a baseball dropped from 1000 meters. With no air drag it would hit the ground at 313 mph. In fact any object would hit the ground at 313 mph. What differences should there be between raindrops and a baseball? How about any sphere? We could slightly revise the ΔV_D and the terminal velocity equation so that it will work for any sphere. To do that we only need the mass density, ρ, of the sphere. The equations are:

$$\Delta V_D = \frac{0.5867\left(2.69x10^{25}\right)\left(4.79x10^{-26}\right)\Delta T}{\rho R}V^2$$

$$\Delta V_D = \frac{0.7568\Delta T}{\rho R}V^2$$

$$\Delta V_D = \Delta V_g$$

$$\frac{0.7568\Delta T}{\rho R}V_T^{\ 2} = g\Delta T$$

$$V_T^{\ 2} = \frac{9.81\rho R}{.7568}$$

$$V_T = \sqrt{12.96\rho R}$$

$$V_T = 3.6\sqrt{\rho R}$$

A baseball has a radius of .023 meters. Therefor its volume is $5.097x10^{-5}$ cubic meters. It has a mass of 0.145 kilograms. Its density is 2,844 kilograms per cubic meter. We can use the equation for terminal velocity of a sphere to find that the terminal velocity is 29.1 m/s.

IT'S A LONG DRIVE…

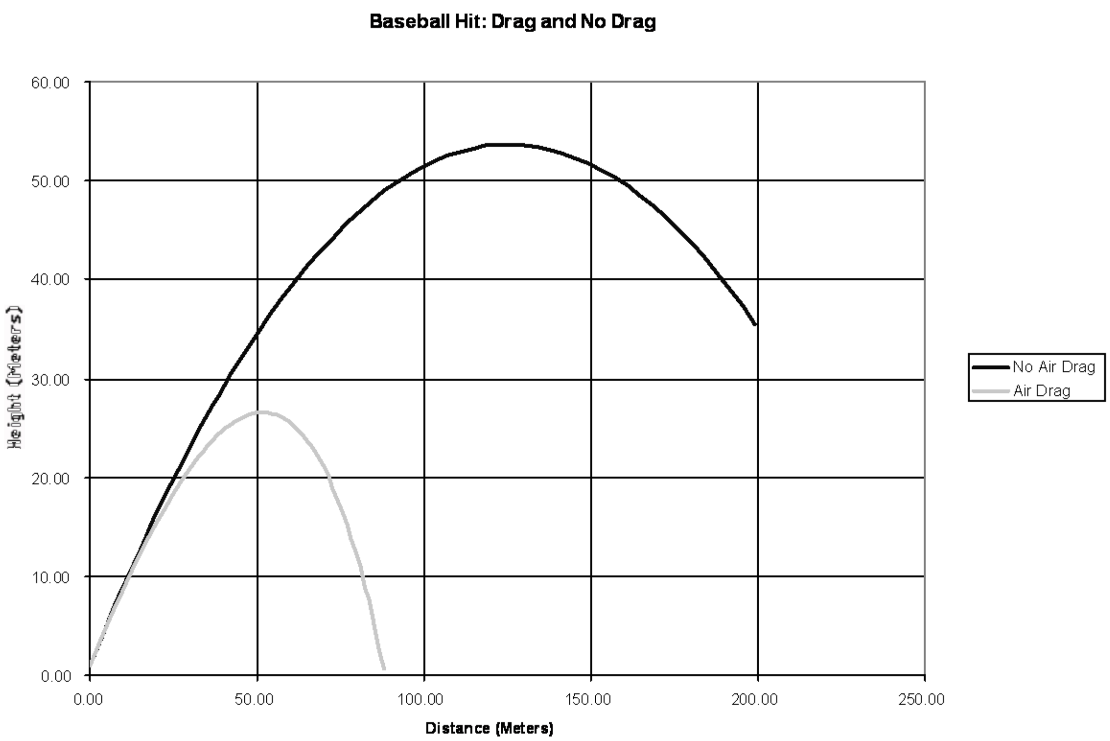
Hitting a baseball at an angle poses a new problem. Air drag occurs along the path of the ball. The equations, however, are in two directions. This means that we have to convert back and forth between the path of the ball and its velocity in two dimensions. The equation that we will use is:

$$\Delta V_D = \frac{0.7568 \Delta T}{\rho R} V^2$$

$$\Delta V_D = \frac{0.7568 \Delta T}{2844(.023)} V^2$$

$$\Delta V_D = .01157 V^2 \Delta T$$

The dramatic effect of the air drag on a baseball can be seen by the graph below. The ball was assumed to be hit at 50 m/s or 111.8 miles per hour. The angle is 40^O. It was hit at a height of 1 meter (3.28 feet). With air drag it traveled 88 meters (288 Feet). Without air drag it would have traveled 249 meters (817 feet).



The graph was expanded to show only the complete flight of the air drag case. The no air drag case is symmetric so that the total travel can by doubling the distance to its highest point. Note that the air drag path is not symmetric. Its maximum point occurred at 54 meters. On the way down it only traveled 34 meters. With air drag, the path is not a parabola.

Below is the case for the air drag model. The no air drag was computed with the simple equations from physics:

$$x = x_o + v\cos(\theta)t$$

$$y = y_o + v\sin(\theta)t - \frac{1}{2}gt^2$$

T	X	Y	Vx	Vy	V	θ	ΔVd
0	0.00	1.00	38.30	32.14	50.00	40.00	5.79
0.2	7.22	6.40	33.87	26.46	42.98	38.00	4.27
0.4	13.65	10.85	30.50	21.87	37.53	35.63	3.26
0.6	19.49	14.50	27.85	18.00	33.17	32.88	2.55
0.8	24.85	17.47	25.72	14.66	29.60	29.69	2.03
1	29.81	19.84	23.95	11.69	26.66	26.02	1.64
1.2	34.46	21.66	22.48	9.01	24.22	21.85	1.36
1.4	38.83	22.99	21.22	6.54	22.20	17.14	1.14
1.6	42.96	23.85	20.13	4.25	20.57	11.91	0.98
1.8	46.89	24.28	19.17	2.08	19.28	6.20	0.86
2	50.64	24.29	18.31	0.03	18.31	0.08	0.78
2.2	54.22	23.91	17.54	-1.94	17.64	-6.30	0.72
2.4	57.66	23.15	16.82	-3.82	17.25	-12.79	0.69
2.6	60.96	22.04	16.15	-5.63	17.10	-19.21	0.68
2.8	64.12	20.57	15.51	-7.37	17.17	-25.41	0.68
3	67.16	18.77	14.89	-9.04	17.42	-31.25	0.70
3.2	70.08	16.65	14.29	-10.63	17.82	-36.65	0.73
3.4	72.88	14.23	13.70	-12.16	18.32	-41.58	0.78
3.6	75.57	11.51	13.12	-13.60	18.90	-46.03	0.83
3.8	78.13	8.53	12.55	-14.97	19.54	-50.03	0.88
4	80.59	5.28	11.98	-16.26	20.20	-53.61	0.94
4.2	82.93	1.80	11.42	-17.46	20.86	-56.81	1.01
4.4	85.16	-1.91	10.87	-18.58	21.53	-59.67	1.07

The results in the table above are slightly different than the results of the graph. That is because a time step of .2 seconds was used in the table to shorten it. The graph was made with a time step of .1 seconds. The smaller the time step, the more accurate the results. The compromise is between accuracy and time of calculations. With the advent of high-speed personal computers, the accuracy can be greatly enhanced.

Below is the set of iterative equations used:

Initial Conditions

$X = X_O$	Initial Distance from some Given Point (usually 0)
$Y = Y_O$	Initial Height
$V = V_O$	Initial Speed
$\theta = \theta_O$	Initial Angle above the Horizon
ΔT	Time Step (Between Calculations)

Loop

$$\Delta V_D = .01157 V^2 \Delta T$$

$$V_{NEW} = V_{OLD} - \Delta V_D$$

$$V_{XNEW} = V_{NEW}\cos(\theta)$$

$$V_{YNEW} = (V_{YNEW} - g\Delta T)\sin(\theta)$$

$$X_{NEW} = X_{OLD} + \left(\frac{V_{XOLD} + V_{XNEW}}{2}\right)\Delta T$$

$$Y_{NEW} = Y_{OLD} + \left(\frac{V_{YOLD} + V_{YNEW}}{2}\right)\Delta T$$

$$\theta = Tan^{-1}\left(\frac{V_{YNEW}}{V_{XNEW}}\right)$$

$$V_{OLD} = V_{NEW}$$

$$V_{XOLD} = V_{XNEW}$$

$$X_{OLD} = X_{NEW}$$

$$Y_{OLD} = Y_{NEW}$$

Repeat Loop until X< 0 (Ball Hit Ground)

10,9,8,7,6,5,4,3,2,1, LIFT OFF…

The model for a rocket introduces four new concepts. The first of these is that the flight is powered. Also, both gravity and the mass of the atmosphere both vary with altitude. The final concept is that the shape of the object is not spherical. These new concepts along with the corresponding equations are explained below.

Rocket power is based on the Conservation of Momentum. The momentums created by gases that leave the back of the rocket cause a corresponding change in momentum of the rocket. In the equation below, Δm_F represents the amount of fuel leaving a rocket in the time period ΔT; V_F represents the speed that the gases leave the rocket; m_T represents the total mass of the rocket and the fuel that has not yet been burned; and ΔV_R represents the increase in speed of the rocket. The equation for the increase in speed becomes:

$$m_T \Delta V_R = \Delta m_F V_F$$

$$\Delta V_R = \frac{\Delta m_F V_F}{m_T}$$

Both the amount of fuel burned per time period and the velocity of the fuel are constant. The total mass of the rocket, however, is decreasing as the fuels burns. Therefor, the change in velocity is constantly increasing during the flight.

The mass of the molecules per cubic meter also varies. As the altitude of the rocket increases, the density decreases. This will cause a decrease in the air drag as the rocket gains altitude. If we measure the altitude in terms of z meters, we can use an approximation to represent the number of molecules. The approximation that we will use is the Isothermic Model. It assumes that the temperature remains constant as the altitude increases. This is not true. Our model, however, is a simplification of reality. Our simplification will make the problem more understandable. The equation that we will use assumes that the atmosphere is a constant 15^{O} Celsius. The equation becomes:

$$N = N_O e^{-\frac{z}{8420}}$$

N_O is the number of molecules per cubic meter at ground level. This number (2.69×10^{25}) when multiplied by the mass of an average molecule (4.79×10^{-26} kg) gives the mass of a cubic meter of molecules at ground level (1.29 kilograms). Thus the mass of a cubic meter of molecules will decrease from 1.29 kilograms as the altitude increases. The chart below shows how dramatic a decrease this can be:

Altitude (meters)	Altitude (feet)	Mass per Cubic Meter Kilograms
0	0	1.29
5,000	16,400	0.712
10,000	32,800	0.394
20,000	65,600	0.120
50,000	164,000	0.003
100,000	328,000	0.000009
200,000	656,000	0.00000000006
500,000 (Shuttle Orbit)	1,640,000	2.1×10^{-26}

Note that even in orbit there is a slight mass of atmosphere that creates some air drag. We need to use a new equation for the air drag. This equation looks complicated but it is easy to plug into a spreadsheet for calculations. Thee new air drag equation becomes:

$$\Delta V_D = D_P \frac{A_O N m_m \Delta T}{m_T} V^2$$

$$\Delta V_D = D_P \frac{A_O 1.29 e^{\frac{z}{8420}} \Delta T}{m_T} V^2$$

Gravity is not a constant it varies with the distance from the center of the Earth. The radius of the Earth is $6.375x10^6$ meters. Therefore the distance form the center of the Earth is R = (is $6.375x10^6$ +Z). The equation involves a Universal Gravitational Constant (6.67×10^{11}) and the mass of the Earth (5.98×10^{24}). Using these numbers for the Earth, equation for g becomes:

$$g = \frac{GM}{R^2}$$

$$g = \frac{(6.67x10^{-11})(5.98x10^{24})}{(6.375x10^6 + Z)^2}$$

$$g = \frac{3.99x10^{14}}{(6.375x10^6 + Z)^2}$$

The table below shows how g varies with altitude:

Altitude (meters)	Altitude (feet)	g (m/s^2)
0	0	9.81
5,000	16,400	9.80
10,000	32,800	9.79
20,000	65,600	9.76
50,000	164,000	9.67
100,000	328,000	9.52
200,000	656,000	9.23
500,000 (Shuttle Orbit)	1,640,000	8.44

Note that g exists at close to 9.81 even in shuttle orbit. There is a common misconception that there is zero gravity in orbit. An object would not be in orbit without gravity.

There are now three changes in velocity, ΔV_R, ΔV_D, and ΔV_g. The total change in velocity is $\Delta V_R - \Delta V_D - \Delta V_g$. The iterative equation becomes long. Rather than list them, I would rather list some interesting results.

We will begin our example with a rocket that weighs 10,000 kilograms; has 90,000 kilograms of fuel to start and has the exhaust velocity of the fuel be 3,000 m/s. We will make the rocket have a parabolic nose cone and be 5 meters in radius.

The table below illustrates results for without gravity, with gravity and no air drag, and with gravity and air drag. We will also vary the rate at which the fuel burns in terms of how long it takes to burn all of the fuel.

Exhaust Velocity = 3,000 m/s

Time of Burn	No Gravity No Air Drag	Gravity No Air Drag	Gravity Air Drag
200	V_F =34,728 m/s Z_F = 2,230,372 m	V_F = 32,790 m/s Z_F = 2,047,594 m	V_F = 29,049 m/s Z_F = 1,049,044 m
400	V_F = 17,239 m/s Z_F = 2,230,372 m	V_F = 13, 698 m/s Z_F = 1,480,208 m	V_F = 12,158 m/s Z_F = 1,043,881 m
600	V_F = 11,493 m/s Z_F = 2,230,372 m	CAN'T LIFT OFF	CAN'T LIFT OFF
524 Maximum Time	V_F =13,160 m/s Z_F = 2,230,372 m	V_F = 8,728 Z_F = 908, 045	V_F = 7,051 Z_F = 576,099

Exhaust Velocity = 2,000 m/s

Time of Burn	No Gravity No Air Drag	Gravity No Air Drag	Gravity Air Drag
200	V_F = 22,985 m/s Z_F = 1,486, 914 m	V_F = 21,209 m/s Z_F = 1,299,495 m	V_F = 18,363 m/s Z_F = 866,681 m
400	V_F = 11,493 m/s Z_F = 1,486,914 m	V_F = 7,745 m/s Z_F = 716,029	V_F = 6,404 m/s Z_F = 424,195 m
428 Maximum Time	V_F = 10,741 m/s Z_F = 1,486,914	V_F = 6,686 m/s Z_F = 599,278 m	V_F = 5,267 m/s Z_F = 318,549 m

Exhaust Velocity = 1,000 m/s

Time of Burn	No Gravity No Air Drag	Gravity No Air Drag	Gravity Air Drag
200	V_F = 11,493 m/s Z_F = 743,457 m	V_F = 9,612 m/s Z_F = 550,887 m	V_F = 7,342 m/s Z_F = 275,345
302 Maximum Time	V_F = 7,611 m/s Z_F = 743,457 m	V_F = 4,701 m/s Z_F = 298,794 m	V_F = 2,531 m/s Z_F = 93,056 m

Exhaust Velocity = 500 m/s

Time of Burn	No Gravity No Air Drag	Gravity No Air Drag	Gravity Air Drag
200	V_F = 5,746 m/s Z_F = 371,299 m	V_F = 3,807 m/s Z_F = 176,357 m	V_F = 861 m/s Z_F = 37,298 m
211 Maximum time	V_F = 5,447 m/s Z_F = 371,299 m	V_F = 3,396 m/s Z_F = 154,206 m	V_F = 582 m/s Z_F = 29,876 m

The maximum time in the tables above indicates that for that exhaust velocity, any slower burn will not allow the rocket to lift off with gravity and air drag. This time of burn decreased with decreasing exhaust velocity.

The effect of air drag increased dramatically as the exhaust velocity decreased. At burn times of 200 seconds, the percentage decrease in speed with air drag vs. no air drag went from 11.4% at an exhaust velocity of 3,000 m/s to 77.3% at an exhaust velocity of 500 m/s.

The distance traveled remained constant for the no gravity and no drag case at each exhaust velocity. This means that when gravity and air drag are not factors, the rocket travels the same distance during burn. The final velocity, however, decreases with increasing burn time.

From the results of these tables, it can be seen that the exhaust velocity and the time of burn can have a major effect on both the velocity and the final height that a rocket reaches. The major limitation in the real world is the structural stress put on a rocket by high acceleration forces (g forces).

LESSON PLANS

The lesson plans are listed as exercises by topic. They are suggested exercises that depend on the amount of equipment available.

Atmospheric Modeling

Several neat experiments can be performed with an air table. Start with a large number of pennies. Place them in a matrix with small spaces between them. Make all of the pennies be heads up. Slide a penny that is tails up into them. See what happens to the original penny and the matrix of pennies. Try the experiment several times, starting with the same matrix configuration. The results should vary considerably. This is the basis of Chaos Theory.

Now try the same experiment with a fifty-cent piece being the object originally in motion. The fifty-cent piece could represent a raindrop going through the air. Again try the experiment several times. What happens to the fifty-cent piece? What happens to the pennies?

Repeat this last experiment with a large puck. How are the results different?

Terminal Velocity

Get several ping pong balls Inject each ball with a varying amount of liquid. Number each ball and record its mass. Try dropping them from as a high a point as possible.

Record the various times of flights. Graph the mass vs. time of flight. Also graph the average velocity vs. mass. Use the equation below to determine what the time of flight would be without air friction:

$$t = \sqrt{2gh}$$

The average velocity is:

$$V_{AVERAGE} = \frac{h}{t}$$

Flight Path

The flight path of a particle that is affected by air drag can be demonstrated using a water hose and a large wooden protractor (Every geometry teacher should have one.) The water hose should be adjusted to emit a thin stream of water. The water hose is set at various angles on the ground. The protractor measures the angles. The distance that the water travels, the maximum height that it travels, and the distance it travels to maximum height is measured. Graph the results at angles from 10 to 80 degrees. The shapes of the paths can demonstrate the effects of air drag.

Rocket Flight

This experiment should be performed by the use of model rockets. The rockets should be fired with various cones of different shapes and sizes. The height of flight can be measured with simple devices available at stores who sell rockets. The height and time to the highest altitude should be measured. This will give a qualitative look at air drag on rockets.

ANNOTATED BIBLIOGRAPHY

The units were created using basic concepts to create simplified models. Therefore, no works were cited. The following sources were used to understand basic concepts; they also provide the capability of enriching parts of the unit. The sources have been divided into topics to simplify their use:

Modeling

Ford, Andrew. Modeling the Environment: An Introduction to Systems Dynamics Modeling of Environmental Systems. Washington, D.C.: Island Press, 1999

> This book studies modeling at a systems level. It is an excellent source to understand the broader principles of modeling. It is written at a level that could be followed by students. It is general enough to cover almost any aspect of modeling.

Richmond, Peterson, et al. An Introduction to Systems Thinking Hanover, New Hampshire: High Performance Systems, Inc., 1997

> This manual comes with Stella™ Software. The software has the capability of modeling complex systems by a graphical approach. It allows students to understand the concept of differential equations without needing the ability to solve them. This manual provides the rationale for placing problem solving at the forefront of education.

Molecular Motion

"Atomic Size", http://mychemistrypage.future.easyspace.com/Inorganic/periodicity/size.htm

> This web-site and its links provide the size of molecules and their mass. The site could be used for a more complex but accurate model of the atmosphere.

"Kinetic Theory of Gases: A Brief Review" http//www.phys.virginia.edu/classes /252/kinetic_theory.html

> This is an excellent source of the equation of motion of gases. The web-site article derives the probability distribution of velocities of molecules.

Nambu, John, Translator. Transnational College of NEX What is Quantum Mechanics? A Physics Overview. Boston, Ma.: Language Research Foundation, 1997

> This is a quite interesting book. Students at the Transnational College of Japan originally published it in Japanese. The students treat science as a language. The book looks at a very complex concept and makes it into a cartoon story. The beginning gives a very good insight in quantum mechanics. The mechanisms that cause changes in the atmosphere can only be understood from a quantum mechanics perspective.

Wolf, Fred Alan, Taking the Quantum Leap, New York: Harper & Row, 1989

The subtitle for this book is "The New Physics for Non-Scientists". It is an apt title. The book turns quantum mechanics into a history lesson. This would be my first choice for students (or teachers) to start into the subject.

Gleick, James, Chaos: Making a New Science, Toronto, Canada: Penguin Books, 1987.

As I was carrying this book with me to read, I ran across a number of people in all fields who have read it. It is similar to Wolf's book above in that it traces the history of Chaos Theory to cover the key concepts. This is an excellent starting book on Chaos Theory. This topic is another essential element in understanding the atmosphere.

Williams, Jack, The Weather Book. New York: Vintage Books, 1997

This book has the look and feel of *USA Today,* which created the illustrations. It offers a very easy to understand explanation of the atmosphere from a meteorologist's viewpoint.

Terminal Velocity

"Raindrops" http://www.shortsmeyer.com/wxfaqs/float/rdtable.html

This web-site and links give terminal velocities for various size raindrops and hailstones. There is also an explanation of why the size of raindrops is limited. There is even an explanation of the shape of a water molecule and how it operates in clouds.

Projectile Motion

"The Drag Equation". http://www.lerc.nasa.gov/WWW/k-12/airplane/drageq.html

This web-site supplies a simple equation for air drag in airplanes.

"Using Spreadsheets for Projectile Motion".
http://www.phys.virginia.edu/classes/581/ ProjectilesExcel.html

This web-site provides a step b y step approach to creating a spreadsheet model for projectile motions with air drag. It is a good way to learn to use spreadsheets in modeling.

Rocket Equations

"Rocket Equations". http://exepc.com/~culp/rockets/rckt_txt.htm/#Method

This web-site and its links provide all of the information that you need to model the flight of small model rockets. Its links provide all types of other information. This is an excellent web-site.

Lee, Wayne. To Rise from Earth. New York: Checkmark Books, 2000

This is an easy-to understand book that covers numerous topics on rockets and space flight. It should be the first book for students who want to do research in this area.

Thomson, William Tyrell. Introduction to Space Dynamics. New York: Dover Publications, 1986

This is a high level book for students who are serious about space physics. The student should be taking calculus before taking on this book.

Bate, Mueller, and White. Fundamentals of Astrodynamics. New York: Dover Publications, 1971

This book was written as a text for the United States Air Force Academy. It is a fairly high level text, but it has less calculus than the text above.