Let's Patternize It

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Introductory Discussion

I teach Algebra at Austin High School in the Houston Independent School District. I have taught at Austin High School for two years and this is also where I started my teaching career. The school population is about 2400 with about 99% Hispanic and about 1% African American. Most of my students are first time ninth graders just completing 8th grade.

The Algebra I teach is an extended Algebra class. That means that they are taking a two- year Algebra course in a period of one year because we are on an accelerated block schedule. Some of my students are eager to learn whereas some of them have no desire to be there and is just letting time pass them by. I like to approach my students in an interesting way by letting them apply Algebra to real world situations. This seems to switch some of the students over to the eager category from the just letting time pass by category.

I chose patterns as my topic because mathematics has been described as the study of patterns. We see patterns everywhere: in wallpaper, tiles, traffic, and even television schedules. Police investigators study case files to find the modus operandi, or pattern of operation, when a series of crimes is committed. Their discovery of it, sometimes aided by a computer, does not necessarily find the criminal, but it may provide clues to do so. Scientists look for patterns in order to isolate variables so that they can reach valid conclusions in their research.

This curriculum will consist of two units with the first being an introduction to patterns and the second being an introduction to patterns, symmetry and design.
The students will be introduced to new vocabulary words and will have a chance to experiment with activities that will be appropriate for their grade level. The students will learn some history pertaining to the mathematical terms learned throughout the unit; how to model situations using oral, written, concrete, pictorial, and algebraic methods; how to reflect on and clarify their own thinking about mathematical ideas and situations; how to develop common understandings of mathematical ideas, including the role of definitions; how to use the skills of reading, listening, and viewing to interpret and evaluate mathematical ideas; how to discuss mathematical ideas and make conjectures and convincing arguments; and how to appreciate the value of mathematical notation and its role in the development of mathematical ideas.

This unit is meant to last a period of two weeks in a class that meets daily for ninety minutes. These units are geared towards an Algebra class. At the end of the unit the students will do a detailed report on the history noted in the unit.

UNIT 1

To first introduce my students to patterns I will do two exercises with them. With both exercises they will have to figure out the patterns but yet still have fun while participating in the exercises.

Exercise 1

I will place students two per group and let them make a deck of cards numbered 1 through 10. I will then pose the question, "In what order should the cards be arranged in the deck so that when they are placed on the table according to the following procedure, the pile or cards on the table is in the correct numerical order 1 through 10?"

Procedure

Place the top card on the table.
Place the second card on the bottom of the deck.
Place the third card on the table.
Place the fourth card on the bottom of the deck, and so on, until all cards are on the table.

One person should deal the cards and another should record the results of the various investigations.

Investigation

The students should be able to realize that every other card up to the number 5 should be in sequence.

Example: cards	1	____	2	____	3	____	4	____	5	____
	T	D	T	D	T	D	T	D	T	D

(T means placed on table and D means placed underneath the deck)

In conclusion the students should come up with a pattern of 1,6,2,10,3,7,4,9,5,8.

Exercise 2

In a class of 30 the students are asked to shake each students hand one time. After shaking each student's hand I will pose the question, "How many handshakes just took place in this room with each of the thirty students shaking each others hand only once.

Procedure

The students will actually have to go around and shake every ones hand. This could get very confusing; therefore I will have only 10 students to demonstrate before we could go on to the bigger picture of 30 students shaking hands. I will encourage the students to devise some kind of system or chart to keep up with the data that they are collecting.

The students should realize that it should take one handshake for two students, 3 handshakes for 3 students, 6 handshakes for 4 students and 10 handshakes for 5 students. The students should be able to come up with the fact that there should be 45 handshakes between the 10 students. We will figure out the pattern to this problem later in the lesson because this is just an introduction to patterns and we are not yet ready to solve the pattern sets yet.

Lecture about Patterns

When investigating number patterns you will learn that a number sequence is a list of numbers having a first number, a second number, a third number, and so on, called the terms of the sequence. To indicate that the terms of a sequence continue past the last term written, we use three dots (an ellipsis).

There are different types of sequences. The different types of sequences that will be approached will be arithmetic, geometric, and Fibonacci sequence. An arithmetic sequence has a common difference between successive terms, while a geometric sequence has a common ratio between successive terms. These two sequences will be discussed first.

We will use successive difference to try to determine the next term in the sequence. (The student must understand that difference means to subtract).
I will write this sequence on the overhead and ask the students to try to figure out the next number in the pattern.

2,6,22,56,114...

Since the next term is not obvious, subtract the first term from the second term, the second from the third, the third from the fourth and so on.

2 6 22 56 114

6 - 2 = 4 22 - 6 = 16 56 - 22 = 34 114 - 56 = 58

Now repeat the process with the sequence 4,16,34,58 and continue repeating until the difference is a constant value. (The student must understand that a constant value means the same number.)

2 6 22 56 114

4 16 34 58

12 18 24

6 6

For this pattern to continue, another 6 should appear in line (4), meaning that the next term in line (3) would have to be 24 + 6 = 30. The next term in line (2) would have to be 58 + 30 = 88. Finally, the next term in the given sequence would be 114 + 88 = 202. The final scheme is shown below.

2		6		22		56		114		202
	4		16		34		58		88	
		12		18		24		30		
			6		6		6			

The constant value is determined to 6. To find the next term we must add 6 to 30 to get 36 then add 36 to 88 to get 124 and then add 124 to 202 to get 326. That will show us that 326 will be the next term in our sequence. I will let students know that this is an example of an arithmetic sequence.

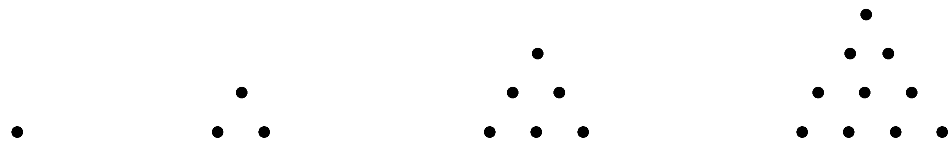
I will now introduce some history to the students and give them more definitions and examples. I will read this story out a loud to them and only write the definitions on the overhead or the chalkboard. Later the students will be expected to research and find out more about historians in mathematics.

Pythagoras lived during the sixth century BC. He and his fellow mathematicians formed the Pythagorean brotherhood devoted to the study of mathematics and music. The Pythagorean investigated the figurate numbers. They investigated connections between mathematics and music as well, and discovered that musical tones are related to the length of stretch strings by ratios of counting numbers. You can test this on a cello. Stop any string midway, so that the ratio of the whole string to the part is 2/1. If you pluck the free half of the string, you get the octave above the fundamental tone of the whole string. The ratio 3/2 gives you the fifth above the octave, and so on. Pythagoras noted that simple ratios of 1,2,3,4 give harmonious musical intervals. he claimed that the intervals between planets must also be ratios of counting numbers. The idea came to be called “music of the spheres.” (The planets are believed to orbit around the earth on crystal spheres.)

Figurate numbers are numbers of geometric arrangements of points, such as triangular numbers, square numbers, and pentagonal numbers. The figurate numbers possess numerous interesting patterns. With every square number greater than 1 is the sum of two consecutive triangular numbers. (For example, 9 = 3 + 6 and 25 = 10 + 15.) Every pentagonal number can be represented as the sum of a square number and a triangular number. (For example, 5 = 4 + 1 and 12 = 9 + 3.)

In the expression T_n, n is called a subscript. T_n is read “T sub n,” and it represents the triangular number in the nth position in the sequence. For example,

$T_1 = 1$ $T_2 = 3$, $T_3 = 6$, $T_4 = 10$.



I then show them the formula:

$$T_n = \frac{n\,(n + 1)}{2},$$

Example:

(a) Find the seventh triangular number

$$T_7 = \frac{7(7 + 1)}{2} = \frac{7(8)}{2} = \frac{56}{2} = 25$$

(b) Find the twelfth triangular number

$$T_{12} = \frac{12(12+1)}{2} = \frac{12(13)}{2} = \frac{156}{2} = 78$$

These problems will be placed on the overhead or the chalkboard and the students will be expected to work the problems out on their own. The will need to be aware of the fact that they will let the subscript number substitute the *n* in the equation.

Now I will go back and approach the problem of the students shaking hands with each student in the classroom. I will make a chart and show my work to find the *n*th term of the sequence.

# of People	# of Handshakes	Difference	Difference
2	1		
3	3	2	
4	6	3	1
5	10	4	1
n	$n(n-1)/2$		

If you will note, we completed the sequence until we found a constant, which ended up being the number 1. Since the constant difference was one that should be a clue to me that I will need to subtract one when trying to find the *n*th term? I also pointed out that the number of people plus the number of handshakes will give me the next number of handshakes. Example: 2 people plus 1 handshake will give me the next number of handshakes, which would be 3. 3 people plus 3 handshakes will give me the next number of handshakes, which would be 6. Therefore if I take the number of people and multiply it times the number of people minus one and divides all of that by two, which will give me the number of handshakes.

I will have the students complete this worksheet of exercises to see if they are grasping the idea of finding patterns by way of successive differences. The worksheet will be given in the form of a handout, which is located on the next page.

Exercise:

Use the method of successive difference to determine the next number in each sequence.

1. 2,8,16,26...

2. 15,24,39,60...

3. 21,34,51,72,97...

4. 11,18,29,44,63...

5. 5,28,87,200,385...

6. 11,34,81,164,295...

Use patterns to complete the table below. Draw a diagram of the first three illustrations of the sequence.

Figurate Number	1st	2nd	3rd	4th	5th	7th	8th
Triangular	1	3	6	10	15		
Square	1	4	9	16	25		
Pentagonal	1	5	12	22			

Next, the question will be posed, "How many toothpicks do you need to make 10 triangles in a row? 100 triangles? 1,000,000?" Patterns can help you answer questions like these and write variable expressions. The students will need to note that this exercise is different from the figurate numbers. The students will need to derive at another pattern to work this activity.

ACTIVITY

-Work in groups of two's
-Will need about 30 toothpicks

Make as many triangles in a row as you can. Copy and extend the table. Then answer the questions below.

Number of triangles	Number of toothpicks
1	3
2	5
3	7
4	?
?	?

<u>Questions</u>

1. Describe any number patterns you see. What is the relationship between the number of triangles and the number of toothpicks?

2. How many toothpicks do you need to make 10 triangles? 100 triangles? 1,000,000 triangles.   How do you get your answers?

3. How many toothpicks do you need to make *n* triangles? Write you answer as a variable expression. The class will then compare their answers with other groups of the class.

In discussion the class should realize that there is a constant variable of 2. Therefore they should realize that when finding the formula for my *n*th term they would use the number two in there somewhere. It turns out to be that my formula is 2n+1.

I will continue talking about patterns and giving more examples of pattern finding. The next example that I will give them can really be applied to real world situations for them. The subject is about going to a dance and at this age most of them has or will be attending a dance sooner or later.

Example

Problem:

Jay is going dancing. The dance club charges $8 at the door and $1.25 for each soft drink. Write a variable expression for the amount of money Jay should bring; based on the number of soft drinks he buys.

Solution:

Problem Solving Strategy: Identify a pattern.

Number of soft drinks Jay buy	Money needed (dollars)

Step 1 Try different numbers to see if you can find a pattern.	0	8 + 1.25 * 0 = 8.00
	1	8 + 1.25 * 1 = 9.25
	2	8 + 1.25 * 2 = 10.50
	3	8 + 1.25 * 3 = 11.75
	4	8 + 1.25 * 4 = 13.00
Step 2 after a pattern appears, test a larger number	10	8 + 1.25 * 10 = 20.50
Step 3 Use a pattern to write a variable expression	Let *d* = the number of soft drinks Jay buys. $8 + 1.25d$	

If Jay wants to buy *d* soft drinks, then he needs to bring $8 + 1.25d$ dollars.
I then ask the students to put themselves in Jay's shoes and how many drinks will they buy if they were at the dance.

I will give the students another worksheet and let them complete the worksheet on their own. The students will have a chance to work hands on and this will be used as a small project for the students.

Exercise

Each student will need about 30 toothpicks.

a. Write a variable expression for the number of toothpicks needed to make a row of *N Square*. Use a table to support your answer.

b. Write a variable expression for the number of toothpicks needed to make a row of *n* pentagons (five sides). Use a table to support your answer.

c. Write a paragraph comparing the variable expressions you wrote for squares and for pentagons. Compare these expressions with the ones that were obtained with the triangles.

Now I will talk about geometric sequences. I will ask the students to consider the possibility that a child has 2 biological parents, 4 grandparents, 8 great grandparents, 16 great-great grand parents, and so on. The number of ancestors forms the geometric sequence 2,4,8,16,32…. Each successive term of a geometric sequence is obtained from its predecessor by multiplying by a fixed number, the ratio. In the example in the table below, both the 1st term and the ratio are 2. To find the *n*th term we will need to examine the table.

Number of Term	Term
1	$2 = 2^1$
2	$4 = 2*2 = 2^2$

3	$8 = (2*2)*2 = 2^3$
4	$16 = (2*2*2)*2 = 2^4$
n	n^2

It is possible to find the *n*th term of any geometric sequence when given the first term and the ratio. If the 1st term is *a* and the ratio is *r*, then the terms are as listed in the table below. The *n*th term is ar^{n-1}. For $n = 1$, we have $ar^{1-1} = ar^0$. Because the 1st term is a then $ar^o = a$. This implies that $r^0 = 1$. This is true for all numbers r not equal to zero. Thus when $n = 1$ and *r* not equal to zero, we have $ar^o = a(1) = a$. For the geometric sequence 3,12,48,192…, the first term is 3 and the ratio is 4, and so the *n*th term is given by $ar^{n-1} = 3*^{4n-1}$.

Number of Term	Term
1	a
2	ar
3	ar^2
4	ar^3
5	ar^4
.	.
.	.
n	ar^{n-1}

Exercise

Find the next two terms in the sequence 3,6,12,24,48,…

Now to continue on with the last sequence that I will discuss in this unit. The name of this sequence is the Fibonacci sequence. Before jumping directly into the lesson I will read the students a little history about Fibonacci and how he derived at his sequence.

A man put a pair of rabbits in a cage. During the first month the rabbits produced no offspring, but each month thereafter produced one new pair of rabbits. If each new pair thus produced reproduces in the same manner, how many pairs of rabbits will there be at the end of one year?

This problem is a famous on in the history of mathematics and first appeared in *Liber Abaci*, a book written by the Italian mathematician Leonardo Pisano (also known as Fibonacci) in the year 1202. Fibonacci discovered the sequence named after him in a problem on rabbits. Fibonacci (son of Bonaccio) is on of several names for Leonardo of

Pisa. His father managed a warehouse in present-day Bougie (or Bejaia) in Algeria. Thus it was that Leonardo Pisano studied with a Moorish teacher and learned the “Indian” numbers that the Moors and other Moslems brought with them in their westward drive.

Fibonacci wrote books on algebra, geometry, and trigonometry. These contain Arabian mathematics as well as his own work. I will ask the student the question, “How many pairs of rabbits will there be after on year?” To help the students answer the question we will complete a table to find out.

Month	# of pairs at start	# of new pairs produced	# of pairs at end of month
1^{st}	1	0	1
2^{nd}	1	1	2
3^{rd}	2	1	3
4^{th}	3	2	5
5^{th}	5	3	8
6^{th}	8	5	13
7^{th}	13	8	21
8^{th}	21	13	34
9^{th}	34	21	55
10^{th}	55	34	89
11^{th}	89	55	144
12^{th}	144	89	233

According to our table there will be 233 pairs of rabbits at the end of the first year. If you would note, to get the next sequence you needed to add the number of pairs at start plus the number of pairs produced. Example, 1^{st} month, number of pairs at start which is 1 plus 0 pairs produced will give me 1 pair at the end of the month. 2^{nd} month, 1 pair to start plus one pair produced will give me two pairs at the end of the month.

The Fibonacci sequence has many other interesting properties. Now I will choose any term of the sequence after the first and square it. Then I will multiply the terms on either side of it, and subtract the smaller result from the larger. The difference is always 1. For example, choose the sixth term in the sequence, 8. The square of 8 is 64. Now multiply the terms on either side of 8: 5 x 8 = 65. Subtract 64 from 65 to get 65-64 = 1. This pattern continues throughout the sequence.

Unit 2

This unit is about the decorative ornamental ironwork. This unit is intended to fun and interesting for the students. Prior to beginning this lesson the students will be expected to go out and look for different types of patterns and symmetry. I will give them suggestions, look at different car wheels, at butterflies, buildings, and I will let them tell me about some of the patterns of design and symmetry that they did find out in the world.

Before continuing on I will give the students a small history story about the wrought iron coming to New Orleans.

Wrought iron was first brought to New Orleans from Spain in 1790. During the next twenty years, a number of free, mixed-race Haitians fled the Haitian slave revolts and entered the southern ports of Savannah, Charleston, and New Orleans. The Haitian refugees who came to Louisiana between 1791 and 1809 were better trained and better educated than were the inhabitants of the Louisiana territory, and "their influence insured that the state would have a Creole flair for years to come.

One such influence is the decorative wrought-iron work that is the most universally admired feature of Haitian workmanship in New Orleans architecture. The Haitian artisans constructed elegant iron-lace balconies displaying patterns replicated in ornamental wrought-iron doors. The patterns are evident in mathematical transformations, which can be used to connect mathematics with the real world, as recommended by the NCTM's Curriculum and Evaluation Standards for School Mathematics (1989).

The following investigation will guide students in detecting vertical and horizontal lines of symmetry as well as symmetry in the decorative designs of wrought-iron doors. As an introduction to the study of transformations, this investigation is well suited for students in Algebra because the multiple ways presented for thinking about transformations afford teachers the flexibility to expand or delete activities on the basis of the level of their students.

INVESTIGATION:

Prior to the class I will gather pictures of wrought-iron doors and designs that may be seen as wrought-iron doors. I will ask the students to describe ornamental doors that they have seen, and review the preceding historical references. I will place examples of wrought-iron doors on the board. The students will be asked to ignore the doorknobs and any extra space needed for the doors' installation, and to assume that the doors had a uniform border and that the designs extended beyond the doorknobs to the border. I will ask them to describe any patterns that they saw.

I will review the basic properties of reflections and translations. Afterwards I will have students find objects in their environment with each type of transformation. For another opportunity for students to see and think about these transformations, I will use an overhead transparency with templates of some of the doors or designs I have chosen. The essential properties of a rotational symmetry of 180 degrees easily surface when students use the transparencies as a tool. I will give them the transparency and a copy of the activity sheet to have them align the two. I will instruct them to put their index finger near the center of the horizontal midline of the door and to rotate the transparency slowly until its bottom matches the top of its copy. Once they have straighten the transparency, the will see that the doors align.

ACTIVITIES:

Using a few enlarged pictures of the doors, I will have the students find lines of symmetry and discuss modification that can be make to change the number of lines of symmetry in the doors. I will then distribute the pictures that I have collected of the wrought-iron doors and have students determine which doors have which lines of symmetry. This activity will help me assess how well students understand the concept of reflection, as well as how careful they are in verifying that an object and its image do indeed coincide.

Activity 2 requires the students to construct a wrought-iron door for a house. Students will have the opportunity to create their own designs for their doors and will be required to share, describe, and name the symmetries of their design.

Activity 3 will require that the student determine the cost of their door and compute the total price with sales tax.

The Symmetry of Decorative Wrought-Iron Doors

Prior to introducing the lesson I plan to gather brochures from wrought-iron companies, and enlarge some pictures of the ornamental doors to show the designs. I will ask the students to describe the ornamental doors that they have seen, and review the preceding historical references.

I will review basic properties of reflections and translations, and have students find objects in their environment with each type of transformation. For opportunities for students to see and think about these transformations I will use an overhead transparency with templates of some of the doors that I collected. We will investigate the horizontal lines of symmetry and the vertical lines of symmetry and also flip the transparency to that the left side is on the right and will reveal its vertical line of symmetry.

Student activities will include arranging the doors according to their type of symmetry, designing their own wrought-iron door, and computing the cost of their wrought-iron door.

Final Projects

I will give students a choice of a final project. These projects are intended to be an individual project for the students. Their choices will be to 1) Research the history of an ethnic group in our area of the country and to develop mathematical questions that are based on their work or 2) Pick one person in history that has contributed to our mathematical society by analyzing or developing some form of patterns.