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LEARNING FROM THE EXPECTATIONS OF OTHERS

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This paper explores the equilibrium properties of boundedly rational heterogeneous agents under adaptive learning. In a modified cobweb model with a Stackelberg framework, there is an asymmetric information diffusion process from leading to following firms. It turns out that the conditions for at least one learnable equilibrium are similar to those under homogeneous expectations. However, the introduction of information diffusion leads to the possibility of multiple equilibria and can expand the parameter space of potential learnable equilibria. In addition, the inability to correctly interpret expectations will cause a "boomerang effect" on the forecasts and forecast efficiency of the leading firms. The leading firms' mean square forecast error can be larger than that of following firms if the proportion of following firms is sufficiently large.

Keywords: Adaptive Learning, Expectational Stability, Information Diffusion, Cobweb Model, Heterogeneous Expectations

1. INTRODUCTION

The rational expectations hypothesis [Muth (1961); Lucas (1972, 1973)] has revolutionized how economists conceptualize and model economic phenomena. Currently, rational expectations (RE) represents a key component in the study of macroeconomic problems [Frydman and Phelps (1983); Haltiwanger and Waldman (1985)]. Under RE, agents are assumed to act as if they can take conditional (mathematical) expectations of all relevant variables. However, for all its analytical traction, it is also well known that RE rests on a strong assumption.

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Sargent (1993), for example, points out that agents with RE are even more sophisticated than the economist who sets up the economic model.

With this theoretical and empirical challenge in mind, one line of inquiry has been to determine whether a rational expectations equilibrium (REE) can be achieved under the assumption that agents form expectations using less sophisticated mechanisms [Bray (1982); Bray and Savin (1986); Evans (1983); Evans and Honkapohja (2001)]. This line of inquiry allows agents to achieve the REE within the context of a stochastic (updating) process that is typically represented via adaptive learning. Agents do not initially obtain the REE, but they attempt to learn the stochastic process by updating their forecasts (expectations) over time as new information becomes available.

In more technical terms, adaptive learning is used so that agents update parameters of a forecasting rule—perceived law of motion (PLM)—associated with the stochastic process of the variable in question to learn an REE. This process requires a condition establishing convergence to the REE—the E-stability condition. The E-stability condition determines the stability of the equilibrium in which the PLM parameters adjust to the implied actual law of motion (ALM) parameters.

Evans (1989) and Evans and Honkapohja (1992) show that the mapping from the PLM to the ALM is generally consistent with the convergence to REE under least squares learning. This correspondence is called the E-stability principle.[1] This principle also possesses additional attributes. If the equilibrium is E-stable, then the RE method may be an appropriate technique for solving long run equilibria. Moreover, E-stability conditions can be an important selection criteria (i.e., determining stable solutions) when a model has multiple equilibria.

In this paper, we extend adaptive learning methods described earlier to a scenario involving heterogeneous information levels and social interaction. Prior research linking adaptive learning procedures to heterogeneous information levels has not made use of social interaction. This previous research generally assumes that agents forecast independently and solely gather their own information.

For example, Evans and Honkapohja (1996) relax the assumption of homogeneity and allow for N different groups of agents who may form different expectations. Agents are allowed to have different parameter estimates in the same structural forecasting rule. They use a general cobweb-type model and show that the E-stability condition remains the same as in the case of a homogeneous expectations learning model. In addition, Giannitsarou (2003) allows heterogeneous adaptive learning in an economy with a homogeneous structure. She finds that different types of heterogeneity may result in different stability conditions compared to homogeneous learning.

Others have relaxed the representative agent assumption in the learning process [Honkapohja and Mitra (2006)]. They find that such structural heterogeneity alters E-stability conditions in different macroeconomic models. Finally, Guse (2005) allows heterogeneity in the forecasting models used to form expectations—in a model with two equilibria. He finds that the E-stability conditions of each equilibrium are determined by the proportion of agents using each forecasting model. Furthermore, the two equilibria "exchange" stability at the smallest proportion of heterogeneity where the mean-square forecast error (MSE) of the two forecasting models are equal.

Currently, there is no study analyzing how agents' interactions would affect model equilibria under adaptive learning. Although standard adaptive learning models provide important extensions of the RE framework, the assumption of agent forecast independence can be relaxed. Information diffusion (or interaction) among different groups of agents could occur especially when people do not interpret the public information in an identical manner [Kandel and Zilberfarb (1999)]. Carroll (2003) finds, for example, statistical evidence of information diffusion where professional inflation forecasts Granger-cause household forecast accuracy.[2]

Against this theoretical and empirical background, we present a modified Muthian cobweb model [Muth (1961)] to allow for both information heterogeneity and information diffusion. We assume a Stackelberg framework, where there are two types of agents—first and second moving firms. The first-moving firms (leading firms) make the initial forecasts of an aggregate price level according to exogenous information in a market while second-moving firms (following firms) form their forecasts based on the forecasts made by the leading firms. Although the following firms obtain the leading firms' forecasts, they are unable to accurately interpret the content of information because there is some miscommunication between firms. Thus, observational errors due to misinterpretation of leading firms' expectations would naturally occur in the information acquisition process.[3]

With the assumption of social/information interactions, we first examine conditions for a unique (real) equilibrium in our cobweb model. In contrast to a simple cobweb model (without social interaction), which has a unique REE, there may exist multiple mixed expectations equilibria (MEE) in the "interactive" cobweb model.[4] We show that if the variance of the observational error is not sufficiently large, then there exists a unique MEE for the parameter space that Evans and Honkapohja (2001) show to be stable under adaptive learning. However, if the variance of the observational error is large, then there may exist three MEE.

When there is a unique equilibrium the E-stability conditions for the MEE—where there is information diffusion—are identical to the conditions under no information diffusion (homogeneous expectations). When there exists three equilibria, the "high" and "low" solutions are E-stable, whereas the "middle" solution is not E-stable for the E-stable parameter space discussed in Evans and Honkapohja (2001). Interestingly, it turns out that the "low" equilibrium can be E-stable outside of this parameter space.

Next, we find that the degree of information diffusion affects the stochastic (equilibrium) process of the aggregate price level. We argue that the inability to fully share information and the inability to observe observational errors by the leading firms will cause a "boomerang effect" on the leading firm's forecasts and their forecast efficiency. As a result, the leading firms learn a stochastic process different than the REE due to the following firms' misinterpretation(s). Not only is the REE unobtainable, but the MSE for the leading firms is larger than it would be under the REE. In addition to these findings, we also examine the relation between the size of the boomerang effect and the proportion of leading and following firms in the model. We show that, under certain conditions, the leading firms' MSE can be even larger than that of following firms if the proportion of following firms is sufficiently large.

This paper is organized as follows. Section 2 introduces the cobweb model, which includes interactions between leading and following firms. In this section, we also show the conditions of uniqueness and multiplicity of MEE. In Section 3, we study the E-stability conditions of the model. Section 4 demonstrates the boomerang effects, and Section 5 concludes.

2. A SIMPLE INTERACTIVE COBWEB MODEL

The cobweb model has been used extensively in the macroeconomic and learning literature [see Muth (1961); Arifovic (1994); Evans and Honkapohja (2001); Branch and McGough (Forthcoming)]. It is assumed that there are n firms in a competitive market producing a homogeneous product. The firms produce an optimal quantity of their good to maximize their expected profits in accordance with their (rational or nonrational) expectations of the market price in the next period.

The reduced form of the model can be presented as follows:[5]

$$y_t = \beta E^*_{t-1} y_t + \gamma x_{t-1} + \eta_t, \tag{1}$$

where y_t is the price level at time t, $E^*_{t-1} y_t$ is the expectation (not necessarily rational) of y_t formed at the end of time $t-1$, and $\eta_t \sim iid(0, \sigma^2_\eta)$. x_{t-1} is a exogenous observable following a stationary AR(p) process driven by a white noise shock. We assume $Ex_t = 0$ and $Ex_t^2 = \sigma_x^2$. Under the standard cobweb model with a single good, it must be that $\beta < 0$. However, there exist variants of the cobweb model such that $\beta \in (-\infty, \infty)$.[6]

We modify the cobweb model into a Stackelberg setup where it contains two types of firms. Assume that there is a continuum of firms located on the unit interval [0, 1] of which a proportion of $1-\mu$, where $\mu \in [0, 1)$, are first-moving (or leading) firms who form expectations of the market price based on the information (x_{t-1}) observed in the market.

Following the adaptive learning literature, firms will act like econometricians and forecast y_t by running least-squares regressions of y_t based on their past information. Assume that the leading firms (Type-L firms) form expectations using a forecasting model consistent with the form of the minimum state variable (MSV) REE under homogeneous expectations.[7] The PLM for the Type-L firms and their expectations of y_t are given as

$$\begin{aligned} y_t &= bx_{t-1} + \varepsilon_t, \\ y^e_{L,t} &= bx_{t-1}, \end{aligned} \tag{2}$$

where $y^e_{L,t}$ presents the expectations of y_t for the Type-L firms at time $t-1$.

The remaining μ firms are assumed to be second-moving (following) firms who observe the Type-L firms' expectations to form their expectations of market price. However, the following (Type-F) firms may interpret (or even misinterpret) the Type-L firms' expectations differently among themselves or may not be able to obtain the exact information from the Type-L firms. We impose a distribution of observational errors, v_{t-1}, which indicates the degree of misinterpretation of other firms' actions.[8] The PLM for the Type-F firms and their expectations of y_t are

$$\begin{aligned} y_t &= c\left(y^e_{L,t} + v_{t-1}\right) + \varepsilon_t, \\ y^e_{F,t} &= c\left(bx_{t-1} + v_{t-1}\right), \end{aligned} \tag{3}$$

where $y^e_{F,t}$ presents the expectations of y_t for the Type-F firms at time $t-1$ and $v_{t-1} \sim iid(0, \sigma^2_v)$ represents the observational errors which are uncorrelated with ε_t and x_{t-1}. Under this setup, we assume that v_{t-1} is unobservable by the Type-L firms and the Type-F firms are either unable or unwilling to observe $y^e_{L,t}$ and v_{t-1} separately.

In equation (3), we also assume that the objective of the Type-F firms is to minimize their MSE by choosing c. If Type-F firms instead are assumed to completely adopt the noisy information from the Type-L firms as their own expectations (by setting $c = 1$ *a priori*), then Type-F firms' MSE would not be minimized based on their PLM in equation (3). However, Type-L firms would obtain the REE coefficient of b. Given that Type-F firms cannot disentangle Type-L firms' expectations ($y^e_{L,t}$) from the observational errors or excess noise (v_{t-1}), Type-F firms would find that their MSE can be minimized when c is not equal to one.[9]

Furthermore, the assumption that v_{t-1} is not observed separately by the Type-F firms is crucial in our analysis. If the Type-F agents were able to form expectations using an alternative PLM:

$$y_t = d_1 y^e_{L,t} + d_2 v_{t-1} + \varepsilon_t,$$

then they would choose to ignore the unimportant noise, v_{t-1} ($d_2 = 0$) and fully use the expectations of the Type-F firms ($d_1 = 1$). This case would become equivalent to the cobweb model under homogeneous expectations where all firms are Type-L.

Next, based on the proportions of the Type-L and Type-F firms, the average market price expectation is

$$E^*_{t-1} y_t = \mu[c(bx_{t-1} + v_{t-1})] + (1-\mu)bx_{t-1}.$$

The ALM is obtained by substituting average expectations of next period's market price into equation (1):

$$\begin{aligned} y_t &= \{\beta b[\mu(c-1)+1] + \gamma\}x_{t-1} + \beta\mu c v_{t-1} + \eta_t \\ &= (\xi_b + \xi_c b)x_{t-1} + \xi_c v_{t-1} + \eta_t, \end{aligned} \tag{4}$$

where $\xi_b \equiv \beta(1-\mu)b + \gamma$ and $\xi_c \equiv \beta\mu c$.

Note that the form of each group's PLM is inconsistent with the ALM. However, following Evans and Honkapohja (2001), one can obtain a projected ALM associated with a particular PLM. The projected ALM is an ALM projected onto the same class of a particular PLM. It also represents "the best description of the process within the permitted class of PLMs considered" [see Evans and Honkapohja (2001, p. 322)]. We define the projected ALM in our model as follows:

DEFINITION 1. *For Type-j firms, where* $j \in \{L, F\}$, *the Type-j projected ALM is* $T_j(\phi)' z_{j,t-1} + \varepsilon_t$, *where* $T_j(\phi)'$ *is obtained from the linear projection of equation* (4) *on* $z_{j,t-1}$, $z_{j,t-1}$ *is the information used in* PLM_j, *and* ϕ *is a vector representing the parameters used in each* PLM.

The projected ALMs are obtained by computing the following linear projections:

$$E\{x_{t-1}[(\xi_b + \xi_c b)x_{t-1} + \xi_c v_{t-1} + \eta_t - T_b x_{t-1}]\} = 0$$

$$E\{(bx_{t-1} + v_{t-1})[(\xi_b + \xi_c b)x_{t-1} + \xi_c v_{t-1} + \eta_t - T_c(bx_{t-1} + v_{t-1})]\} = 0,$$

where

$$y_t = T_b x_{t-1} + \varepsilon_t,$$

is the projected ALM associated with the PLM of the Type-L firms and

$$y_t = T_c\,(bx_{t-1} + v_{t-1}) + \varepsilon_t,$$

is the projected ALM associated with the PLM of the Type-F firms.

Due to the above linear projections, the forecasts associated with each PLM must satisfy the least-squares orthogonality condition where the regressors are uncorrelated with the forecast errors. This projection gives the following T-mapping from the two PLMs to their associated projected ALMs:

$$T\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} T_b(b,c) \\ T_c(b,c) \end{pmatrix} = \begin{pmatrix} \xi_b + \xi_c b \\ \xi_c + \dfrac{b\sigma_x^2}{b^2\sigma_x^2 + \sigma_v^2}\xi_b \end{pmatrix}$$

$$= \begin{pmatrix} [(1-\mu) + \mu c]\beta b + \gamma \\ \mu c \beta + \dfrac{b\sigma_x^2}{b^2\sigma_x^2 + \sigma_v^2}((1-\mu)\beta b + \gamma) \end{pmatrix}. \tag{5}$$

For a model where agents have a choice of using one of several forecasting models, Guse (2005, 2006) refers to a resulting stochastic equilibrium defined by the ALM and a fixed point of the T-map as a "mixed expectations equilibrium" (MEE).[10] In our model, a MEE is a stochastic process following the (unprojected) ALM:

$$y_t = \{\beta\bar{b}[\mu(\bar{c} - 1) + 1] + \gamma\}x_{t-1} + \beta\mu\bar{c}v_{t-1} + \eta_t, \tag{6}$$

where

$$\begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix} = T \begin{pmatrix} \bar{b} \\ \bar{c} \end{pmatrix}.$$

Similar to a REE, the coefficients in a MEE are such that each PLM is consistent with its associated projected ALM. We will refer to such an equilibrium as $\bar{q} = (\bar{b}, \bar{c})$, which implies the stochastic process in equation (6) given $\bar{b}$ and $\bar{c}$. The MEE coefficients are the following in this model:[11]

$$\bar{b} = \frac{\gamma}{1 - \beta(1 - \mu + \mu\bar{c})} \qquad \textbf{(7)}$$

$$\bar{c} = \frac{\bar{b}^2 a}{\bar{b}^2 a + (1 - \beta\mu)},$$

where

$$a = \frac{\sigma_x^2}{\sigma_v^2}.$$

One can think of a as a ratio of noise in important information over noise in "unimportant" information. Note that as $a \to \infty$, the MEE value for $\bar{c} \to 1$ and the MEE value of $\bar{b} \to \frac{\gamma}{1-\beta}$, which is equal to the REE value of b.

Although firms misspecify their forecasting models, the MEE are optimal relative to the restricted information set used by the firms. Due to the orthogonality condition, firms cannot detect a misspecification unless they step outside of their forecasting models. There is some concern about MEE because variables in the ALM not included in a forecast will be correlated with the forecast errors, making it possible to easily detect misspecification. However, the Type-L firms will be unable to detect their misspecification, as they do not observe the variable v_{t-1}. Therefore, the forecast error correlation problem should not be a concern in our model.[12]

From solution (7), we observe that the functions representing $\bar{b}$ and $\bar{c}$ are nonlinear (cubics) and there may exist multiple equilibria for a open set of parameters when $\sigma_v^2 > 0$. The conditions for a unique real MEE and multiple real MEE when $\beta \in (-\infty, \frac{1}{\mu})$ are given in Proposition 1:[13]

PROPOSITION 1. *Define* $a_{\max} = \frac{27\mu^2(8-7\mu)}{\gamma^2(8+\mu)^3}$ *and* $a_{\min} = \frac{(1-\mu)\mu^2}{4\gamma^2}$.

1. *A unique real MEE exists if* $\beta < \frac{8}{8+\mu}$.
2. *For* $\beta \in (\frac{8}{8+\mu}, \frac{1}{\mu})$ *there are* 3 *cases to consider for multiple equilibria:*
   (a) *If* $a > a_{\max}$, *then there exists a unique real MEE.*
   (b) *If* $a \in (a_{\min}, a_{\max})$, *there exists a unique real MEE if* $\beta \notin (\beta_1, \beta_2)$ *and multiple* (3) *real MEE if* $\beta \in (\beta_1, \beta_2)$ *where* $\beta_1 \in (\frac{8}{8+\mu}, 1)$, $\beta_2 \in (\frac{8}{8+\mu}, 1)$, *and* $\beta_2 > \beta_1$.
   (c) *If* $a \in (0, a_{\min})$, *there exists a unique real MEE if* $\beta \notin (\beta_1, \beta_2)$ *and multiple* (3) *real MEE if* $\beta \in (\beta_1, \beta_2)$ *where* $\beta_1 \in (\frac{8}{8+\mu}, 1)$ *and* $\beta_2 \in (1, \min(\frac{1}{\mu}, \frac{1}{1-\mu}))$.

The proof is given in Appendix A. Figure 1 shows two functions, $\beta_1(a)$ and $\beta_2(a)$, that can be generated using the Implicit Function Theorem where

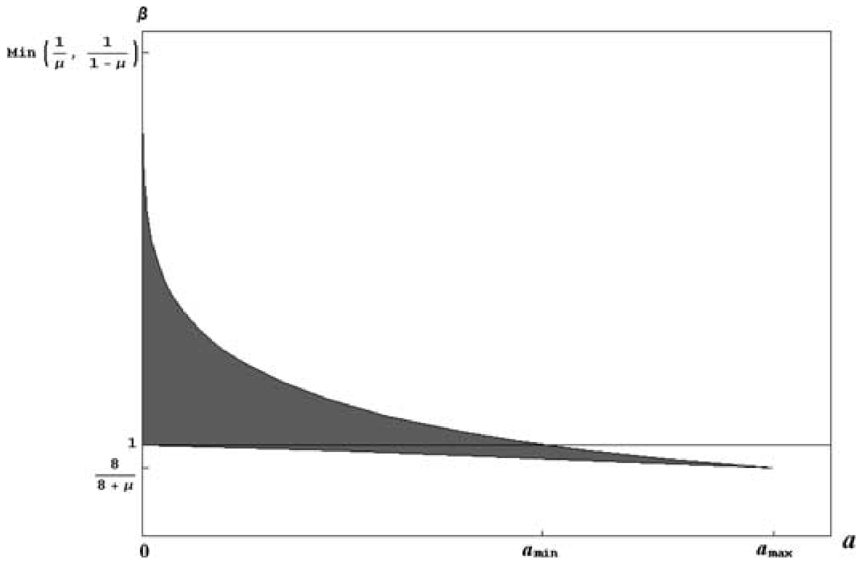

FIGURE 1. Multiple real MEE exist within the shaded region for a given μ and γ.

$\beta_1(a) < \beta_2(a)$ for all $a \in (0, a_{\max})$. β_1 and β_2 in Proposition 1 are defined by these functions for a given a where $\beta_1(a) \in (\frac{8}{8+\mu}, 1)$ for $a \in (0, a_{\max})$, $\beta_2(a) \in (\beta_1(a), 1)$ for $a \in (a_{\min}, a_{\max})$, and $\beta_2(a) \in [1, \min(\frac{1}{\mu}, \frac{1}{1-\mu}))$ for $a \in (0, a_{\min}]$. If $\beta < \frac{1}{\mu}$, then, for a given μ and γ, there exists multiple real MEE only when (a, β) is inside of the shaded region defined by $\beta_1(a)$ and $\beta_2(a)$ in Figure 1.

Under our setup here, the standard cobweb model with $\beta < 0$ will continue to have a unique solution for any $a \geq 0$. However, the stochastic process will be different than that of an REE as discussed later. Interestingly, we note that in a cobweb model where $\beta \in (\frac{8}{8+\mu}, \frac{1}{\mu})$, multiple equilibria can exist[14] if σ_v^2 is sufficiently large so that $a \in (0, a_{\max})$. For $\beta \in (-\infty, \frac{1}{\mu})$, if there are multiple equilibria, then we refer to each equilibrium as $\bar{q}_i = (\bar{b}_i, \bar{c}_i)$, where $0 < \bar{c}_1 < \bar{c}_2 < \bar{c}_3 < 1$ and $\bar{b}_i$ is obtained from (7) given $\bar{c}_i$.

Next, we explore some important properties of the MEE,which will prove useful when discussing the learnability of the MEE. The properties are expressed in the following lemma:

LEMMA 1. *$\bar{b}$ and $\bar{c}$ have the following properties*:

1. $\bar{c} \in (0, 1]$ *if* $\beta < \frac{1}{\mu}$, $\sigma_x^2 > 0$, *and* σ_v^2 *is finite.*
2. *sign*$(\bar{b}) =$ *sign*(γ) *if* $\beta < 1$.
3. (a) *If $\bar{c}$ is unique, then $\bar{c}$ is monotonically increasing in β for $\beta < 1$, monotonically decreasing in β for $\beta \in (1, \tilde{\beta})$ for some $\tilde{\beta} \in (1, \frac{1}{\mu})$, and monotonically increasing in β for $\beta \in (\tilde{\beta}, \frac{1}{\mu})$.*
   (b) *If there exists multiple MEE, then*

i $\bar{c}_1$ *is monotonically increasing in* β *for* $\beta \in (\beta_1, \beta_2)$,
ii $\bar{c}_2$ *is monotonically decreasing in* β *for* $\beta \in (\beta_1, \beta_2)$, *and*
iii $\bar{c}_3$ *is monotonically increasing in* β *for* $\beta < 1$, *monotonically decreasing in* β *for* $\beta \in (1, \tilde{\beta})$ *for some* $\tilde{\beta} \in (1, \frac{1}{\mu})$, *and monotonically increasing in* β *for* $\beta \in (\tilde{\beta}, \frac{1}{\mu})$.

The proof is given in Appendix C. Part 1 of Lemma 1 states that the Type-F firms will always use some information from the Type-L firms in equilibrium. If $\sigma_v^2 = 0$, then the Type-F firms have the same information as the Type-L firms and thus the equilibrium level of c will be $\bar{c} = 1$. As σ_v^2 increases, the information from the Type-L firms becomes less useful to the Type-F firms and thus $\bar{c} \to 0$ as $\sigma_v^2 \to \infty$. Part 2 of the Lemma states that the sign of $\bar{b}$ must be consistent with the sign of γ for $\beta < 1$. However, if $\beta > 1$ this is not always the case. We show below that under our model, an MEE where $\text{sign}(\bar{b}) \neq \text{sign}(\gamma)$ is never stable under learning.

Part 3 of the Lemma states that for a unique solution, information from the Type-L firms is more useful for larger values of β (for a fixed $a > 0$) when $\beta < 1$. From equation (4), we see that the effect from the observational error on the ALM is $\beta\mu\bar{c}v_{t-1}$. For the standard cobweb model with $\beta < 0$, there is a negative feedback effect and prices will respond in the opposite direction of the observational error, v_{t-1}. As a result, as β takes larger negative values (for a fixed σ_v^2) information from the Type-L firms becomes less informative and thus $\bar{c}$ will decrease. Under the case of a positive expectational feedback, the price will move in the same direction as the observational error. Consequently, the uninformative v_{t-1} feeds into the ALM to actually become "important" information as β increases, thereby causing $\bar{c}$ to increase. When $\beta = 1$, the response to x_{t-1} is infinite (if $\bar{c}$ is unique) and, therefore, the v_{t-1} shock is relatively unimportant. In this case, agents will fully use the information from the Type-L agents and thus $c \to 1$ as $\beta \to 1$. For multiple equilibria, the same intuition should follow; however, we see that the equilibrium $\bar{c}_2$ moves in the opposite direction of β. We show later that this nonintuitive equilibrium is actually not E-stable; firms will not coordinate to this MEE under learning.

3. EXPECTATIONAL STABILITY OF THE MEE

Evans and Honkapohja (2001) discuss the E-stability condition of the cobweb model under homogeneous expectations. Assuming that all agents have the forecasting rules as equation (2), they show that the REE is E-stable if $\beta < 1$. Evans and Honkapohja (1996) relax the assumption of homogeneous expectations learning, allowing for N different groups of agents forming different expectations. Based on this framework, they show that the condition for E-stability is the same as that for the case of homogeneous expectations learning.

In this section, we explore E-stability conditions in a cobweb model allowing for interactions among agents. It turns out that if there exists a unique real MEE for some $\beta \in (-\infty, 1)$, then this MEE is E-stable. This result is equivalent to the E-stability condition of $\beta < 1$ under the cases of homogeneous and heterogeneous expectations learning discussed in Evans and Honkapohja (1996, 2001). Recall that there may exist multiple MEE according to solution (7). "High" and "low" MEE are E-stable and the "middle" MEE is E-unstable if $\beta < 1$. Interestingly, if there exists multiple equilibria for some $\beta \in (1, \min(\frac{1}{\mu}, \frac{1}{1-\mu}))$, then the "low" MEE is E-stable while the "middle" and "high" MEE are E-unstable. Our current setup of heterogeneity thus expands the E-stability space of the specific model. Finally, we conclude that there does not exist an E-stable MEE for $\beta > 2$. The results here generalize the previous E-stability results of Evans and Honkapohja (2001) to a Stackelberg type setting.

To show the E-stability condition, consider the following ordinary differential equation (ODE):

$$\frac{d\phi}{d\tau} = T(\phi) - \phi,$$

where: T is the mapping from the PLM, ϕ, to the implied ALM, $T(\phi)$ and τ denotes "notional" or "artificial" time. In this case, $T(\phi)$ is represented by equation (5) and:

$$\phi = \begin{pmatrix} b \\ c \end{pmatrix}.$$

Evans and Honkapohja (2001) define an equilibrium (stochastic process defined by the ALM and a fixed point of the ODE) to be E-stable if the ODE is stable when evaluated at the equilibrium values.

We present the E-stability conditions in the following proposition:

PROPOSITION 2. *E-stability conditions for the above interactive cobweb model*:

1. *If* $a > a_{\max}$, *then the unique MEE is E-stable if* $\beta < 1$ *and E-unstable for all* $\beta > 1$.
2. *If* $a \in (a_{\min}, a_{\max})$, *then the unique MEE is E-stable for all* $\beta \in (-\infty, \beta_1)$ *and all* $\beta \in (\beta_2, 1)$ *and is E-unstable for all* $\beta > 1$, $\bar{q}_1$ *and* $\bar{q}_3$ *are E-stable for all* $\beta \in (\beta_1, \beta_2)$, *and* $\bar{q}_2$ *is E-unstable for all* $\beta \in (\beta_1, \beta_2)$.
3. *If* $a \in (0, a_{\min})$, *then the unique MEE is E-stable for all* $\beta \in (-\infty, \beta_1)$ *and E-unstable for all* $\beta > \beta_2$, $\bar{q}_1$ *is E-stable for all* $\beta \in (\beta_1, \beta_2)$, $\bar{q}_2$ *is E-unstable for all* $\beta \in (\beta_1, \beta_2)$, *and* $\bar{q}_3$ *is E-stable for all* $\beta \in (\beta_1, 1)$ *and E-unstable for all* $\beta \in (1, \beta_2)$.

The proof is given in Appendix D. Similar to Figure 1, we illustrate the conditions of uniqueness, multiplicity, and E-stability in Figure 2, which plots $\beta_1(a)$ and $\beta_2(a)$ in (a, β) space. Region I represents the area where there does not exist E-stable MEE. As shown in Proposition 1, regions II and III represent the areas for the existence of multiple real MEE. However, not all multiple real MEE are E-stable. If $\beta < 1/\mu$, then, for a given μ and γ, $\bar{q}_1$ is the only E-stable MEE in both regions, while $\bar{q}_3$ is E-stable in region III but E-unstable in region II. Moreover, $\bar{q}_2$ is E-unstable in both regions. In Figure 2, we see that region IV is the area where the unique MEE is E-stable.

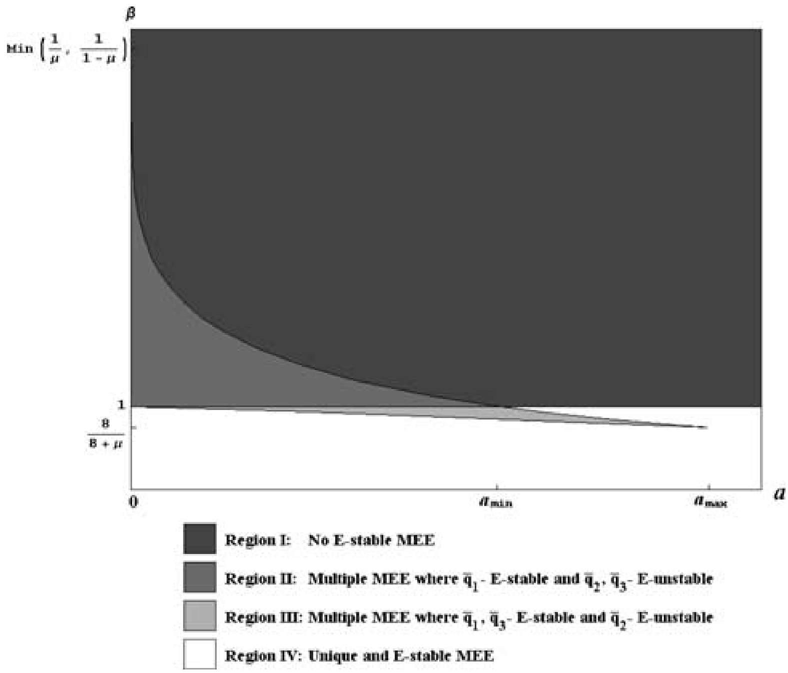

FIGURE 2. Regions of uniqueness, multiplicity, and E-stability for a given μ and γ.

To illustrate the expansion of the E-stability region under heterogeneity, we provide a simulation of 15,000 periods of the model (1) with the reduced form parameters: $\beta = 1.05 > 1$, $\gamma = 2$, and $\mu = 0.5$. In this simulation, the observable follows an AR(1) process such that

$$x_t = 0.9x_{t-1} + noise,$$

where $\sigma_x^2 = 4$. v_{t-1} and η_t are assumed to be white noise processes such that $\sigma_v^2 = 800$ and $\sigma_\eta^2 = 1$.[15] As the E-stability condition is a concept of the local stability, we assume that the Type-L and Type-F firms obtain the initial values close to the E-stable MEE, ($b_0 = 6.13$ and $c_0 = 0.28$).

The objective of the simulation is to show that both Type-L and Type-F firms are able to learn a MEE in the case of $\beta = 1.05 > 1$. For this simulation, we assume that in each period, agents will estimate the parameter in their PLM using standard recursive least squares (RLS):

$$\begin{aligned} \phi_{j,t} &= \phi_{j,t-1} + \lambda_t R_{j,t-1} z_{j,t-1}(y_t - z'_{j,t-1}\phi_{j,t-1}), \\ R_{j,t} &= R_{j,t-1} + \lambda_t (z_{j,t-1} z'_{j,t-1} - R_{j,t-1}), \end{aligned}$$

where $\phi_{L,t} = b_t, \phi_{F,t} = c_t, z_{L,t-1} = x_{t-1}, z_{F,t-1} = (b_{t-1}x_{t-1} + v_{t-1})$, $R_{j,t}$ represents the estimated second moment of $z_{j,t-1}$, and λ_t is a sequence of nonincreasing "gains."[16]

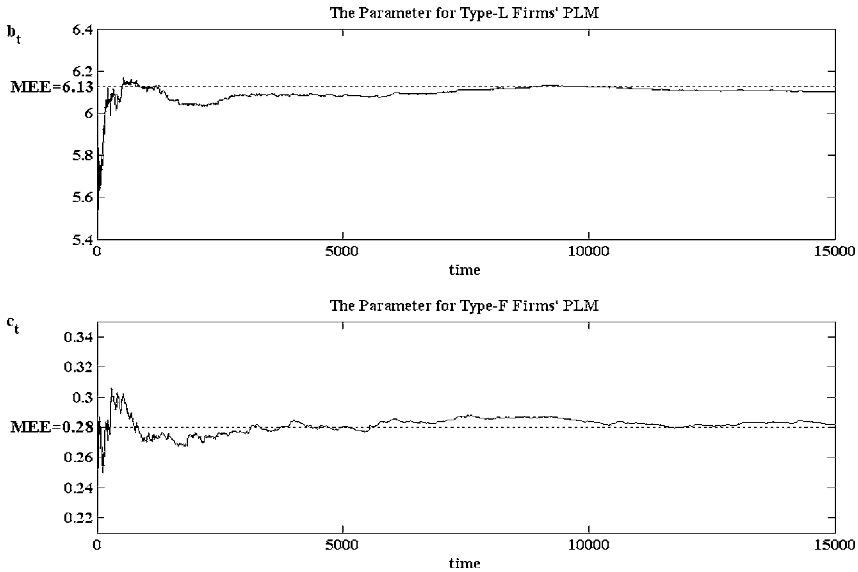

FIGURE 3. Simulations for Type-L and Type-F Firms' PLM's When $\beta > 1$.

Figure 3 demonstrates the learning process for the Type-L (upper panel) and the Type-F (lower panel) firms given $\beta = 1.05$. The Y axis represents the estimated parameter value of a firm's PLM and the X axis represents the real-time learning periods. Figure 3 shows that firms adjust their PLM parameters in the first 5,000 periods. After period 5,000, the parameters become more stable and convergent. When $t = 15{,}000$, Type-L and Type-F firms' parameters converge to $b = 6.1000$ and $c = 0.2816$, respectively, which are close to the solutions of $\bar{q}_1$. This simulation demonstrates that $\bar{q}_1$ is locally stable under least-squares learning.[17]

Recall that Evans and Honkapohja (2001) show that the unique REE is E-stable if $\beta < 1$ in the cobweb model with homogeneous expectations. In this model under the process of information diffusion, we find that $\beta < 1$ is still an important condition for obtaining E-stable solution(s). We present the following corollary:

COROLLARY 1. $\beta < 1$, *then there exists at least one E-stable MEE under the above setup of information diffusion.*

The proof comes directly from Proposition 2. Proposition 2 shows that the unique MEE is E-stable when $\beta < 1$. Parts 2 and 3 of Proposition 2 then point out that $\bar{q}_1$ and $\bar{q}_3$ are E-stable if $\beta < 1$. Thus, we conclude that there exists at least one E-stable MEE—either a unique or multiple MEE—when $\beta < 1$.

Proposition 2 also shows that if σ_v^2 is sufficiently large [$a \in (0, a_{\min})$], then the "region" of at least one E-stable MEE is expanded to be $\beta < \beta_2$ for $\beta_2 \in (1, 2)$. Previous work on heterogeneous expectations [Giannitsarou (2003) and Honkapohja and Mitra (2006)] has shown general results where the E-stability space is contracted due to heterogeneity, but to our knowledge, there has not been a general result where heterogeneity can *expand* the E-stability space of a specific model. Under our setup, the upper bound on the region of E-stability depends on μ. In other words, a sufficient condition for E-instability is $\beta > 2$ as shown in the following corollary:

COROLLARY 2. *If $\beta > 2$, then there does not exist an E-stable MEE.*

The proof is straightforward. As $\beta_2 \in (\frac{8}{8+\mu}, \min(\frac{1}{\mu}, \frac{1}{1-\mu}))$, the maximum value of β_2 is obtained when $\mu = \frac{1}{2}$ so that $\max(\beta_2) = 2$. Because no MEE is E-stable when $\beta > \beta_2$, then it must always be the case that no MEE is E-stable when $\beta > 2 = \max(\beta_2)$.

There seems to be a relationship between the sign of the MEE value of $\bar{b}$ and the E-stability of an MEE. Suppose that x_{t-1} is a shock to productivity and therefore, under standard economic theory, it must be that $\gamma < 0$. Under homogeneous expectations ($\mu = 0$), $\text{sign}(b) = \text{sign}(\gamma)$ for $\beta < 1$ and $\text{sign}(b) \neq \text{sign}(\gamma)$ for $\beta > 1$. When $\beta > 1$, under RE, agents unreasonably produce less output when their productivity is increased and more when their productivity is decreased. However, this nonsensical equilibrium is not E-stable and thus not stable under learning. We show that these nonsensical equilibria continue to be E-unstable under the interactive cobweb model in the following corollary:

COROLLARY 3. *There does not exist an E-stable MEE where* $\text{sign}(\bar{b}) \neq \text{sign}(\gamma)$.

The proof is given in Appendix E. For $a \in (0, a_{\min})$ and $\beta \in (1, \beta_2)$, $\bar{c}_1$ is small enough such that $\text{sign}(\bar{b}_1) = \text{sign}(\gamma)$. Therefore, under an adaptive learning rule, agents will only learn a stochastic process such that their response to an x_{t-1} shock will be consistent with economic theory. As all MEE where $\text{sign}(\bar{b}) \neq \text{sign}(\gamma)$ are E-unstable, we can think of this as a sufficient condition for E-instability.

Recall another nonintuitive result earlier where $\bar{c}_2$ was monotonically decreasing in β meaning that as v_{t-1} became more "informative," the Type-F firms would make less use of it. We have also shown that $\bar{c}_3$ or the unique MEE $\bar{c}$ is also decreasing in β for $\beta \in (1, \tilde{\beta})$ where $\tilde{\beta} \in (1, \frac{1}{\mu})$. As these MEE are always E-unstable, another sufficient condition for E-instability for this model is that $\bar{c}$ is decreasing in β.

4. THE BOOMERANG EFFECT

In this section, we discuss the comparative statics of an equilibrium (MEE) and its forecast accuracy (i.e., mean-squared forecast error) with respect to the observational errors (v_{t-1}). The inability to fully share information and the inability to observe v_{t-1} by the Type-L firms will cause a "boomerang effect" on the forecasts and forecast efficiency of the Type-L firms. First, as $\bar{c} \in (0, 1)$ for $\sigma_v^2 > 0$, the temporary equilibrium:

$$y_t = \{\beta b_{t-1}[\mu(c_{t-1} - 1) + 1] + \gamma\}x_{t-1} + \beta\mu c_{t-1}v_{t-1} + \eta_t,$$

will typically be different than it would be under the case of homogeneous expectations (i.e., when $\mu = 0$). Therefore, if $\sigma_v^2 > 0$, then the Type-L firms will eventually learn a stochastic equilibrium such that they will respond differently to a x_{t-1} shock than under the REE with homogenous expectations. We refer to this as the *Boomerang Effect on Expectations*.

Second, the observational errors will enter the ALM, but as the Type-L firms are unable to observe this shock, it becomes additional noise. Therefore, the observational error has a negative effect on forecast accuracy. In equilibrium, MSE is higher for the Type-L firms than under the REE since the observational error introduces excess volatility. We refer to this as the *Boomerang Effect on the MSE*.

Finally, we examine the relationship between the boomerang effect on the MSE and the proportion of Type-F firms in the model, μ. We find that if μ is sufficiently large, then the Type-L firms' MSE can actually be larger than that of Type-F firms.

4.1. The Boomerang Effect on Expectations

In MEE (7) and Lemma 1, the observational error, v_{t-1}, plays a very important role in the model. How much the Type-F firms use the observed expectations from the Type-L firms depends on how accurately the Type-F firms interpret the Type-L firms' expectations. This accuracy is represented by the variance of the observational error, σ_v^2. According to the previous discussion of part 1 of Lemma 1, we see that if $\beta \in (-\infty, \frac{1}{\mu})$, then every real $\bar{c}$ is between zero and one depending on the size of σ_v^2. If the Type-F firms fully understand and make use of the Type-L firms' expectations (i.e., $\sigma_v^2 = 0$), then $\bar{c} = 1$. It implies that both types of firms' expectations become homogeneous and therefore they are able to achieve the REE.

Next, consider the case of $\sigma_v^2 > 0$, where the Type-F firms misinterpret the Type-L firms' expectations. Although the Type-L firms use the existing exogenous observable, x_{t-1}, to form their expectations, the ex-post observational error created by the Type-F firms eventually confounds the Type-L firms. Instead, the Type-L firms obtain the MEE $\bar{b}$ rather than the REE $\bar{b}^{\mathrm{REE}}$ (where $\mu = 0$ or $\sigma_v^2 = 0$). This boomerang effect on expectations is summarized in the following proposition:

PROPOSITION 3 (*Boomerang Effect on Expectations*). *For a finite σ_v^2, the E-stable MEE $|\bar{b}| \in (\frac{|\gamma|}{1-\beta(1-\mu)}, \frac{|\gamma|}{1-\beta})$ for $\beta \in [0, 1)$ and $|\bar{b}| \in (\frac{|\gamma|}{1-\beta}, \frac{|\gamma|}{1-\beta(1-\mu)})$ for $\beta \in (-\infty, 0)$.*

The proof of this proposition is straightforward. According to Lemma 1, $\bar{c} \in (0, 1]$ for all finite σ_v^2; therefore, from equation (7), we see that $|\bar{b}| \in (\frac{|\gamma|}{1-\beta(1-\mu)}, \frac{|\gamma|}{1-\beta})$ for $\beta \in (0, 1)$ and $|\bar{b}| \in (\frac{|\gamma|}{1-\beta}, \frac{|\gamma|}{1-\beta(1-\mu)})$ if β is negative. When expectations are not involved in the model ($\beta = 0$), it turns out that $\bar{b} = \gamma$.

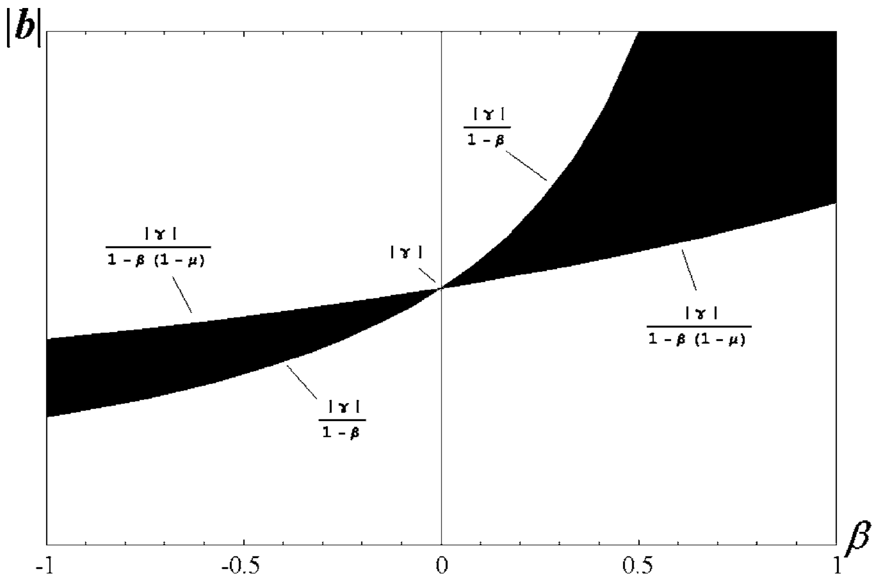

FIGURE 4. The Region of the Boomerang Effect on the MEE.

The shaded region in Figure 4 shows possible values of $\bar{b}$ for a given β when γ and μ are fixed. The lower boundary for $\beta < 0$ and the upper boundary for $\beta > 0$ represent the REE of b when the Type-F firms accurately observe the forecasts from the Type-L firms (i.e., $\sigma_v^2 = 0$ and $c = 1$), whereas other boundaries represent a case where the Type-F firms do not use any of the forecasts given by the Type-L firms (i.e., $\sigma_v^2 \to \infty$ and $c \to 0$). Any MEE $\bar{b}$ will be located within the shaded region and determined by $a > 0$. If the Type-F firms accurately observe the expectations of the Type-L firms ($\sigma_v^2 = 0$), the Type-L firms obtain the E-stable REE ($\bar{b}^{\text{REE}}$). However, if the Type-F firms are unable to perfectly observe the Type-L expectations (a finite σ_v^2), then $\bar{b}$ for the Type-L firms would move to the inside of the shaded region. This result represents the Type-L firms' failure to obtain the REE when the Type-F firms misinterpret the forecasts of the Type-L firms under the process of information diffusion. Finally, note from earlier that $\bar{b}$ can take on positive or negative values for $\beta > 1$ and we do not generally demonstrate a relation between the MEE and the REE. However, one can typically say that $\bar{b} \neq \bar{b}^{\text{REE}}$ meaning that the Type-L agents are still unable to obtain the REE for $\beta > 1$.

To illustrate that the Type-L firms fail to obtain the REE with finite values of σ_v^2 we provide a simulation that is similar to Figure 3. We use the following reduced form parameters: $\beta = -0.5 < 1$, $\gamma = 2$, $\mu = 0.5$, $\sigma_x^2 = 4$, $\sigma_v^2 = 25$ and $\sigma_\eta^2 = 1$.[18] Assume that the Type-L and Type-F firms initially obtain the REE, ($b_0 = \frac{4}{3}$ and $c_0 = 1$).[19] In Figure 5, under the process of information diffusion with a finite misinterpretation error variance (generated by the Type-F firms),

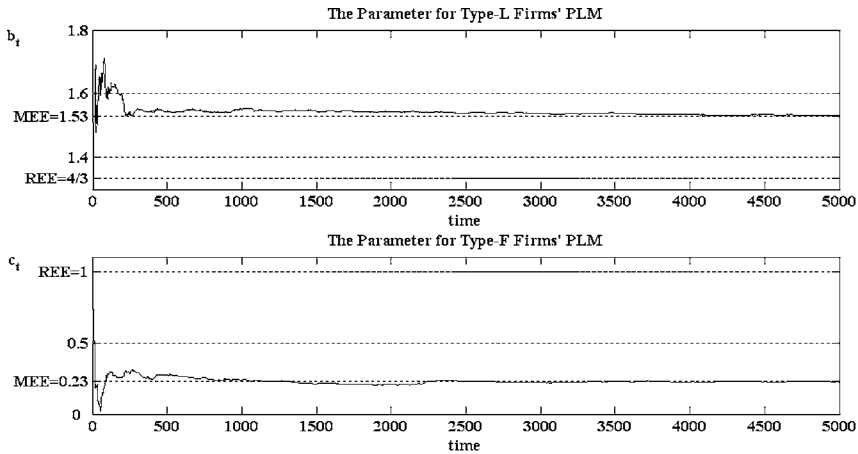

FIGURE 5. Simulations for Type-L and Type-F Firms' PLMs.

the parameters for both firms' PLM's do not converge to the REE.[20] Part 1 of Lemma 1 is shown numerically in the lower panel of Figure 5. With $\sigma_v^2 = 25$, the Type-F firms make partial use of the expectations formed by the Type-L firm. The result is the Type-F firms' PLM parameter, c, converges to MEE $\bar{c} = 0.23 \in [0, 1]$.

More important, the upper panel of Figure 5 describes the boomerang effect on the Type-L firms' forecasts. Although the Type-L firms are initially at the REE and obtain exogenous observables, x_{t-1}, to make forecasts, they fail to stay at the REE and instead eventually learn the MEE when they interact with the Type-F firm (with a finite σ_v^2). For time periods between one to 150, the value of b fluctuates and gradually adjusts. After period 1,000, the parameter b becomes more stable and converges to the MEE $\bar{b} = 1.53$, which is different from the REE $\bar{b}^{\text{REE}} = \frac{4}{3}$.

4.2. The Boomerang Effect on the MSE

Next, we consider how both types of firms' forecast accuracy are affected from miscommunication or misinterpretation of information. To show this, we calculate the (equilibrium) MSE for the forecasts of the Type-F and Type-L firms. The MSEs are derived in Appendix F. The MSE for the Type-F firms is given as

$$\text{MSE}_F = \bar{c}(1 - \beta\mu)(1 - \beta\mu\bar{c})\sigma_v^2 + \sigma_\eta^2, \tag{8}$$

and the MSE for the Type-L firms is given as

$$\text{MSE}_L = (\beta\mu\bar{c})^2\sigma_v^2 + \sigma_\eta^2. \tag{9}$$

The MSE for the Type-F firms in (8) shows that when the Type-F firms accurately observe the expectations from the Type-L firms ($\sigma_v^2 = 0$), the Type-F firms obtain the minimum MSE ($\text{MSE}_F = \sigma_\eta^2$). However, the finite σ_v^2 reduces the Type-F firms' predictive accuracy where $\text{MSE}_F > \sigma_\eta^2$. More interestingly, the results for the Type-L firms indicate that only $\sigma_v^2 = 0$ or $\sigma_v^2 \to \infty$ produce the most efficient outcome, $\text{MSE}_L = \sigma_\eta^2$. However, if there exists a finite σ_v^2, the Type-L firms' forecasts become less efficient (i.e., larger MSE). We call this result the boomerang effect on the MSE:

PROPOSITION 4 (*The Boomerang Effect on the* MSE). *The finite variance of the Type-F firms' observational errors* (σ_v^2) *generates a higher MSE relative to the REE for the Type-L firms where* $\text{MSE}_L = (\beta\mu\bar{c})^2\sigma_v^2 + \sigma_\eta^2 > \sigma_\eta^2$.

The proof is trivial as it comes directly from MSE_L. The MSE of a correctly specified linear forecasting model is typically equal to the variance of the unforecastable noise. Under the REE where $\mu = 0$ or $\sigma_v^2 = 0$, the unforecastable noise is η_t and therefore, the $\text{MSE}_L = \sigma_\eta^2$. However, when $\mu > 0$ and $\sigma_v^2 > 0$, the Type-L firms are not able to directly observe the (mis)interpretation error, v_{t-1}, making it additional unforecastable noise. Under the ALM, this additional noise is equal to $\beta\mu c v_{t-1}$. Therefore, MSE_L is increased by $(\beta\mu\bar{c})^2\sigma_v^2$ as shown in Proposition 4.

Next, the Type-L firms' forecasts can actually have a higher MSE than the Type-F firms' forecasts under certain values of μ. We first present the following proposition:

PROPOSITION 5. $\text{MSE}_L > \text{MSE}_F$ *if* $\beta > 0$ *and* $\bar{c} > \frac{1-\beta\mu}{\beta\mu}$.

The proof is given in Appendix G. This proposition states that it may be possible that $\text{MSE}_L > \text{MSE}_F$ if $\bar{c}$ is large enough. However, for $\beta < \frac{1}{\mu}$, this cannot always be the case since $\bar{c}$ is constrained to be between zero and one. The following corollary states that for $\text{MSE}_L > \text{MSE}_F$, it must be that μ and β are sufficiently large:

COROLLARY 4. *Conditions for* $\text{MSE}_L > \text{MSE}_F$:

1. *If* $\mu < \frac{1}{2}$, *then there does not exist an E-stable* $\bar{q}$ *such that* $\text{MSE}_L > \text{MSE}_F$.
2. *If there exists a unique MEE and* $\mu \in (1/2, 1)$, *then there is a* $\hat{\beta} \in (1/2, 1)$ *such that if* $\hat{\beta} < \beta < 1$, *then* $\text{MSE}_L > \text{MSE}_F$.
3. *If* $\mu \in (\frac{1}{2}, 1)$, *there exists an* $a \in (0, a_{\min})$, *and a* $\beta_1 < \hat{\beta} < \beta_2$, *such that* $\text{MSE}_L > \text{MSE}_F$ *for all* $\beta \in (\hat{\beta}, \beta_2)$ *under the E-stable* $\bar{q}_1$.
4. *If* $\mu \in (\frac{8}{13}, 1)$ *and* $a \in (0, a_{\max})$, *then under the E-stable* $\bar{q}_3$, $\text{MSE}_L > \text{MSE}_F$ *for all* $\beta \in (\beta_1, 1)$.
5. *If* $\mu \in (\frac{2}{3}, 1)$, *there exists an* $a \in (a_{\min}, a_{\max})$, *and a* $\beta_1 < \hat{\beta} < \beta_2 < 1$, *such that* $\text{MSE}_L > \text{MSE}_F$ *for all* $\beta \in (\hat{\beta}, \beta_2)$ *under the E-stable* $\bar{q}_1$.

The proof is given in Appendix H. In Proposition 5 and Corollary 4, when the fraction of the Type-F firms is larger than that of the Type-L firms (i.e., for some

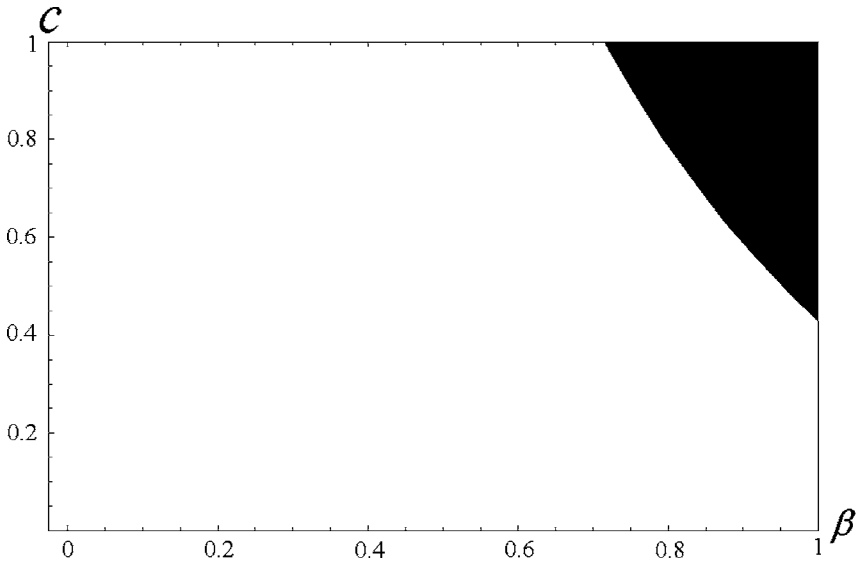

FIGURE 6. The Region of $\mathrm{MSE}_L > \mathrm{MSE}_F$.

$\mu > 1/2$), it is possible that the MSE of the Type-L firms is larger than that of the Type-F firms for some $\beta \in (\frac{1}{2}, \min(\frac{1}{\mu}, \frac{1}{1-\mu}))$. We assume that $a > a_{\max}$ and $\mu = 0.7 > \frac{1}{2}$ to illustrate a scenario when there exists a unique MEE in Figures 6 and 7. The shaded region in Figure 6 represents the values of $\bar{c}$ and β such that $\bar{c} > \frac{1-\beta\mu}{\beta\mu}$ so that $\mathrm{MSE}_L > \mathrm{MSE}_F$. From Proposition 5, the shaded region becomes larger as μ approaches to one.

As $0 < \bar{c} < 1$ for $\beta < 1$ and $\bar{c} = 1$ is an MEE for $\beta = 1$, then, provided that β and μ are large enough, it is possible to have an MEE $\bar{c}$ which lies in the critical region where $\mathrm{MSE}_L > \mathrm{MSE}_F$. In Figure 7, for $a > a_{\max}$ we combine the shaded region in Figure 6 with a function representing a unique $\bar{c}$ given values of β.[21] For a large β, there exists some values of the MEE $\bar{c}$ which lie in the shaded region. In Figure 7, the numerical example shows that the function of the MEE intersects the boundary of the shaded region at $\beta = 0.74$.[22] It implies that when the proportion of Type-F firms sufficiently outweighs that of Type-L firms (i.e., $\mu = 0.7$), we can obtain a MEE $\bar{c}$ such that $\mathrm{MSE}_L > \mathrm{MSE}_F$ for $\beta \in (0.74, 1)$. However, when $\mu < 0.5$, the critical region does not exist for any $\beta < 1$. Therefore, when $\mu < 0.5$ and $\beta < 1$, it is impossible to obtain a MEE $\bar{c}$ where $\mathrm{MSE}_L > \mathrm{MSE}_F$.

As noted earlier, the size of the boomerang effect is related to μ. In particular, when the proportion of the Type-F firms becomes large (i.e., a large μ), equation (4) shows that the weight of the misinterpretation error (v_{t-1}) generated by the Type-F firms increases. For $\beta < 0$, v_{t-1} is always misinformation as the price level reacts in the opposite direction. Therefore, when $\beta < 0$, $\mathrm{MSE}_L < \mathrm{MSE}_F$ for all $\mu \in [0, 1]$ and $a \in (0, \infty)$. However, for $\beta > 0$, the price level responds in the same direction as v_{t-1} and it actually becomes “important” information. In

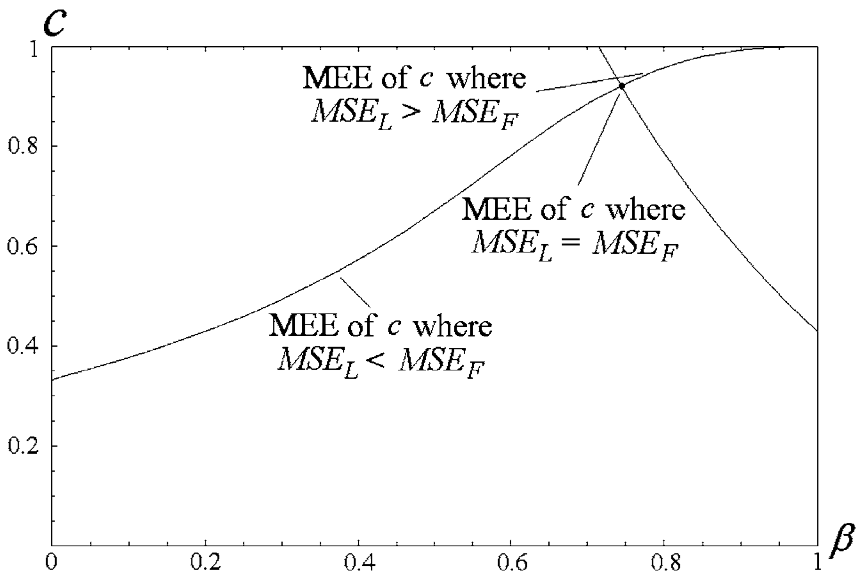

FIGURE 7. The MEE of c satisfying $\text{MSE}_L > \text{MSE}_F$.

fact, given that $\mu > \frac{1}{2}$, for $a \in (0, a_{\max})$ and $\beta \in (\hat{\beta}, \max\{1, \beta_2\})$ or for $a > a_{\max}$ and $\beta \in (\hat{\beta}, 1)$, the temporary equilibrium resembles that of the Type-F PLM. Therefore, Type-F firms actually have more information (through "dumb luck") about the process of the price level due to the form of the ALM. For expectation formation, v_{t-1} seems to be unimportant information except that it feeds directly into the actual law of motion. Consequently, if μ and β are sufficiently close to 1 such that $\bar{c} > \frac{1-\beta\mu}{\beta\mu}$, then v_{t-1} becomes important information for forecasting y_t. The variations of v_{t-1} would have a more significant effect (larger inaccuracy) for the Type-L firms' PLM. Eventually, when learning, the MSE for the Type-L firms would turn out be larger than that for the Type-F firms. This would occur when the proportion of the Type-F firms sufficiently exceeds that of the Type-L firms.

5. CONCLUSION

In this paper, we introduce a process of information diffusion in a modified Muthian cobweb model where agents—firms—form their expectations in accordance with an adaptive learning process. There are two types of firms following a Stackelberg process in the market. The leading firms (Type-L) form initial forecasts while the following firms (Type-F) observe (and use) the leading firms' forecasts (with noise) when forming their own expectations.

In this modified cobweb model, there may exist multiple MEE. However, if the variance of the observational errors is sufficiently small and Type-F firms attempt to minimize their MSE (given their observations of Type-L expectations), then there will exist a unique MEE for $\beta < 1$. When there is a unique MEE for all $\beta < 1$, then the E-stability condition in the modified model is identical to the E-stability condition under homogeneous expectations. However, if the variance of the observational error is sufficiently large, there exists three MEE where the "high" and "low" MEE are E-stable for $\beta < 1$ and the "middle" MEE is E-unstable for $\beta < 1$. Furthermore, if there exist three solutions for some range of $\beta \in (\beta_1, \beta_2)$ where $\beta_2 \in (1, 2)$, then the "low" solution is E-stable for all $\beta \in (\beta_1, \beta_2)$ and the "high" and "middle" solutions are E-unstable for all $\beta \in (1, \beta_2)$. We, therefore, have shown a general result in a well-known model where the E-stability parameter space can actually be expanded through heterogeneity.

It turns out that the inability to fully share information and the inability to observe v_{t-1} by the Type-L firms will cause a "boomerang effect" on the forecasts and forecast efficiency of the Type-L firms. We focus particular attention on the equilibrium properties and forecasting accuracy of the model. We introduce and find evidence for the boomerang effect, which we define as a situation in which the inaccurate forecasts of the Type-F firms confound the Type-L firms' forecasts. Furthermore, the MSE of the Type-L firms can possibly exceed that of the Type-F firms when the proportion of the Type-F firms is larger than the proportion of the Type-L firms.

In the current setting of our model, heterogeneity—the proportion of the Type-L and Type-F firms—is assumed to be exogenous. However, endogenizing heterogeneity would be an important future research challenge [see Brock and Hommes (1997); Evans and Ramey (1998)]. One specific issue to consider is the degree to which heterogeneity exists for alternative types of firms if firms optimally choose to become either Type-L or Type-F firms based on the trade-off between forecast accuracy and the cost of acquiring forecast information.

The framework in this paper also can be extended to monetary policy issues [see Bernanke and Woodford (1997)]. There are implications for the overall performance of an inflation-stabilizing monetary policy [see Granato and Wong (2006)]. If we substitute the public for the Type-F firms and the monetary authority for the Type-L firms and also assume that the information disadvantage resides in the public's limited understanding of economic events, then a plausible consequence (based on our model's findings) is that information diffusion creates a boomerang effect for the policy makers. Because the equilibrium forecasts in an economy are an aggregation of agents' forecasts, a large boomerang effect can cause policy makers themselves to make inaccurate forecasts of economic conditions. The inaccurate forecasts can eventually cause additional economic volatility and failed stabilization policies.[23]

To alleviate the boomerang effect, one normative policy suggestion is that policymakers should be more transparent about policy information. Greater transparency may make it possible for the public to better understand how the policy will work and hence make more accurate use of information from better informed sources.[24] More precision in information acquisition may reduce boomerang effects, improve policy effectiveness, and help with overall economic performance.

NOTES

1. See Evans and Honkapohja (2001) for further details about E-stability.

2. Information diffusion has been documented in many areas of research. For example, financial economists have studied explanations for herding behavior in which rational investors demonstrate some degree of behavioral convergence [see Devenow and Welch (1996)]. Most recently, studies of monetary economics are exploring how information diffusion influences economic forecasting behavior. The standard monetary view from the "credibility" literature holds that policy makers have superior information to citizens [Romer and Romer (2000)] and hence can choose how much information to disseminate for better stabilization outcomes [see Backus and Driffill (1985); Barro and Gordon (1983)].

3. The terms "observational errors," "(mis)interpretation errors," and "(mis)communication errors" are used interchangeably in the text.

4. The MEE concept will be discussed further in the following sections.

5. Lucas's (1972) model shares the same reduced form as (1) with $0 < \beta < 1$. Thus, our findings would also apply to his model.

6. Honkapohja and Mitra (2003) present such a variant in their appendix. This is a model of interrelated markets where the supply of one of the goods is affected by a production lag and the supply of the other good is not. It turns out that $\beta > 0$ if both the demand and the supply curves of the good without the production lag are relatively steep.

7. McCallum (1983, 1999) discusses the MSV concept at length, interpreting it as a fundamental solution that includes no bubble or sunspot components. McCallum proposes a solution procedure that generates a unique solution in a very wide class of linear RE models.

8. Kandel and Zilberfarb (1999) argue that people do not interpret existing information in an identical way. Using Israeli inflation forecast data, they show that the hypothesis of identical-information interpretation is rejected. In addition, Bernanke and Woodford (1997) study "inflation forecast" targeting policy rule where policy makers are assumed to conduct monetary policy by targeting private-sector forecasts of inflation. In their model, they also suggest a similar argument regarding the error misinterpretation by private-sector forecasts. The authors argue that some private-sector agents may be "incompetent" at using their information to produce optimal forecasts (p. 659).

9. An equilibrium value of c is obtained via a linear projection. Hamilton (1994, p. 74) points out that the linear projection gives the smallest MSE among a specific class of forecasting rules.

10. Although MEE is precisely a restricted perceptions equilibrium (RPE) in a heterogeneous expectations setting, we employ the term MEE here because the initial definition of RPE in Evans and Honkapohja (2001) does not have heterogeneous perceptions and MEE was initially introduced to emphasize that. However, it is natural to generalize RPE to incorporate heterogeneous misspecified perceptions. See also Adam (2005), Adam, Evans, and Honkapohja (2006), Branch and Evans (2006), and Guse (Forthcoming).

11. To obtain the MEE coefficient $\bar{b}$, one can solve $\bar{b} = T(\bar{b})$. To obtain $\bar{c}$, one can solve $\bar{c} = T(\bar{c})$ and use the expression of $\bar{b}\gamma = \bar{b}^2(1 - \beta(1 - \mu + \mu\bar{c}))$ from the equilibrium $\bar{b}$.

12. We note here that because the observational errors are not observed by the first-moving agents, the MEE in our model are similar to the exuberance equilibria discussed in Bullard, Evans, and Honkapohja (2005).

13. In Appendix B, we restrict the discussion of equilibria for $\beta > \frac{1}{\mu}$ in Proposition B as there does not exist an E-stable equilibrium when $\beta > \frac{1}{\mu}$.

14. If $a \in (0, a_{\max})$ and $\beta = \beta_1$ or $\beta = \beta_2$, then there exists 2 real MEE as one of the two solutions is repeated (due to an increase in β, two of the solutions have either become real or they are about to become imaginary).

15. Note that $a = \frac{\sigma_x^2}{\sigma_v^2} = \frac{4}{800} = 0.005$ and $a_{\min} = \frac{(1-\mu)\mu^2}{4\gamma^2} = 0.0078$. Given $\beta = 1.05$, we find that $a(\beta) = 0.00556 > a = 0.005$. Therefore, there exist 3 real MEE: $\bar{q}_1 \equiv (\bar{b}_1, \bar{c}_1) = (6.1336, 0.2836)$,

$\bar{q}_2 \equiv (\bar{b}_2, \bar{c}_2) = (10.8685, 0.5542)$, and $\bar{q}_3 \equiv (\bar{b}_3, \bar{c}_3) = (-57.0022, 0.9715)$. As the model is located in Region II in Figure 2, $\bar{q}_1$ is E-stable while $\bar{q}_2$ and $\bar{q}_3$ are E-unstable.

16. In this simulation, we specify the gain as follows: $\lambda_t = \frac{\lambda_{t-1}}{1+\lambda_{t-1}}$ and set $\lambda_1 = 0.02$.

17. We find that the speed of convergence is slow in this simulation. The convergence for all $\beta \in (1, \beta_2)$ cases seem to have a similar pattern.

18. Note that $a = \frac{\sigma_x^2}{\sigma_v^2} = \frac{4}{25} = 0.16$ and $a_{\max} = \frac{27*(0.5)^2*(8-7*0.5)}{2^2*(8+0.5)^3} \approx 0.0124$. As $a > a_{\max}$, there exists a unique MEE.

19. In this simulation, we assume that $\sigma_v^2 = 0$ $(a = \infty)$ and expectations have converged to the REE for $t = \{-\infty, \ldots, 0\}$ (i.e. $b_0 = b^{\text{REE}}$ and $c_0 = 1$). Therefore, Type-F firms perfectly understand the expectations made by Type-L. We assume that after time $t = 0$, there is a permanent shock to the information diffusion process so that $\sigma_v^2 > 0$ $(a \in (0, \infty))$ for $t = \{1, \ldots, \infty\}$.

20. In the following simulation, we specify the gain as follows: $\lambda_t = \frac{\lambda_{t-1}}{1+\lambda_{t-1}}$ and set $\lambda_1 = 0.1$.

21. We assume $\mu = 0.7$, $a = \frac{1}{2}$, and $\gamma = 1$.

22. Therefore, at $\beta = 0.74$, for this MEE, $\text{MSE}_L = \text{MSE}_F$.

23. A similar implication is also suggested by Bomfim (2001).

24. There is research supporting this commonsense suggestion. Bernanke et al. (1999) notes that when information about the plans, objectives, or decisions of the monetary authorities are carefully explained, the public can more easily understand the contents of a policy announcement.

25. Given $\sigma_x^2 > 0$, as $\sigma_v^2 \to \infty$, $a \to 0$, then the function is linear. As discussed in part 1 of Lemma 1, the information from the Type-L firms becomes less useful when σ_v^2 increases. In this case, as $\sigma_v^2 \to \infty$, $a \to 0$, there exists a unique MEE where $c \to 0$ and $b \to \gamma/(1-\beta(1-\mu))$.

26. This is algebraically straightforward. A proof is available upon request.

27. For $\beta = \frac{1}{\mu} + \xi$, such that there are multiple real solutions, one solution is close to 1 while the other solutions are near ∞ and $-\infty$ respectively.

28. We omit this as it can easily be shown graphically.

29. This requires finding the $\bar{\beta}$ such that $c_{TR} = c_L$. One then substitutes this expression into the Cardano function to determine if there exists multiple or a unique real solution under this $\bar{\beta}$. If $\mu \in (0, 1/2)$, then $\bar{\beta} \in (\beta_3, \beta_4)$ and if $\mu \in (1/2, 1)$, then $\bar{\beta} \in (1/\mu, \beta_3)$.

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APPENDIX A

PROOF OF PROPOSITION 1

The model is in equilibrium if the following holds:

$$F(b) = Ab^3 + Bb^2 + Cb + D = 0, \tag{A.1}$$

where $A = (1-\beta)a$, $B = -a\gamma$, $C = [1-\beta(1-\mu)](1-\beta\mu)$, $D = -\gamma(1-\beta\mu)$. We remark that (i) the solution to equation (A.1) also yields a solution for c in view of equation (7) and (ii) that the subsequent arguments consider the nature of solutions to characterize the set of equilibria for b and c and their values in particular cases. According to Cardano's approach [see Nickalls (1993)], there exists a single real root in a cubic function if $G^2 + 4H^3 > 0$, where $G = A^2 D - ABC/3 + 2(B/3)^3$ and $H = AC/3 - (B/3)^2$. We substitute A, B, C, and D into Cardano's solution and get what we refer to as the "Cardano function":

$$R(a, \beta, \mu, \gamma) = G^2 + 4H^3 = \frac{a^3}{729}(4M^3 + a\gamma^2 N^2),$$

where $M = -a\gamma^2 + 3(1-\beta)\{1-\beta[1+\beta\mu(1-\mu)]\}$ and $N = 2(9+a\gamma^2) - 9\beta(4+3\mu-\beta\{2+\mu[5+\mu-\beta(2+\mu)]\})$. Also, let $\delta = -8 + \beta(8+\mu)$.

We now find

$$G^2 + 4H^3 = 0,$$

for $a = 0$, $a = a_1(\beta)$, and $a = a_2(\beta)$, where[25]

$$a_1(\beta) = \frac{1}{8\gamma^2}(1-\beta\mu)\{\delta[1-\beta(1-\mu)] + 27\beta\mu(1-\beta) - \sqrt{\beta\mu\delta^3}\},$$

$$a_2(\beta) = \frac{1}{8\gamma^2}(1-\beta\mu)\{\delta[1-\beta(1-\mu)] + 27\beta\mu(1-\beta) + \sqrt{\beta\mu\delta^3}\}.$$

Note that $a_1(\beta)$ and $a_2(\beta)$ are real for $\beta > \frac{8}{8+\mu}$. Therefore, there exists a unique real MEE if $\beta < \frac{8}{8+\mu}$.

It can be shown[26] that $a_1(\beta)$ is monotonically decreasing in β for $\beta \in (\frac{8}{8+\mu}, 1)$, $a_2(\beta)$ is monotonically decreasing in β for $\beta \in (\frac{8}{8+\mu}, \min(\frac{1}{\mu}, \frac{1}{1-\mu}))$, and $a_2(\beta) > a_1(\beta)$ for all $\beta \in (\frac{8}{8+\mu}, \frac{1}{\mu})$.

One can show that, for $\beta < \frac{1}{\mu}$, the third derivative of the Cardano function with respect to β is positive and the first and second derivatives with respect to β are equal to zero when evaluated at $a = 0$. This implies that the Cardano function is increasing at $a = 0$, decreasing at $a = a_1(\beta)$, and increasing at $a = a_2(\beta)$. Therefore, we can say for $\delta > 0$, there is a single MEE when $a \in [0, a_1(\beta))$ or when $a \in (a_2(\beta), \infty)$, and there are multiple equilibria when $a \in [a_1(\beta), a_2(\beta)]$.

Finally, as $a_1(\beta)$ and $a_2(\beta)$ are monotonically decreasing in β, one can obtain the maximum and minimum a such that there may be multiple equilibria for some β. a_1 is minimized when $\beta = 1$ at $a_1 = 0$ and a_2 is maximized when $\beta = \frac{8}{8+\mu}$ at

$$a_{\max} = \frac{27\mu^2(8 - 7\mu)}{\gamma^2(8 + \mu)^3}.$$

When $\beta = 1$, we have

$$a_2 = a_{\min} = \frac{(1 - \mu)\mu^2}{4\gamma^2}.$$

Now, one can implicitly solve for the $\beta \in [\frac{8}{8+\mu}, 1)$ that gives the solution to $a_1(\beta) = a$ and the $\beta \in [\frac{8}{8+\mu}, \min(\frac{1}{\mu}, \frac{1}{1-\mu}))$ that gives the solution to $a_2(\beta) = a$ for some $a \in (0, a_{\max}]$. Let $\beta_1 \in [\frac{8}{8+\mu}, 1)$ be the solution to

$$a_1(\beta) = a,$$

and $\beta_2 \in [\frac{8}{8+\mu}, \min(\frac{1}{\mu}, \frac{1}{1-\mu}))$ be the solution to

$$a_2(\beta) = a,$$

where $\beta_1 \leq \beta_2$. ■

APPENDIX B

DISCUSSION OF EQUILIBRIA FOR $\beta > \frac{1}{\mu}$

PROPOSITION B. *For $\beta > \frac{1}{\mu}$ and $a > 0$, there are three cases to consider for multiple equilibria*:

1. *If $\beta \in (\frac{1}{\mu}, \beta_3)$, then there are 3 MEE for some $\beta_3 \in (\frac{1}{\mu}, \infty)$.*
2. *If $\beta \in (\beta_3, \beta_4)$, then there exists a single real MEE for some $\beta_4 \in (\max(\frac{1}{\mu}, \frac{1}{1-\mu}), \infty)$ where $\beta_3 < \beta_4$.*
3. *If $\beta > \beta_4$, then there exists 3 MEE.*

Proof. It can be shown that $a_1(\beta) > 0$ is monotonically increasing in β for $\beta > \frac{1}{\mu}$, $a_2(\beta) > 0$ is monotonically increasing in β for $\beta > \max(\frac{1}{\mu}, \frac{1}{1-\mu})$, and $a_1(\beta) > a_2(\beta)$ for all $\beta > \frac{1}{\mu}$. Therefore, using the Implicit Function Theorem, one can obtain a $\beta_3 \in (\frac{1}{\mu}, \infty)$ and a $\beta_4 \in (\max(\frac{1}{\mu}, \frac{1}{1-\mu}), \infty)$ that give the respective results:

$$a_1(\beta_3) = a,$$

$$a_2(\beta_4) = a,$$

where $\beta_3 < \beta_4$. We find when $\beta = \frac{1}{\mu}$ there is a unique solution of $c = 1$. For $\beta \in (\frac{1}{\mu}, \beta_3)$, the Cardano function is negative and thus multiple solutions exist. Call these solutions: $\hat{c}_1 < \hat{c}_2 < \hat{c}_3$. where $\hat{c}_1 \in (-\infty, \frac{\mu-1}{\mu})$, $\hat{c}_2 \in (1, c_H)$, and $\hat{c}_3 \in (c_H, \infty)$ for $\beta \in (\frac{1}{\mu}, \beta_3)$. For $\beta \in (\beta_3, \beta_4)$, the Cardano function is greater than zero and thus there exists a unique real solution, $\hat{c}_1 \in (-\infty, \frac{\mu-1}{\mu})$. For $\beta \in (\beta_4, \infty)$, the Cardano function is less than zero and thus there exists multiple solutions. Here, $\hat{c}_1 \in (-\infty, \frac{\mu-1}{\mu})$, $\hat{c}_2 \in (\frac{\mu-1}{\mu}, c_L)$, and $\hat{c}_3 \in (c_L, 0)$. We finally note that $\lim_{\beta\to\infty} \hat{c}_1 = \lim_{\beta\to\infty} \hat{c}_2 = \frac{\mu-1}{\mu}$, and $\lim_{\beta\to\infty} \hat{c}_3 = 0$. ■

APPENDIX C

PROOF OF LEMMA 1

For (1), consider the MEE for $\bar{c}$:

$$\bar{c} = \frac{\bar{b}^2\sigma_x^2}{\bar{b}^2\sigma_x^2 + (1-\beta\mu)\sigma_v^2}.$$

As $1 - \beta\mu > 0$ for all $\beta < \frac{1}{\mu}$, it must be that $\bar{c} \in (0, 1]$.

For (2), consider the MEE for $\bar{b}$:

$$\bar{b} = \frac{\gamma}{1 - \beta(1 - \mu + \mu\bar{c})}.$$

As $1 - \beta(1 - \mu + \mu\bar{c}) > 0$ for $\beta < 1$, $\bar{b}$ and γ must have the same sign.

For (3a), consider the following function:

$$F(c, \beta) = c(1 - \beta\mu)[1 - \beta + (1 - c)\beta\mu]^2 - a(1 - c)\gamma^2.$$

$F(c, \beta)$ is obtained by inserting the expression for the MEE $\bar{b}$ into the expression for the MEE $\bar{c}$ and multiplying the result by $(1 - \beta + (1 - c)\beta\mu)^2$. The zeros of $F(c, \beta)$ are fixed points to the T-map. Note that the solution(s), $\bar{c}$ is (are) not continuous in β if $(1 - \beta + (1 - c)\beta\mu)^2 = 0$. The solution is not continuous in β for $\beta = 1$ and $\bar{c} = 1$. Furthermore, the Cardano function shows that the MEE are not continuous at the break where $\beta = \frac{1}{\mu}$.[27]

It follows that

$$F_\beta(c, \beta) = -c[1 - \beta - (1 - c)\beta\mu]\{2 - \mu[1 - 2c + 3\beta - 3(1 - c)\beta\mu]\}.$$

This derivative is equal to zero at the following values for c:

$$c = 0$$
$$c = \frac{1 - \beta(1 - \mu)}{\beta\mu}$$
$$c = 1 - \frac{1}{\mu} + \frac{1}{3\beta\mu - 2}.$$

The second solution gives a solution of $\bar{b} = \infty$. If $\bar{c}$ is continuous in β for some range, then when $\bar{c} < \frac{1-\beta(1-\mu)}{\beta\mu}$ for some β in that range, it must be that $\bar{c} < \frac{1-\beta(1-\mu)}{\beta\mu}$ for that entire range. The third solution is undefined if $\beta = \frac{2}{3\mu}$. It can be shown that if $\beta < \frac{2}{3\mu}$, then $F_{\beta,c}(c, \beta) > 0$ for $c = -\infty$ and $c = \infty$ and if $\beta > \frac{2}{3\mu}$, then $F_{\beta,c}(c, \beta) < 0$ for $c = -\infty$ and $c = \infty$. Furthermore,

$$\frac{1 - \beta(1 - \mu)}{\beta\mu} < 1 - \frac{1}{\mu} + \frac{1}{3\beta\mu - 2},$$

for $\beta \in (\frac{2}{3\mu}, \frac{1}{\mu})$. This implies

$$F_\beta(c, \beta) < 0,$$

for $c \in (0, \frac{1-\beta(1-\mu)}{\beta\mu})$. Note also that $\frac{1-\beta(1-\mu)}{\beta\mu} > 1$ for all $\beta < 1$. As a consequence, the function $F(c, \beta)$ is monotonically decreasing in β for $\beta < 1$. One also can show that $F_c(c, \beta) > 0$ for $c = -\infty$ and $c = \infty$ when $\beta < 1$. Therefore, if there is one solution, $\bar{c}$, to equation

$$F(c, \beta) = 0,$$

then it is monotonically increasing in β for $\beta < 1$. Note that because $\bar{c} = 1$ at $\beta = 1$, $\bar{c}$ is monotonically decreasing in β. However, as $\bar{c} = 1$ for $\beta = \frac{1}{\mu}$, it must mean that there is some $\tilde{\beta} \in (1, \frac{1}{\mu})$ such that $c_L < \bar{c} = 1 - \frac{1}{\mu} + \frac{1}{3\beta\mu-2}$. For $\beta \in (\beta_5, \frac{1}{\mu})$, $\bar{c}$ will be monotonically increasing in β.

For (3b), if there are three solutions: $c_1 < c_2 < c_3$, then it follows directly from the discussion above that c_1 and c_3 are monotonically increasing in β while c_2 is monotonically decreasing in β for $\beta < 1$. Consider the solution c_1 for $\beta \in (1, \beta_2)$. One can show that $c_L < \frac{1-\beta(1-\mu)}{\beta\mu}$ for all $\beta \in (\frac{8}{8+\mu}, \beta_2)$, so it follows that $c_1 < \frac{1-\beta(1-\mu)}{\beta\mu}$ for all $\beta \in (\beta_1, \beta_2)$. Therefore, c_1 is monotonically increasing in β for $\beta \in (\beta_1, \beta_2)$. Next, consider the solution c_2 for $\beta \in (1, \beta_2)$. For $\beta = 1$, $c_2 < 1$, and therefore, c_2 is decreasing in β. c_2 will only start to increase if $c_2 = \frac{1-\beta(1-\mu)}{\beta\mu}$. In addition, $\frac{1-\beta(1-\mu)}{\beta\mu}$ is monotonically decreasing in β and that $\frac{1-\beta(1-\mu)}{\beta\mu} = c_L$ when $\beta = \min(\frac{1}{\mu}, \frac{1}{1-\mu})$. As $\beta_2 < \min(\frac{1}{\mu}, \frac{1}{1-\mu})$, it follows that $c_2 < \frac{1-\beta(1-\mu)}{\beta\mu}$ for all $\beta \in (\beta_1, \beta_2)$. Therefore, c_2 is monotonically decreasing in β for $\beta \in (\beta_1, \beta_2)$. Finally, consider the solution c_3. Note that because $c_3 = 1$ at $\beta = 1$, c_3 is monotonically decreasing in β. However, as $c_3 = 1$ for $\beta = \frac{1}{\mu}$, it must mean that there is some $\tilde{\beta} \in (1, \frac{1}{\mu})$ such that $c_L < c_3 = 1 - \frac{1}{\mu} + \frac{1}{3\beta\mu-2}$. For $\beta \in (\beta_5, \frac{1}{\mu})$, c_3 will be monotonically increasing in β. ■

APPENDIX D

PROOF OF PROPOSITION 2

A bivariate ODE is stable if the trace of the Jacobian matrix is less than zero and the determinant of the Jacobian matrix is greater than zero. From (5), the bivariate ODE is

$$\frac{d\phi}{d\tau} = h(\phi) = T(\phi) - \phi.$$

One can solve for the Jacobian matrix:

$$Dh(\phi) = \begin{bmatrix} -1+\beta+(-1+c)\beta\mu & b\beta\mu \\ \dfrac{\sigma_x^2\left[\sigma_v^2\gamma - b^2\sigma_x^2\sigma_v^2 + 2b\sigma_v^2\beta(1-\mu)\right]}{\left(\sigma_v^2+b^2\sigma_x^2\right)^2} & -1+\beta\mu \end{bmatrix}.$$

The Trace of the Jacobian matrix is the following:

$$-2+\beta(1+c\mu).$$

This is negative for $\beta < 1$ as $c \in (0, 1)$ or for $\beta > 1$, it must be that

$$c < c_{TR} = \frac{1}{\mu}\left(\frac{2-\beta}{\beta}\right).$$

We consider case of $\beta > 1$ below. Next, if we enter the following into the determinant:

$$\gamma = b[1-\beta(1-\mu+\mu c)],$$

$$b = \sqrt{\frac{(1-\beta\mu)c\sigma_v^2}{(1-c)\sigma_x^2}},$$

then we get the following expression for the determinant:

$$\frac{\left(\sigma_v^2\right)^2(1-\beta\mu)(1-c\beta\mu)\{1-\beta[1-(1-c)(1-2c)\mu]\}}{(1-c)^2}.$$

For $\beta < \frac{1}{\mu}$, this expression is greater than zero if:

$$\{1-\beta[1-(1-c)(1-2c)\mu]\} > 0.$$

Therefore, when

$$c \in [c_L, c_H],$$

where

$$c_H = \frac{3}{4} + \sqrt{\frac{\beta\mu^2 - 8\mu(1-\beta)}{16\beta\mu^2}},$$

$$c_L = \frac{3}{4} - \sqrt{\frac{\beta\mu^2 - 8\mu(1-\beta)}{16\beta\mu^2}},$$

then the determinant is negative and the given MEE is not E-stable.

Now put c_H into the cubic expression for c. This expression is equal to zero only if

$$a = a_1(\beta),$$

and if one substitutes c_L into the cubic expression for c, this expression is equal to zero only if

$$a = a_2(\beta),$$

where $a_2(\beta) \geq a_1(\beta)$. Note that these are the values of a such that the Cardano function is equal to zero, meaning that the solutions can be in $[c_L, c_H]$ only if there may exist multiple MEE for some β.

First, consider the case of a unique equilibrium. If $\beta \in [\frac{8}{8+\mu}, 1)$ it must be that $c > c_H$ as $c \in [0, 1]$ and is continuous in β. As $c \notin [c_L, c_H]$, the unique MEE is E-stable for all $\beta < 1$. Furthermore, $c = c_H = 1$ for $\beta = 1$ and $c \in (c_L, c_H)$ for $\beta \in (1, \frac{1}{\mu})$. If there exists a unique MEE for $\beta \in (1, \frac{1}{\mu})$, it is not E-stable. Consequently, we can say that if c is unique, then it is E-stable for $\beta < 1$.

Next, consider the case where there are multiple equilibria if $\beta \in [\beta_1, \beta_2]$ where $\beta_1 \in [\frac{8}{8+\mu}, 1)$ and $\beta_2 \in [\frac{8}{8+\mu}, \frac{1}{\mu})$. As imaginary solutions come in conjugate pairs, it must mean that 2 solutions must be equal at $\beta = \beta_1$ and $\beta = \beta_2$. Because each real c is continuous in β for $\beta \in (\beta_1, \beta_2)$, it must be that $\bar{c}_2 = \bar{c}_3 = c_H$ at $\beta = \beta_1$, $\bar{c}_1 = \bar{c}_2 = c_L$ at $\beta = \beta_2$, and $\bar{c}_3 = 1$ when $\beta = 1$. Therefore, for $\beta < \beta_1$, the unique real solution is $\bar{q}_1$ and for $\beta_2 < \beta < \frac{1}{\mu}$, the unique real solution is $\bar{q}_3$. As each solution is continuous in β, it must be that $\bar{c}_2 \in [c_L, c_H]$ for all $\beta \in [\beta_1, \beta_2]$. Therefore, the MEE $\bar{q}_2$ is not E-stable for all $\beta \in [\beta_1, \beta_2]$. Due to continuity, it must also be that $1 > \bar{c}_3 > c_H$ for all $\beta \in (\beta_1, 1)$. If $\beta_2 > 1$, and since $\bar{c}_3 \in (c_L, 1)$ and $c_H > 1$ for $\beta > 1$, $\bar{q}_3$ is not E-stable for $\beta > 1$. When $\beta \in (\beta_2, \frac{1}{\mu})$, $\bar{q}_3$ is the unique real solution. However, by continuity it must be that $c_3 \in (c_L, 1)$ and therefore, the MEE is not E-stable. Finally, for $\bar{q}_1$, it must be that $\bar{c}_1 < c_L$ for all $\beta_1 < \beta < \beta_2$. Therefore, the determinant condition is satisfied; however, if $\beta_2 > 1$, it is possible that the trace condition may fail. Recall that for $\beta > 1$, the trace condition will fail if

$$c < c_{TR} = \frac{1}{\mu}\left(\frac{2-\beta}{\beta}\right).$$

The value of β such that $c_{TR} = c_L$ is the following:

$$\beta_{TR} = \frac{8 + 5\mu - \sqrt{16\mu - 7\mu^2}}{2(2 + 2\mu + \mu^2)}.$$

If $\beta_2 > \beta_{TR}$, then the trace condition for $\bar{q}_1$ may fail. However, one can show that the maximum value for $\beta_2 = \min(\frac{1}{\mu}, \frac{1}{1-\mu}) < \beta_{TR}$ for all $\mu \in [0, 1]$.[28] This means

$$\bar{c}_1 < c_{TR},$$

for all $\beta \in [\beta_1, \beta_2]$. As the trace condition is satisfied and $\bar{c}_1 < c_L$, $\bar{q}_1$ is E-stable for all $\beta \in [\beta_1, \beta_2]$. Therefore, there exists the possibility of an E-stable solution for $\beta > 1$ in the above cobweb model.

Next, we consider the conditions for E-stability when $\beta > 1/\mu$. For $\beta > 1/\mu$, the determinant condition can be simplified to

$$(1 - c\beta\mu)\{1 - \beta[1 - (1 - c)(1 - 2c)\mu]\} < 0.$$

For $\beta > 1/\mu$, we have $c_{TR} < 1/\beta\mu$ and $c_{TR} < c_H$ and thus for a solution to be E-stable, it must be that

$$\bar{c} \in (c_L, c_{TR}).$$

If $c_{TR} < c_L$, then there can not exist an E-stable solution. For $\beta \in (1/\mu, \beta_3)$ we have $\bar{c} \notin (c_L, c_{TR})$ for all three solutions. For $\beta \in (\beta_3, \beta_4)$, as $\hat{c}_1 \in (-\infty, \frac{\mu-1}{\mu})$, it must be that $\hat{c}_1 \notin (c_L, c_{TR})$ as

$$\lim_{\beta \to \infty} c_L > \frac{\mu - 1}{\mu}.$$

Finally, for $\beta \in (\beta_4, \infty)$, the only candidate for E-stability is $\hat{c}_2 \in (c_L, 0)$. However, we have $c_{TR} < c_L$ for $\beta > \beta_4$.[29] $\hat{c}_2$ is E-unstable for $\beta \in (\beta_4, \infty)$, and if $\beta > \frac{1}{\mu}$, all solutions are E-unstable. ■

APPENDIX E

PROOF OF COROLLARY 3

Recall the MEE for b:

$$b = \frac{\gamma}{1 - \beta(1 - \mu + \mu c)}.$$

Under homogeneous expectations ($\mu = 0$), when $\beta > 1$, b and γ will have opposite signs. This destabilizes the system and makes the REE unstable under learning. Recall that we have shown that b and γ share the same sign for $\beta < 1$. Under our heterogeneous framework when $\beta > 1$, b and γ will have the same sign if the following holds:

$$c < c_s = \frac{1 - \beta(1 - \mu)}{\beta\mu}.$$

Next, we show that $c_s > c_L$ for all $\beta \in [\frac{8}{8+\mu}, \min(\frac{1}{\mu}, \frac{1}{1-\mu})]$. Subtract c_s from c_L to get the following:

$$S(\beta) = c_L - c_s = -\frac{4 - \beta[4 - \mu(1 + \sqrt{\delta/\beta\mu})]}{4\beta\mu}.$$

For $\beta \in [\frac{8}{8+\mu}, \min(\frac{1}{\mu}, \frac{1}{1-\mu})]$, this is minimized at $\beta = 1$. One just needs to look at the end points below:

$$S\left(\frac{8}{8+\mu}\right) = -\frac{3}{8},$$

$$S\left(\frac{1}{1-\mu}\right) = 0,$$

and, therefore, $S(\min(\frac{1}{\mu}, \frac{1}{1-\mu})) \leq 0$. The MEE value of b given c_1 can be expressed as

$$b_1 = \frac{\gamma}{1 - \beta(1 - \mu + \mu c_1)}.$$

As $0 < c_1 < c_L < c_s$, it must be the case that

$$\text{sign}(b_1) = \text{sign}(\gamma).$$

As there are no other E-stable MEE for $\beta > 1$, we have shown that there does not exist an E-stable MEE such that

$$\text{sign}(\bar{b}) \neq \text{sign}(\gamma).$$ ■

APPENDIX F

DERIVATIONS OF MSEs

First, we derive the MSE of the leaders. For each forecasting model, i, its corresponding MSE_i is defined as the following:

$$\text{MSE}_i = E\left[\left(y_t - y_{i,t}^e\right)^2\right].$$

For the leaders, we get the following:

$$\text{MSE}_L = E\left[(y_t - \bar{b}x_{t-1})^2\right],$$

where

$$y_t = \{\beta\bar{b}[\mu(\bar{c}-1)+1]+\gamma\}x_{t-1} + \beta\mu\bar{c}v_{t-1} + \eta_t.$$

This can be simplified to

$$\text{MSE}_L = (\beta\mu\bar{c})^2\,\sigma_v^2 + \sigma_\eta^2.$$

The MSE for the followers is expressed as follows:

$$\text{MSE}_F = E\{[y_t - \bar{c}(\bar{b}x_{t-1} + v_{t-1})]^2\}.$$

This can be simplified to

$$\text{MSE}_F = (1-\bar{c})^2\,\bar{b}^2\sigma_x^2 + [(1-\beta\mu)\,\bar{c}]^2\,\sigma_v^2 + \sigma_\eta^2.$$

From the equilibrium for $\bar{c}$, we can find an expression for $\bar{b}^2\sigma_x^2$:

$$\bar{b}^2\sigma_x^2 = \frac{(1-\beta\mu)\,\bar{c}\sigma_v^2}{(1-\bar{c})}.$$

Therefore, we have

$$\text{MSE}_F = \bar{c}\,(1-\beta\mu)\,(1-\beta\mu\bar{c})\,\sigma_v^2 + \sigma_\eta^2.$$

APPENDIX G

PROOF OF PROPOSITION 5

Consider the difference between MSE_L and MSE_F:

$$\begin{aligned}\text{MSE}_L - \text{MSE}_F &= (\beta\mu\bar{c})^2\,\sigma_v^2 - \bar{c}\,(1-\beta\mu)\,(1-\beta\mu\bar{c})\,\sigma_v^2 \\ &= (-\bar{c} + \beta\mu\bar{c}^2 + \beta\mu\bar{c})\sigma_v^2.\end{aligned}$$

If $\beta > 0$, the expression can be positive if

$$\bar{c} > \frac{1-\beta\mu}{\beta\mu}.$$

If $\beta < 0$, the expression can be positive if

$$\bar{c} < \frac{1-\beta\mu}{\beta\mu} < 0.$$

Because $\bar{c} \in (0, 1]$, this cannot occur. ■

APPENDIX H

PROOF OF COROLLARY 4

(1) In our Proposition 5, we show that if

$$c > c_m = \frac{1-\beta\mu}{\beta\mu},$$

then $\mathrm{MSE}_L > \mathrm{MSE}_F$. For $\mu = \frac{1}{2}$, we have $c_m = 1$ at $\beta = 1$ and because c_m is decreasing in μ and $c \in (0, 1)$ for $\beta < 1$, it must mean that $c_m > \bar{c}$ for $\beta < 1$. Therefore, $\mathrm{MSE}_L < \mathrm{MSE}_F$ for $\beta < 1$ and $\mu \in (0, 1/2)$. For $\beta > 1$, there is only one possible E-stable solution, c_1. It can be shown that $c_1 \in (0, c_L)$. Therefore, if $c_m > c_L$ for all $\beta \in (\beta_1, \beta_2)$, then under any c_1, it must be that $\mathrm{MSE}_L < \mathrm{MSE}_F$. We also find that $c_L = c_m$ when

$$\beta = \beta_{m1} = \frac{2}{3\mu - \sqrt{(2-3\mu)\mu}},$$

where $\beta_{m1} > \frac{1}{1-\mu}$ for $\mu < \frac{1}{2}$. As c_L and c_m are decreasing in β, it must mean that $c_L < c_m$ for $\beta < \beta_{m1}$ and $c_L > c_m$ for $\beta > \beta_{m1}$. As $\beta_2 \in (\frac{8}{8+\mu}, \frac{1}{1-\mu})$ for $\mu < \frac{1}{2}$, it must be that $c_1 < c_m$. Therefore, when $\mu < \frac{1}{2}$, $\mathrm{MSE}_L < \mathrm{MSE}_F$ under c_1 for all $\beta \in (\beta_1, \beta_2)$.

(2) As c_m is monotonically decreasing in β, $c_m < 1$ if $\beta\mu > 1/2$, and when $\beta = 1$:

$$c_m = \frac{1-\mu}{\mu}.$$

Assume that $\mu > 1/2$, so we have the following:

$$c_m < 1,$$

for $\beta = 1$. If $\bar{c}$ is unique for all $\beta < \frac{1}{\mu}$, then because $\bar{c}$ is continuous in β and $\bar{c} = 1$ at $\beta = 1$, it must be that there exists a $\hat{\beta} \in (1/2, 1)$ such that $c_m = \bar{c}$. Therefore, for $\hat{\beta} < \beta < 1$, we will have

$$\bar{c} > \frac{1-\beta\mu}{\beta\mu},$$

and thus

$$\mathrm{MSE}_L > \mathrm{MSE}_F.$$

(3) For $\mu > \frac{1}{2}$, $\beta_{m1} < \frac{1}{\mu}$. As $\beta_2 \in (\frac{8}{8+\mu}, \frac{1}{\mu})$ for $\mu > \frac{1}{2}$, there will exist a small $a > 0$, such $\beta_2 > \beta_{m1}$ and therefore, for all $\beta \in (\beta_{m1}, \beta_2)$ it must be that $c_L > c_m$. As c_1 is continuous in β, there must exist a $\beta_{m1} \leq \hat{\beta} < \beta_2$ such that $c_1 = c_m$ and $c_1 > c_m$ for all $\beta \in (\hat{\beta}, \beta_2)$. Therefore, for $\beta \in (\hat{\beta}, \beta_2)$, $\mathrm{MSE}_L > \mathrm{MSE}_F$ under c_1.

(4) Note that c_H (and thus a possible value for c_3) is minimized at $\beta = \frac{8}{8+\mu}$ where $c_H = 3/4$. If $\beta = \frac{8}{8+\mu}$, then $c_m = c_H$ if $\mu = 8/13$. As c_H is increasing in β, and c_m is decreasing in β and μ, it must be that $c_H > c_m$ for all β if $\mu > 8/13$. As $c_3 > c_H$ for all $\beta > \frac{8}{8+\mu}$, it follows that $c_3 > c_m$ for all $\beta > \frac{8}{8+\mu}$. Therefore, $\mathrm{MSE}_L > \mathrm{MSE}_F$ under an E-stable c_3 for $\beta \in (\beta_1, 1)$.

(5) When $\mu = \frac{2}{3}$, the curve representing c_m for all β is tangent to the curve representing c_L. This point of tangency is at $\beta = 1$. As c_m is decreasing in μ, it must be that $c_m < c_L$ for all $\beta > \frac{8}{8+\mu}$ when $\mu \in (\frac{2}{3}, 1]$. Therefore, when $\mu \in (\frac{2}{3}, 1]$, we can choose a small $a > 0$, such that β_2 is sufficiently close to 1. As c_1 is continuous in β for $\beta \in (\beta_1, \beta_2)$, then there must exist a $\hat{\beta}$ such that $c_1 = c_m$ and $c_1 > c_m$ for all $\beta \in (\hat{\beta}, \beta_2)$. Therefore, for $\beta \in (\hat{\beta}, \beta_2)$, $\mathrm{MSE}_L > \mathrm{MSE}_F$ under c_1. ■