Lecture Notes

1 Alesina and Rosenthal (1995)

- The model of economic growth is based on an expectations augmented aggregate supply curve:

$$\hat{y}_t = \hat{y}^n + \gamma\left(\pi_t - \pi_t^e\right) + \epsilon_t,$$

where $\hat{y}_t$ = economic growth rate at time t, $\hat{y}^n$ = natural rate of economic growth, π_t = inflation rate at time t.

- The growth model is extended by including an extra component, called competence, which cannot be observed by voters. Therefore,

$$\epsilon_t = \eta_t + \xi_t,$$

where η_t = the level of competence at time t, and ξ_t = stochastic shocks which are beyond administration control.

- Assume that competence follows an MA(1) process:

$$\eta_t = \mu_t + \rho\mu_{t-1},$$

where $\mu_t \sim \text{iid}\big(0, \sigma_\mu^2\big)$.

- We also assume that voters predict inflation with no systematic errors: $\pi_t^e = \pi_t$. As a result, economic growth performance is associated with voters' uncertainty:

$$\hat{y}_t - \hat{y}^n = \epsilon_t = \eta_t + \xi_t.$$

If the actual economic growth rate $(\hat{y}_t)$ is greater than its natural rate $(\hat{y}^n)$, that is, $\hat{y}_t > \hat{y}^n$. Therefore, $\epsilon_t > 0$, which implies that $\eta_t + \xi_t > 0$.

- However, voters are faced with uncertainty in distinguishing the incumbent's competence (η_t) from the stochastic economic shocks (ξ_t). Since competence can persist, voters use this property for making forecasts:

$$\begin{aligned}
\hat{y}_t - \hat{y}^n &= \epsilon_t = \eta_t + \xi_t \\
&\Rightarrow \epsilon_t = \mu_t + \rho\mu_{t-1} + \xi_t \\
\Rightarrow \hat{y}_t - \hat{y}^n - \rho\mu_{t-1} &= \mu_t + \xi_t \\
\Rightarrow \mu_t + \xi_t &= \hat{y}_t - \hat{y}^n - \rho\mu_{t-1}.
\end{aligned}$$

- The previous equation suggests that the votes observe the composite shock $(\mu_t + \xi_t)$ based on the observable variables: $\hat{y}_t$, $\hat{y}^n$, and μ_{t-1}. Therefore, voters optimally forecast the level of competence for the next period based on the observable factors:

$$\begin{aligned}
E_t\left(\eta_{+1}\right) &= E_t\left(\mu_{t+1} + \rho\mu_t\right) \\
&= E_t\left(\mu_{t+1}\right) + \rho E\left(\mu_t \left| \hat{y}_t - \hat{y}^n - \rho\mu_{t-1}\right.\right) \\
&= E_t\left(\mu_{t+1}\right) + \rho E\left(\mu_t \left| \mu_t + \xi_t\right.\right) \\
&= \rho E\left(\mu_t \left| \mu_t + \xi_t\right.\right),
\end{aligned}$$

where $E_t\mu_{t+1} = 0$.

- According to the method of recursive projection, we can show that:

$$E\left(\mu_t \left| \mu_t + \xi_t\right.\right) = \frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma_\xi^2}\left(\mu_t + \xi_t\right). \text{ [WHY?]}$$

Since $\mu_t + \xi_t = \hat{y}_t - \hat{y}^n - \rho\mu_{t-1}$, we have:

$$\begin{aligned}
E_t\eta_{t+1} &= \rho E\left(\mu_t \left| \mu_t + \xi_t\right.\right) \\
&= \rho\frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma_\xi^2}\left(\mu_t + \xi_t\right) \\
&= \rho\frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma_\xi^2}\left(\hat{y}_t - \hat{y}^n - \rho\mu_{t-1}\right).
\end{aligned}$$

How to derive $E\left(\mu_t \left|\mu_t + \xi_t\right.\right) = \frac{\sigma_\mu^2}{\sigma_\mu^2+\sigma_\xi^2}\left(\mu_t + \xi_t\right)$?

Voters forecast μ_t according to the following linear projection function:

$$E\left(\mu_t \left|\mu_t + \xi_t\right.\right) = P\left(\mu_t \left|\mu_t + \xi_t\right.\right) = a_0 + a_1\left(\mu_t + \xi_t\right),$$

where:

$$\begin{aligned} a_1 &= \frac{cov\left(\mu_t, \mu_t + \xi_t\right)}{var\left(\mu_t + \xi_t\right)} \\ &= \frac{E\left[\mu_t\left(\mu_t + \xi_t\right)\right]}{E\left[\left(\mu_t + \xi_t\right)\left(\mu_t + \xi_t\right)\right]} \\ &= \frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma_\xi^2}, \end{aligned}$$

where $E\left(\mu_t\xi_t\right) = 0$; $E\left(\mu_t\mu_t\right) = \sigma_\mu^2$, and $E\left(\xi_t\xi_t\right) = \sigma_\xi^2$; and

$$a_0 = E\left(\mu_t\right) - a_1 E\left(\mu_t + \xi_t\right) = 0,$$

where $E\left(\mu_t\right) = E\left(\xi\right) = 0$. As a result, we can show that:

$$\begin{aligned} E\left(\mu_t \left|\mu_t + \xi_t\right.\right) &= a_0 + a_1\left(\mu_t + \xi_t\right) \\ &= 0 + \frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma_\xi^2}\left(\mu_t + \xi_t\right) \\ &= \frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma_\xi^2}\left(\mu_t + \xi_t\right). \end{aligned}$$

2 Clarke and Granato (2004)

- Clarke and Granato (2004) present a model with 3 equations:

$$\begin{aligned} M_t &= a_1 M_{t-1} + a_2 E_{t-1} M_t + a_3 F_t + u_{1t}, \text{ where } a_1, a_2 > 0 \text{ and } a_3 = 1 && (1) \\ F_t &= b_1 F_{t-1} + b_2 A_t + u_{2t} && (2) \\ A_t &= c_1 A_{t-1} + c_2\left(M_t - M^*\right) + c_3 F_{t-1} && (3) \end{aligned}$$

  Now we plug equation (3) into equation (2), we have:

$$\begin{aligned} F_t &= b_1 F_{t-1} + b_2\left[c_1 A_{t-1} + c_2\left(M_t - M^*\right) + c_3 F_{t-1}\right] + u_{2t} \\ \Rightarrow F_t &= b_1 F_{t-1} + b_2 c_1 A_{t-1} + b_2 c_2\left(M_t - M^*\right) + b_2 c_3 F_{t-1} + u_{2t} \\ \Rightarrow F_t &= \left(b_1 + b_2 c_3\right) F_{t-1} + b_2 c_1 A_{t-1} + b_2 c_2\left(M_t - M^*\right) + u_{2t} && (4) \end{aligned}$$

- Plug equation (4) into equation (1):

$$\begin{aligned} M_t &= a_1 M_{t-1} + a_2 E_{t-1} M_t + F_t + u_{1t} \\ M_t &= a_1 M_{t-1} + a_2 E_{t-1} M_t + \left[\left(b_1 + b_2 c_3\right) F_{t-1} + b_2 c_1 A_{t-1} + b_2 c_2\left(M_t - M^*\right) + u_{2t}\right] + u_{1t} \\ M_t &= a_1 M_{t-1} + a_2 E_{t-1} M_t + \left(b_1 + b_2 c_3\right) F_{t-1} + b_2 c_1 A_{t-1} + b_2 c_2 M_t - b_2 c_2 M^* + u_{2t} + u_{1t} \\ \left(1 - b_2 c_2\right) M_t &= a_1 M_{t-1} + a_2 E_{t-1} M_t + \left(b_1 + b_2 c_3\right) F_{t-1} + b_2 c_1 A_{t-1} - b_2 c_2 M^* + u_{2t} + u_{1t} \\ M_t &= -\left(\frac{b_2 c_2}{1 - b_2 c_2}\right) M^* + \left(\frac{a_1}{1 - b_2 c_2}\right) M_{t-1} + \left(\frac{a_2}{1 - b_2 c_2}\right) E_{t-1} M_t + \left(\frac{b_2 c_1}{1 - b_2 c_2}\right) A_{t-1} + \\ &\quad \left(\frac{b_1 + b_2 c_3}{1 - b_2 c_2}\right) F_{t-1} + \frac{u_{2t} + u_{1t}}{1 - b_2 c_2} \\ M_t &= \Theta_0 + \Theta_1 M_{t-1} + \Theta_2 E_{t-1} M_t + \Theta_3 A_{t-1} + \Theta_4 F_{t-1} + \epsilon_t^*, && (5) \end{aligned}$$

  where $\Theta_0 = -\left(\frac{b_2 c_2}{1 - b_2 c_2}\right) M^*$, $\Theta_1 = \frac{a_1}{1 - b_2 c_2}$, $\Theta_2 = \frac{a_2}{1 - b_2 c_2}$, $\Theta_3 = \frac{b_2 c_1}{1 - b_2 c_2}$, $\Theta_4 = \frac{b_1 + b_2 c_3}{1 - b_2 c_2}$, and $\epsilon_t^* = \frac{u_{2t} + u_{1t}}{1 - b_2 c_2}$.

- To solve the rational expectations equilibrium (REE), we first form an expectation of equation (5) at $t-1$, we have:

$$\begin{aligned} E_{t-1}M_t &= \Theta_0 + \Theta_1 M_{t-1} + \Theta_2 E_{t-1}M_t + \Theta_3 A_{t-1} + \Theta_4 F_{t-1} + E_{t-1}\epsilon_t^* \\ \Rightarrow (1-\Theta_2) E_{t-1}M_t &= \Theta_0 + \Theta_1 M_{t-1} + \Theta_3 A_{t-1} + \Theta_4 F_{t-1} \\ E_{t-1}M_t &= \frac{\Theta_0}{1-\Theta_2} + \frac{\Theta_1}{1-\Theta_2} M_{t-1} + \frac{\Theta_3}{1-\Theta_2} A_{t-1} + \frac{\Theta_4}{1-\Theta_2} F_{t-1}, \qquad (6) \end{aligned}$$

where $E_{t-1}\epsilon_t^* = 0$.

- Plug equation (6) into equation (5):

$$\begin{aligned} M_t &= \Theta_0 + \Theta_1 M_{t-1} + \Theta_2 E_{t-1}M_t + \Theta_3 A_{t-1} + \Theta_4 F_{t-1} + \epsilon_t^* \\ M_t &= \Theta_0 + \Theta_1 M_{t-1} + \Theta_2 \left(\frac{\Theta_0}{1-\Theta_2} + \frac{\Theta_1}{1-\Theta_2} M_{t-1} + \frac{\Theta_3}{1-\Theta_2} A_{t-1} + \frac{\Theta_4}{1-\Theta_2} F_{t-1} \right) + \\ &\quad \Theta_3 A_{t-1} + \Theta_4 F_{t-1} + \epsilon_t^* \\ M_t &= \frac{\Theta_0}{1-\Theta_2} + \frac{\Theta_1}{1-\Theta_2} M_{t-1} + \frac{\Theta_3}{1-\Theta_2} A_{t-1} + \frac{\Theta_4}{1-\Theta_2} F_{t-1} + \epsilon_t^*. \qquad (7) \end{aligned}$$

- Note that equations (2) and (3) are written as follows:

$$F_t = b_1 F_{t-1} + b_2 A_t + u_{2t},$$

and

$$A_t = c_1 A_{t-1} + c_2 (M_t - M^*) + c_3 F_{t-1}.$$

Take one period backward, we have:

$$F_{t-1} = b_1 F_{t-2} + b_2 A_{t-1} + u_{2t-1}, \qquad (8)$$

and

$$A_{t-1} = c_1 A_{t-2} + c_2 (M_{t-1} - M^*) + c_3 F_{t-2}. \qquad (9)$$

Now plug (9) into (8):

$$\begin{aligned} F_{t-1} &= b_1 F_{t-2} + b_2 [c_1 A_{t-2} + c_2 (M_{t-1} - M^*) + c_3 F_{t-2}] + u_{2t-1} \\ F_{t-1} &= (b_1 + b_2 c_3) F_{t-2} + b_2 c_1 A_{t-2} + b_2 c_2 M_{t-1} - b_2 c_2 M^* + u_{2t-1}. \qquad (10) \end{aligned}$$

- Now plug equations (9) and (10) into (7):

$$\begin{aligned} M_t &= \frac{\Theta_0}{1-\Theta_2} + \frac{\Theta_1}{1-\Theta_2} M_{t-1} + \frac{\Theta_3}{1-\Theta_2} [c_1 A_{t-2} + c_2 (M_{t-1} - M^*) + c_3 F_{t-2}] + \\ &\quad \frac{\Theta_4}{1-\Theta_2} [(b_1 + b_2 c_3) F_{t-2} + b_2 c_1 A_{t-2} + b_2 c_2 M_{t-1} - b_2 c_2 M^* + u_{2t-1}] + \epsilon_t^* \\ M_t &= \left(\frac{\Theta_0}{1-\Theta_2} - \frac{\Theta_3}{1-\Theta_2} c_2 - \frac{\Theta_4}{1-\Theta_2} b_2 c_2 \right) M^* + \\ &\quad \left(\frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2} c_2 + \frac{\Theta_4}{1-\Theta_2} b_2 c_2 \right) M_{t-1} + \\ &\quad \left(\frac{\Theta_3}{1-\Theta_2} c_1 + \frac{\Theta_4}{1-\Theta_2} b_2 c_1 \right) A_{t-2} + \\ &\quad \left[\frac{\Theta_3}{1-\Theta_2} c_3 + \frac{\Theta_4}{1-\Theta_2} (b_1 + b_2 c_3) \right] F_{t-2} + \\ &\quad \frac{\Theta_4}{1-\Theta_2} u_{2t-1} + \epsilon_t^* \\ M_t &= \Pi_1 + \Pi_2 M_{t-1} + \Pi_3 A_{t-2} + \Pi_4 F_{t-2} + \xi_t'. \end{aligned}$$

- Therefore, we can see that:

$$\Pi_2 \equiv \frac{\Theta_1}{1-\Theta_2} + \frac{\Theta_3}{1-\Theta_2} c_2 + \frac{\Theta_4}{1-\Theta_2} b_2 c_2$$

$$
\begin{aligned}
\Pi_2 &= \frac{a_1/(1-b_2c_2)}{1-[a_2/(1-b_2c_2)]} + \frac{b_2c_1/(1-b_2c_2)}{1-[a_2/(1-b_2c_2)]}c_2 + \frac{(b_1+b_2c_3)/(1-b_2c_2)}{1-[a_2/(1-b_2c_2)]}b_2c_2 \\
\Pi_2 &= \frac{a_1}{1-b_2c_2-a_2} + \frac{b_2c_1c_2}{1-b_2c_2-a_2} + \frac{(b_1+b_2c_3)\,b_2c_2}{1-b_2c_2-a_2} \\
\Pi_2 &= \frac{a_1+b_2c_2(c_1+b_1+b_2c_3)}{1-b_2c_2-a_2} \\
\Pi_2 &= (a_1+b_2c_2(c_1+b_1+b_2c_3))(1-b_2c_2-a_2)^{-1}.
\end{aligned}
$$

- Take the derivative of Π_2 with respect to c_2 :

$$
\begin{aligned}
\frac{\partial \Pi_2}{\partial c_2} &= b_2(c_1+b_1+b_2c_3)(1-b_2c_2-a_2)^{-1} + \\
&\quad [a_1+b_2c_2(c_1+b_1+b_2c_3)](-1)(1-b_2c_2-a_2)^{-2}(-b_2) \\
\frac{\partial \Pi_2}{\partial c_2} &= \frac{b_2(c_1+b_1+b_2c_3)}{1-b_2c_2-a_2} + \frac{b_2[a_1+b_2c_2(c_1+b_1+b_2c_3)]}{(1-b_2c_2-a_2)^2} \\
\frac{\partial \Pi_2}{\partial c_2} &= \frac{b_2(c_1+b_1+b_2c_3)(1-b_2c_2-a_2)+b_2[a_1+b_2c_2(c_1+b_1+b_2c_3)]}{(1-b_2c_2-a_2)^2} \\
\frac{\partial \Pi_2}{\partial c_2} &= \frac{b_2[(c_1+b_1+b_2c_3)(1-b_2c_2-a_2)+a_1+b_2c_2(c_1+b_1+b_2c_3)]}{(1-b_2c_2-a_2)^2} \\
\frac{\partial \Pi_2}{\partial c_2} &= \frac{b_2[(c_1+b_1+b_2c_3)(1-b_2c_2-a_2+b_2c_2)+a_1]}{(1-b_2c_2-a_2)^2} \\
\frac{\partial \Pi_2}{\partial c_2} &= \frac{b_2[(c_1+b_1+b_2c_3)(1-a_2)+a_1]}{(1-b_2c_2-a_2)^2} \\
\frac{\partial \Pi_2}{\partial c_2} &= \frac{b_2[a_1+(1-a_2)(c_1+b_1+b_2c_3)]}{(1-b_2c_2-a_2)^2} \\
\frac{\partial \Pi_2}{\partial c_2} &= \frac{b_2[a_1+(1-a_2)A]}{(1-b_2c_2-a_2)^2},
\end{aligned}
$$

where $A \equiv c_1+b_1+b_2c_3$.