Chapter 5

Strategists and Party Identification

An important debate in political science centers on the persistence of party identification, termed *macropartisanship* (See Erikson, MacKuen, and Stimson (2002: 109-151)). Using Clarke and Granato's (2004) example of an EITM formulation it is assumed political campaign advertisements influence the public's party identification. In particular, party identification persistence can be influenced by a rival political party strategist's use of campaign advertisements. Consequently, shocks to macropartisanship can either be amplified or die out quickly depending on the rival political strategist's actions.

The EITM linkage is the relation between the behavioral concept of expectations and the empirical concept of persistence. Empirical tools for this example require a basic understanding of autoregressive processes. Formal tools include an extended discussion of conditional expectations modeling, difference equations (various orders), their solution procedures, and relevant stability conditions.

5.1 Step 1: Relating Behavioral and Applied Statistical Concepts: Expectations and Persistence

Clarke and Granato (2004) relate agent expectations to the persistence of agent behavior. It is demonstrated how a rival political strategist can use campaign advertisements to influence aggregate persistence in party identification.

The model is based on three equations. Each citizen (i) is subject to an event (j) at time (t). Clarke and Granato then aggregate across individuals and events so the notation will only have the subscript t.

$$M_t = a_1 M_{t-1} + a_2 E_{t-1} M_t + a_3 F_t + u_{1t} \tag{5.1.1}$$

$$F_t = b_1 F_{t-1} + b_2 A_t + u_{2t} \tag{5.1.2}$$

$$A_t = c_1 A_{t-1} + c_2 \left(M_t - M^*\right) + c_3 F_{t-1}. \tag{5.1.3}$$

The first equation (5.1.1) specifies what influences aggregate party identification (M_t). The variable M_{t-1} accounts for the empirical concept of persistence. The behavioral concept in the model is citizen expectations. It is assumed citizens have an expectation of what portion of the population identifier with a particular political party $(E_{t-1}M_t)$. In forming their expectations, citizens use all available and relevant information (up to time $t-1$) as specified in this model (i.e., rational expectations).[1] Further party identification depends on how favorably a citizen views the national party (F_t). Finally, party identification can be subject to unanticipated stochastic shocks (realignments) (u_{1t}) where $u_{1t} \sim N\left(0, \sigma^2_{u_{1t}}\right)$. These relations are assumed to be positive — $a_1, a_2, a_3 \geq 0$. Without loss of generality, we assume $a_3 = 1$ in this case.

Equation (5.1.2) represents citizens' impression and sense of favorability about a political party (F_t). In this equation, favorability is a linear function of the lag of favorability (F_{t-1}) and an advertising resource variable (A_t). u_{2t} is a stochastic shock representing unanticipated events (uncertainty), where $u_{2t} \sim N\left(0, \sigma^2_{u_{2t}}\right)$. The parameter $b_1 \geq 0$, while $b_2 \gtreqless 0$ depending on the tone and content of the advertisement.

Equation (5.1.3) presents the contingency plan or rule that (rival) political strategists use. Clarke and Granato posit that political strategists track their previous period's advertising resource expenditures (A_{t-1}) and react to that period's

[1] Rational expectations is only one type of expectation modeling. It has particular implications for how fast citizen's adjust to new information, which in this case is political adverstisements. See the Appendix, Section 5.5.2 for a discussion on the speed of adjustment.

favorability rating for the (rival) national party (F_{t-1}). The strategists also base their current expenditure of advertisement resources on the degree to which macropartisanship (M_t) approximates a prespecified and desired target (M^*).

Ideally, political strategists want $(M_t - M^*) = 0$. The parameters c_1 and c_3 are positive. The parameter c_2 is countercyclical ($-1 \leq c_2 < 0$): it reflects a willingness to increase or conserve their advertising resources depending on whether macropartisanship is above or below the target.

5.2 Step 2: Analogues for Expectations and Persistence[2]

The reduced form for macropartisanship is determined by substituting (5.1.3) into (5.1.2). Note that there is an autoregressive component ($\Theta_1 M_{t-1}$) in the reduced form for macropartisanship:

$$M_t = \Theta_0 + \Theta_1 M_{t-1} + \Theta_2 E_{t-1} M_t + \Theta_3 A_{t-1} + \Theta_4 F_{t-1} + \varepsilon_t^*, \tag{5.2.1}$$

where:

$$\Theta_0 = -\frac{b_2 c_1 M^*}{1 - b_2 c_2},$$

$$\Theta_1 = \frac{a_1}{1 - b_2 c_2},$$

$$\Theta_2 = \frac{a_2}{1 - b_2 c_2},$$

$$\Theta_3 = \frac{b_2 c_1}{1 - b_2 c_2},$$

$$\Theta_4 = \frac{b_1 + b_2 c_3}{1 - b_2 c_2},$$

$$\varepsilon_t^* = \frac{u_{2t} + u_{1t}}{1 - b_2 c_2}.$$

The system is simplified to a model of macropartisanship that depends on lagged macropartisanship and also a conditional expectation at time $t-1$ of current macropartisanship. This lagged dependent variable is the analogue for *persistence* (See the Appendix, Section 5.5.1). Note that the prior values of advertising and favorability may also have an effect.

Because (5.2.1) possesses a conditional expectations operator we must make it a function of other variables (not operators) (See the Appendix, Section 5.5.2). In this example, "closing the model" and finding the rational expectations equilibrium (REE) involves taking the conditional expectation at time $t-1$ of equation (5.2.1) and then substituting this result back into equation (5.2.1):

$$M_t = \Pi_1 + \Pi_2 M_{t-1} + \Pi_3 A_{t-2} + \Pi_4 F_{t-2} + \xi_t'. \tag{5.2.2}$$

Equation (6.3.1) is the minimum state variable (MSV) solution (McCallum, 1983) for macropartisanship.[3] Macropartisanship (M_t) depends also on its past history, the autoregressive component, (M_{t-1}).

5.3 Step 3: Unifying and Evaluating the Analogues

The persistence of macropartisanship (Π_2) is now shown as dependent on the persistence and willingness of rival political strategists to maintain a rival macropartisanship target (c_2). In other words, the EITM linkage is the MSV with the AR(1) component in (6.3.1).

The linkage is this case is the reduced form AR(1) coefficient expression, Π_2:

$$\Pi_2 = \frac{a_1 + b_2 c_2 (c_1 + b_1 + b_2 c_3)}{1 - b_2 c_2 - a_2}. \tag{5.3.1}$$

[2]In this example, the applied statistical analogue is an autoregressive process and the formal analogue is conditional expectations. The description of theses analogues and the tools to develop these analogues can be found in the Appendix, Sections 5.5.1, and 5.5.2.

[3]Note: $\Pi_1 = \left(\frac{\Theta_0}{1-\Theta_2} - \left[\frac{\Theta_3}{1-\Theta_2} - \frac{\Theta_4}{1-\Theta_2} b_2\right] c_2 Y^*\right)$, $\Pi_2 = \left(\frac{\Theta_1}{1-\Theta_2} + \left[\frac{\Theta_3}{1-\Theta_2} + \frac{\Theta_4}{1-\Theta_2} b_2\right] c_2\right)$,
$\Pi_3 = \left(\left[\frac{\Theta_3}{1-\Theta_2} + \frac{\Theta_4}{1-\Theta_2} b_2\right] c_1\right)$, $\Pi_4 = \left(\frac{\Theta_3}{1-\Theta_2} c_3 + \frac{\Theta_4}{1-\Theta_2} [b_1 + b_2 c_3]\right)$, and $\xi_t' = \left(\frac{\Theta_4}{1-\Theta_2} u_{2t} + \varepsilon_t^*\right)$.

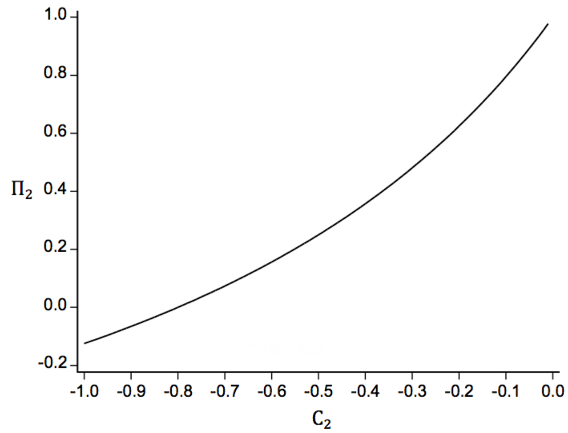

Figure 5.3.1: Simulation Results

Taking the derivative of (5.3.1) with respect to (c_2) and finding the following relation we have:

$$\frac{\partial \Pi_2}{\partial c_2} = \frac{b_2 \left(a_1 + (1 - a_2) A\right)}{\left(1 - a_2 - b_2 c_2\right)^2}, \tag{5.3.2}$$

where $A = (b_1 + c_1 + b_2 c_3)$. Given the assumptions about the signs of the coefficients in the model, the numerator is positive when $a_2 < 1$. Therefore, under these conditions, the relation is positive $\left(\frac{\partial \Pi_2}{\partial c_2} > 0\right)$.[4]

The relation between c_2 and Π_2 is demonstrated in Figure 5.3.1. Using the following values: $a_1 = a_2 = b_1 = b_2 = c_1 = c_3 = 0.5$. The parameter c_2 ranges from 0.0 to -1.0. The value of c_2 is varied between 0.0 and -1.0 and we find that the persistence (autocorrelation) in macropartisanship (Π_2) — all things equal — is zero when $c_2 = -0.8$.

On the other hand, macropartisanship persistence increases ($\Pi_2 \rightarrow 1.0$) when rival political strategists fail to react ($c_2 \rightarrow 0.0$) to deviations from their prespecified target. *A conclusion derived from this model is that negative advertisements from rival political parties can influence the persistence of their opponents national party identification.*

5.4 Leveraging EITM and Extending the Model

Among the ways to extend the model is to use an alternative way to model citizen expectations. In this model the use of RE can limit the complexities of expectation formation. Alternatives could include the use of expectations formation where the public updates at a far slower pace and using information sets that are far more limited. There is also the question of data. Currently, it is difficult to measure and link specific advertisements to response in real time. One way to deal with this particular design concern is to use experiments and ascertain the treatment and response effects with lags of far shorter duration.

5.5 Appendix

To assess the degree of persistence in a variable, autoregressive estimation is the most frequently used technique in empirical research. However, in EITM, persistence is behaviorally based. Persistence can be due to many things including brand loyalty and party loyalty. Persistence might also arise because of habit formation, meaning choosing an option in the present period directly increases the probability of choosing it again in future periods.[5]

While the example in this chapter is about party identification — and an appropriate province of political science — the foundations for developing the tools to model expectations is drawn from economics. Since Muth (1961), a great deal of theoretical research uses RE. RE is a particular equilibrium concept representing the optimal choice of the decision rule used by agents depending on the choices of others. "An RE equilibrium (REE) imposes a consistency condition that each agents' choice is a best response to the choices of others" (Evans and Honkapohja 2001: 11) . Muth (1961) defined expectations to be rational if they were formed according to the model describing the behavior.

[4]Note, for $\frac{\partial \Pi_2}{\partial c_2} > 0 : a_2 > \frac{-A}{1-A}$ is the *necessary condition* and $a_2 < 1$is the *sufficient condition.*

[5]See Shachar (1992) for the role of habit formation in voting decisions.

It is also possible to relate RE to autoregressive processes. Let there be a time series (z_t) generated by a first-order autoregression:

$$z_t = \lambda_0 + \lambda_1 z_{t-1} + \nu_t, \quad for\ z_{t-1} \in I_{t-1} \tag{5.5.1}$$

where ν_t are independent $N(0,\, \sigma_\nu^2)$ random variables, $|\lambda_1| < 1$, and I_{t-1} represents all possible information at period $t-1$. If the agent acts rationally, the equation (5.5.1) is treated as the data generating process (DGP). The mathematical expression of RE corresponding to equation (5.5.1) is:

$$E[z_t|z_{t-1}] = z_t^e = \lambda_0 + \lambda_1 z_{t-1}. \tag{5.5.2}$$

With this simple linkage in mind, the tools in this chapter are used to establish a transparent and testable relation between expectations and persistence. The applied statistical tools provide a basic understanding of:

- Autoregressive processes.

The formal tools include a presentation of:

- Conditional expectations (naive, adaptive, and rational).
- Difference equations.
- Method of undetermined coefficients (minimum state variable procedure).

These tools are used in various applications for models where RE is assumed just as in this chapter.

5.5.1 Empirical Analogues

Autoregressive Processes

An autoregressive process for the time series of Y_t is one in which the current value of Y_t depends upon past values of Y_t and a stochastic disturbance term.[6] A convenient notation for an autoregressive process is $AR(p)$, where p denotes the maximum lag of Y_t , upon which Y_t depends. Note that in an $AR(p)$ process, lags are assumed to be present from 1 through to p.

For simplicity, we now use $AR(1)$ for illustration. An $AR(1)$ process represents a first-order autoregressive process where Y_t depends upon Y_{t-1} and a disturbance term, ε_t:

$$Y_t = \phi_1 Y_{t-1} + \varepsilon_t, \tag{5.5.3}$$

where ε_t is a white noise that has zero mean, constant variance, and zero autocorrelation. The autoregressive parameter ϕ_1 in equation (5.5.3) can take on values with distinct empirical implications bearing on the persistence of a process. In particular, if $\phi_1 > 1$, the process is explosive meaning Y_t will grow without limit.

Random Walk Processes and the Persistence of Random Shocks

A special case arises when $\phi_1 = 1$. In this case, equation (5.5.3) can be written as:

$$Y_t = Y_{t-1} + \varepsilon_t. \tag{5.5.4}$$

Equation (5.5.4) is termed a "pure random walk" process. A pure random walk process is a best guess of Y_{t+1}, given information at period t, is Y_t. The relation between a random walk process and the "persistence" of random shocks is also of importance.

[6] Time series data are discretely ordered by some period. They differ from cross-sectional data in that unlike their cross-sectional cousin, time series are a sequence of data points of the same entity over a period of time. For political science, examples include presidential approval and macropartisanship, while in economics, many macroeconomic data, such as gross domestic product and unemployment rates, are time series. A key property of time series data is stationarity. The consequences of having stationary processes is not trivial. In fact, it is a crucial requirement in that, among other things, most probability moments — the mean, the variance — and all the constituent statistics that derive from these moments are based on the assumption of a stationary time series. No valid inference is achievable absent some assurance that the data and model are stationary. The reason is that non-stationary data affects the moments (mean, variance, for example) of the series and these moments are used in all sorts of inferential statistics such as the t- and F-test. With this in mind, an intuitive definition for stationarity is:

> A data series (or model) is stationary if there is no systematic change in the mean (e.g., no trend), no systematic stochastic variation, and if strict periodic variations (seasonal) are stable. Time plays no role in the sample moments.

To see this relation, consider an $AR(1)$ process with a unit root, $\phi_1 = 1$ (i.e., equation (5.5.4)). If the pure random walk process of equation (5.5.4) starts at $t = 1$, the process then is $Y_1 = Y_0 + \varepsilon_1$. In the next period, $t = 2$, the process is $Y_2 = Y_1 + \varepsilon_2 = (Y_0 + \varepsilon_1) + \varepsilon_2$. Generalizing:

$$Y_t = Y_0 + \sum_{i=1}^{t} \varepsilon_t. \tag{5.5.5}$$

Equation (5.5.5) indicates the impact of a particular shock persists and will never die out. Also, it can be demonstrated from equation (5.5.5) that the mean value of Y_t wanders over time.

Now, using equation (5.5.3), trace how persistence evolves if we have a stationary process, $0 < \phi_1 < 1$. The result is the shock does die out over time. The process is also mean reverting. To illustrate this, consider an $AR(1)$ process with $\phi_1 = 0.5$:

$$Y_t = 0.5Y_{t-1} + \varepsilon_t. \tag{5.5.6}$$

If we start at $t = 1$, the process is: $Y_1 = 0.5Y_0 + \varepsilon_1$. In successive periods we have:

$$\begin{aligned} Y_2 &= 0.5Y_1 + \varepsilon_2 \\ &= 0.5\left(0.5Y_0 + \varepsilon_1\right) + \varepsilon_2 \\ &= 0.5^2 Y_0 + 0.5\varepsilon_1 + \varepsilon_2, \end{aligned}$$

and

$$\begin{aligned} Y_3 &= 0.5Y_2 + \varepsilon_3 \\ &= 0.5\left[0.5^2 Y_0 + 0.5\varepsilon_1 + \varepsilon_2\right] + \varepsilon_3 \\ &= 0.5^3 Y_0 + 0.5^2\varepsilon_1 + 0.5\varepsilon_2 + \varepsilon_3. \end{aligned}$$

In general:

$$Y_t = 0.5^t Y_0 + \sum_{i=1}^{t} 0.5^{t-i} \varepsilon_i. \tag{5.5.7}$$

Equation (5.5.7) indicates the effect of a particular shock, say at period 1, on all the subsequent periods does die out when $t \to \infty$ (i.e, the shock is not persistent).[7]

5.5.2 Formal Analogues

Conditional Expectations

The use of expectations in economic models has a long history. While RE is featured in this particular chapter, there are many ways to model expectations and each method has distinct behavioral implications. Here background is provided on three approaches:

1. Naive or static expectations.
2. Adaptive expectations.
3. Rational expectations.

The solution procedures for RE are then presented. Because the development of expectations modeling was largely the creation of economics, the variables and examples are economic in nature. We stay true to those original examples and the variables used. However, as this chapter demonstrates, the application of these tools can be used for any social science question where expectations are a behavioral and theoretical component.

[7] Autoregressive processes can be estimated using ordinary least squares (OLS). See Box and Jenkins (1970, 1976), for an extensive discussion on the estimation of autoregressive processes. A comprehensive discussion of time series methods can be found in Hamilton (1994). Also, Johnston and DiNardo (1996) provide a basic framework in time series methods, within the broader context of econometric methods.

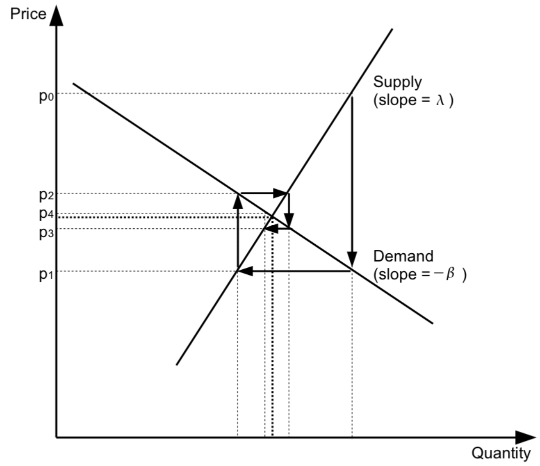

Figure 5.5.1: Cobweb Model with Static Expectation Formation

Static Expectations: The Cobweb Model[8]

Static expectations (also called naive expectations) assume agents form their expectations of a variable based on their previous period $(t-1)$ observation of the variable. An example illustrating the use of static expectations is the traditional cobweb model which was used to determine the dynamic process of prices in agricultural markets.

The cobweb model consists of demand and supply curves, respectively:

$$q_t^d = \alpha - \beta p_t + \epsilon_t^d, \tag{5.5.8}$$

and:

$$q_t^s = \gamma + \lambda p_t^e + \epsilon_t^s, \tag{5.5.9}$$

where $\beta > 0$, $\lambda > 0$, $\alpha > \gamma > 0$. $\epsilon_t^d \sim iid\left(0, \sigma^2_{\epsilon^d}\right)$ and $\epsilon_t^s \sim iid\left(0, \sigma^2_{\epsilon^s}\right)$ are stochastic demand and supply shocks with zero mean and constant variance, respectively.

Equation (5.5.8) is a demand schedule where consumers decide the level of quantity demanded $\left(q_t^d\right)$ given the current price level in the market (p_t) and other stochastic factors $\left(\epsilon_t^d\right)$ at time t. From equation (5.5.9), we assume producers make decisions on the production level (q_t^s) based on *the expected price level* at time t, p_t^e. Since the actual market price p_t is not revealed to producers until goods have been produced in the market, producers make a decision on the level of production by forecasting the market price.

The market equilibrium, where $q_t^d = q_t^s$, gives us the dynamic process of the price level:

$$p_t = \left[\frac{\alpha - \gamma}{\beta}\right] - \left(\frac{\lambda}{\beta}\right) p_t^e + \left[\frac{\epsilon_t^d - \epsilon_t^s}{\beta}\right]. \tag{5.5.10}$$

Equation (5.5.10) is called the *cobweb model*: the current price level (p_t) depends on the expected price level (p_t^e) and a composition of stochastic shocks. Producers form static expectations where they choose the level of production q_t^s at time t by observing the previous price level at time $t-1$ (i.e., $p_t^e = p_{t-1}$). Substituting $p_t^e = p_{t-1}$ into equation (5.5.10):

$$p_t = \left[\frac{\alpha - \gamma}{\beta}\right] - \left(\frac{\lambda}{\beta}\right) p_{t-1} + \left[\frac{\epsilon_t^d - \epsilon_t^s}{\beta}\right]. \tag{5.5.11}$$

Equation (5.5.11) shows the current price level is determined by the past price level and stochastic shocks. Since the initial price level p_t is not in a stationary equilibrium, the price approaches the equilibrium p^* in the long run when certain conditions exist. In this model, $\left|\frac{\lambda}{\beta}\right| < 1$: $\lim_{t\to\infty} p_t = p^*$. The converging process is shown in Figure (5.5.1).

The Use of Difference Equations

The result in Figure 5.5.1 can be demonstrated using stochastic difference equations. Equation (5.5.11) is also called a stochastic first-order difference equation with a constant. The equation (5.5.2) can be presented in a simpler form:

$$p_t = a + bp_{t-1} + e_t, \tag{5.5.12}$$

[8]See Enders (2009) for background material for the following sections.

where $a = \frac{\alpha-\gamma}{\beta}$, $b = -\frac{\lambda}{\beta}$, and $e_t = \frac{\epsilon_t^d - \epsilon_t^s}{\beta}$. To see the sequence of the price level we solve by iteration. Assuming the initial price level is $p_{t=0} = p_0$, the price level at time $t = 1$ is:

$$p_1 = a + bp_0 + e_1.$$

Using the above equation, we solve for p_2:

$$\begin{aligned} p_2 &= a + bp_1 + e_2 \\ &= a + b\left(a + bp_0 + e_1\right) + e_2 \\ &= a + ab + b^2 p_0 + be_1 + e_2. \end{aligned}$$

With a similar substitution, p_3 is:

$$\begin{aligned} p_3 &= a + bp_2 + e_3 \\ &= a + b\left(a + ab + b^2 p_0 + be_1 + e_2\right) + e_3 \\ &= a + ab + ab^2 + b^3 p_0 + b^2 e_1 + be_2 + e_3. \end{aligned} \quad (5.5.13)$$

If we iterate the equation n times, we have (for $n \geq 1$):

$$p_n = a\sum_{i=0}^{n-1} b^i + b^n p_0 + \sum_{i=0}^{n-1} b^i e_{n-i}, \quad (5.5.14)$$

and by extension we can show if $t = n = 3$, then:

$$\begin{aligned} p_3 &= a\sum_{i=0}^{3-1} b^i + b^3 p_0 + \sum_{i=0}^{3-1} b^i e_{3-i} \\ &= a\sum_{i=0}^{2} b^i + b^3 p_0 + \sum_{i=0}^{2} b^i e_{3-i} \\ &= a\left(b^0 + b^1 + b^2\right) + b^3 p_0 + \left(b^0 e_{3-0} + b^1 e_{3-1} + b^2 e_{3-2}\right) \\ &= a + ab + ab^2 + b^3 p_0 + e_3 + be_2 + b^2 e_1. \end{aligned} \quad (5.5.15)$$

Note equations (5.5.13) and (5.5.15) are identical.

Using equation (5.5.14), the current price level p_t depends on the initial level p_0 and the sequence of stochastic shocks $\{e_i\}_{i=1}^t$. Assuming $|b| < 1$, then $\lim_{n\to\infty} b^n = 0$, and $\lim_{n\to\infty}\left(b^0 + b^1 + \cdots + b^n\right) = \lim_{n\to\infty}\sum_{i=0}^n b^i = \frac{1}{1-b}$. Therefore, in the long run, the price level equals:

$$p_{n\to\infty} = \frac{a}{1-b} + \sum_{i=0}^{\infty} b^i e_{n-i}. \quad (5.5.16)$$

Equations (5.5.14) and (5.5.16) show transitory and stationary levels of price, respectively. Using the previous parameter values in the cobweb model: $a = \frac{\alpha-\gamma}{\beta} = 10$, and $b = -\frac{\lambda}{\beta} = -0.8$, we replicate Figure (5.5.2). Based on equations (5.5.14) and (5.5.16) by assuming $p_0 = p^* = 5.56$, $e_1 = 4.44$, and $e_i = 0$ for $i > 1$ we have:

$$\begin{aligned} p_n &= 10\sum_{i=0}^{n-1}(-0.8)^i + (-0.8)^n p_0 + \sum_{i=0}^{n-1}(-0.8)^i e_{n-i} \\ &= 10\sum_{i=0}^{n-1}(-0.8)^i + (-0.8)^n(5.56) + (-0.8)^{n-1}(4.44). \end{aligned}$$

As $t = n = \infty$, we have:

$$p_{n\to\infty} = p^* = 10\sum_{i=0}^{\infty}(-0.8)^i = \frac{10}{1-(-0.8)} = 5.56,$$

where:

$$\lim_{n\to\infty}(-0.8)^n = 0$$

and:

$$\lim_{n\to\infty}\left[(-0.8)^0 + (-0.8)^1 + \cdots + (-0.8)^n\right] = \lim_{n\to\infty}\sum_{i=0}^{n}(-0.8)^i = \frac{1}{[1-(-0.8)]}.$$

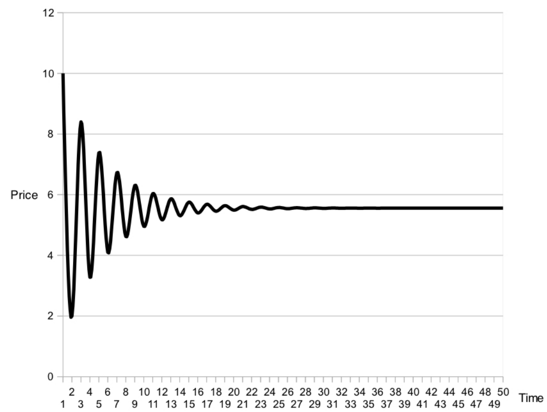

Figure 5.5.2: Price Movements in the Cobweb Model

Expectational Errors: Speed of Adjustment

An important issue in expectations modeling is the speed of adjustment. Naive or static expectation models contain agents who are relatively slow to adjust and update their forecasts. The movement of price over time is assumed by the constant terms and slope coefficients for demand and supply: $\alpha = 20$, $\gamma = 10$, $\beta = 1$, $\lambda = 0.8$, and $|\lambda/\beta| = 0.8 < 1$. Recall the stationary equilibrium p^* is 5.56. At time $t = 1$ assume there is a stochastic shock — to either demand or supply (or both). This moves the price level from $p^* = 5.56$ to $p_1 = 10$. In Figure (5.5.2), we see that the price level fluctuates and approaches the equilibrium $p^* = 5.56$ in the long run.

Intuitively, if there is a one-time shock that shifts the demand or supply curve (or both) producers are assumed to *passively* determine the current level of production by observing the previous price level. A surplus or shortage, would exist while the market price deviated from the equilibrium until $t \to \infty$.

The behavioral implication when agents "naively" form expectations based on the past period's observation are as follows: agents systematically forecast above or below the actual value for an *extensive time period*. McCallum (1989) terms this sluggishness in error correction: *systematic expectational errors*.

Adaptive Expectations

Of course, agents can actively revise their expectations when they realize their forecasting mistakes. This alternative formation of expectations is called adaptive expectations. The revision of current expectations is a function of the difference between the actual observation and the past expectation:

$$\text{Expectational Revision} = \text{Function (Actual Observation - Past Expectation)}.$$

If agents make forecast errors in the previous period, then they revise their current expectations. Mathematically, adaptive expectations can be written as:

$$p_t^e - p_{t-1}^e = (1 - \theta)\left(p_{t-1} - p_{t-1}^e\right), \tag{5.5.17}$$

where $0 < \theta < 1$ represents the degree (or speed) of expectational revision. When $\theta = 0$, $p_{t-1}^e = p_{t-1}$, agents do not revise their expectations: they have static expectations. On the other hand, if $\theta = 1$, $p_t^e = p_{t-1}^e$, then agents form their current expectations (p_t^e) based on the past expectations $\left(p_{t-1}^e\right)$ only. By arranging equation (5.5.17), we have:

$$p_t^e = \theta p_{t-1}^e + (1 - \theta)\, p_{t-1}. \tag{5.5.18}$$

Equation (5.5.18) also shows that the expectation of the current price (p_t^e) is the weighted average of past expectation $\left(p_{t-1}^e\right)$ and the past observation (p_{t-1}). To recover the expected price level at time t, the method of iterations is applied. The expectations at time $t-1$ and $t-2$ is:

$$p_{t-1}^e = \theta p_{t-2}^e + (1 - \theta)\, p_{t-2}, \tag{5.5.19}$$

and:

$$p_{t-2}^e = \theta p_{t-3}^e + (1 - \theta)\, p_{t-3}, \tag{5.5.20}$$

respectively. Substituting (5.5.20) into (5.5.19) and then substituting it back to equation (5.5.18), we have:

$$\begin{aligned} p_t^e &= \theta\left\{\theta\left[\theta p_{t-3}^e + (1-\theta)\,p_{t-3}\right] + (1-\theta)\,p_{t-2}\right\} + (1-\theta)\,p_{t-1} \\ &= \theta^3 p_{t-3}^e + \theta^2(1-\theta)\,p_{t-3} + \theta(1-\theta)\,p_{t-2} + (1-\theta)\,p_{t-1}. \end{aligned} \tag{5.5.21}$$

Iterating equation (5.5.18) n times:

$$p_t^e = \theta^n p_{t-n}^e + (1-\theta)\sum_{i=1}^{n-1}\theta^{i-1}p_{t-i}.$$

If $n \to \infty$, then:

$$p_t^e = (1-\theta)\sum_{i=1}^{\infty}\theta^{i-1}p_{t-i}, \tag{5.5.22}$$

for $|\theta| < 1$.

Equation (5.5.18) shows the current expectation is the weighted average of the last period expectation and observations. An alternative interpretation based on equation (5.5.22) is that the expected price level for the current period is a weighted average of *all price levels* observed in the past (with geometrically declining weights).

Under adaptive expectations, agents make their forecast of a variable by weighting its past behavior (Cagan 1956; Friedman 1957; Nerlove 1958). However, just as with the assumption of static expectations, *systematic expectational errors* can still be generated. Unexpected stochastic shocks have *permanent* effects on future expectations formation.

This result is inconsistent with central tenets in microeconomic theory. If agents know that such errors are systematically generated, they have incentives to avoid them. For example, agents have the incentive to collect other (or even *all* available) information for improving the forecast of the observed variable. Theoretically, one way to avoid the problem of having agents make systematic errors is to assume they have rational expectations (RE) (Muth 1961; Lucas 1972, 1973).

Rational Expectations

Under RE, agents are assumed to take conditional (mathematical) expectations of *all* relevant variables. Agents form their expectations according to all of the information available at time t. The behavioral implications are very different from static or adaptive expectations when it comes to the speed of correcting forecast errors. RE also has very different implications for persistence.

Mathematically, RE can be written as the projection:

$$p_{t+j}^e = E\left(p_{t+j}\left|I_t\right.\right), \tag{5.5.23}$$

where p_{t+j}^e is the subjective expectations of p_{t+j} formed in time t, and $E\left(p_{t+j}\left|I_t\right.\right)$ is a mathematical expectations of p_{t+j} given the information I_t available at time t. Statistically, $E\left(p_{t+j}\left|I_t\right.\right)$ is interpreted as the mean of the conditional probability distribution of p_{t+j} based on available information I_t at time t. Equation (5.5.23) implies agents use *all* information available at time t to forecast the variable of interest for time $t+j$.

More importantly, agents' ability to form a conditional probability distribution of p_{t+j} also implies that agents "know" the structure of the model. For example, agents are able to form a conditional distribution of p_t given the parameters of α, β, γ, and λ as known in equation 5.5.10. It is difficult to imagine agents can know the "true" model in the first place and then construct a probability distribution based on the model.[9]

As mentioned earlier, systematic expectational errors are generated when agents form their adaptive expectations given the related information available. These systematic expectational errors can be eliminated under RE. Defining the expectational error as the difference between actual observation at time $t+1$ and the expectation for time $t+1$:

$$p_{t+1} - p_{t+1}^e.$$

If agents systematically *over*-predict or *under*-predict the variable of interest, then the "average" of the expectational errors is either larger than or less than zero. Under RE there is no systematic forecast error.

To demonstrate this result under RE we calculate the expected value of the expectational errors as:

$$\begin{aligned} E\left(p_{t+1} - p_{t+1}^e\right) &= E\left[p_{t+1} - E\left(p_{t+1}\left|I_t\right.\right)\right] \\ &= E\left(p_{t+1}\right) - E\left[E\left(p_{t+1}\left|I_t\right.\right)\right] \\ &= E\left(p_{t+1}\right) - E\left(p_{t+1}\right) = 0, \end{aligned} \tag{5.5.24}$$

[9] An alternative assumption is agents "learn" the structure of the model over time by least squares to form optimal conditional expectations (Bray 1982; Bray and Savin 1986; Evans 1985; Marcet and Sargent 1986, 1987; Evans and Honkapohja 2001). This is called the *adaptive learning* approach and is discussed in chapters 6 and 7.

where $E\left[E\left(p_{t+1}\left|I_t\right.\right)\right] = E\left(p_{t+1}\right)$ is the unconditional expectation of the conditional expectations of p_{t+1}, given the information set I_t. This is simply the unconditional expectation of p_{t+1}.[10] This result can also be explained by a statistical property called *the law of iterated expectations* [11]

The law of iterated expectations suggests that, given an information set Ω and a subset of information $\omega \subset \Omega$, for a variable of interest x, the conditional expectations of the conditional expectations of x, given a larger information set is just the conditional expectations of x given a subset of information.[12] Mathematically:

$$E\left[E\left(x\left|\Omega\right.\right)\left|\omega\right.\right] = E\left(x\left|\omega\right.\right). \tag{5.5.33}$$

If the conditional expectation of x is formed *over time* according to the available information, then equation (5.5.33) is rewritten as:

$$E\left[E\left(x_{t+1}\left|I_t\right.\right)\left|I_{t-1}\right.\right] = E\left(x_{t+1}\left|I_{t-1}\right.\right),$$

where $I_{t-1} \subset I_t$ for all t.

[10]To show $E\left[E\left(p_{t+1}\left|I_t\right.\right)\right] = E\left(p_{t+1}\right)$, it is necessary to review some important statistical properties. Let us generalize the statements below and use a variable X. Assume X is a random variable where its numerical values are randomly determined. For the discrete case, the variable X has a set of J random numeral values, $x_1, x_2, \ldots, x_J$. The probability of any numerical value, x_j, for $j = 1, 2, \ldots, J$, can be represented by a probability density function $f\left(x_j\right) = Prob\left\{X = x_j\right\} \geq 0$. Note that the sum of the probability for all possible numerical values is $\sum_{j=1}^{J} f\left(x_j\right) = 1$, and $f\left(x_k\right) = 0$, for any $x_k \notin \left\{x_1, \ldots x_J\right\}$. Based on the density function, we calculate the (unconditional) expected value of the random variable X:

$$E\left(X\right) = \sum_{j=1}^{J} x_j f\left(x_j\right). \tag{5.5.25}$$

If $g\left(x_j\right)$ is defined as a function for any random value x_j for all $j = 1, 2, \ldots, J$, then the expected value of $g\left(X\right)$ is:

$$E\left[g\left(X\right)\right] = \sum_{j=1}^{J} g\left(x_j\right) f\left(x_j\right). \tag{5.5.26}$$

Assuming there is another random variable Y which has a set of M random values, $y_1, y_2, \ldots, y_M$. Assuming further that X and Y are jointly distributed random variables such that the joint probability density function is $f\left(x_j, y_m\right) = Prob\left\{X = x_j \text{ and } Y = y_m\right\} \geq 0$, for $j = 1, 2, \ldots J$, and $m = 1, 2, \ldots, M$. Again, note $\sum_{m=1}^{M}\sum_{j=1}^{J} f\left(x_j, y_m\right) = 1$, and $f\left(x_k, y_h\right) = 0$, for any $x_k \notin \left\{x_1, \ldots x_J\right\}$ or $y_h \notin \left\{y_1, \ldots y_M\right\}$. Based on the joint density function $f\left(x_j, y_m\right)$, the single density function can be calculated for the random variable X by summing up all joint probability of $f\left(x_j, y_m\right)$ for any given x_j:

$$f\left(x_j\right) = \sum_{m=1}^{M} f\left(x_j, y_m\right). \tag{5.5.27}$$

Similarly, the single density function can be derived for the random variable Y:

$$f\left(y_m\right) = \sum_{j=1}^{J} f\left(x_j, y_m\right). \tag{5.5.28}$$

In addition, if there is a multivariate function $g\left(x_j, y_m\right)$, then the expected value of $g\left(X, Y\right)$ is:

$$E\left[g\left(X, Y\right)\right] = \sum_{m=1}^{M}\sum_{j=1}^{J} g\left(x_j, y_m\right) f\left(x_j, y_m\right). \tag{5.5.29}$$

The last statistical property introduced is the conditional probability density function. This is defined as the conditional probability density function of y given x (subscripts are dropped for convenience) as:

$$f\left(y|x\right) = \frac{f\left(x, y\right)}{\sum_y f\left(x, y\right)} = \frac{f\left(x, y\right)}{f\left(x\right)}, \tag{5.5.30}$$

for $f\left(x\right) > 0$. As before, the conditional probability density function is the same form but now it is of x given y:

$$f\left(x|y\right) = \frac{f\left(x, y\right)}{\sum_x f\left(x, y\right)} = \frac{f\left(x, y\right)}{f\left(y\right)}, \tag{5.5.31}$$

for $f\left(y\right) > 0$. Equation (5.5.30) shows the probability of any numerical value y_m given a specific value of a random variable X. Therefore, we define the conditional expected value of Y given X as:

$$E\left(Y|X\right) = \sum_y y f\left(y|x\right). \tag{5.5.32}$$

Based on the above statistical properties, we are able to show that $E\left[E\left(Y|X\right)\right] = E\left(Y\right)$ by using the fact that $E\left[g\left(X, Y\right)\right] = \sum_y \sum_x g\left(x, y\right) f\left(x, y\right)$ in equation (5.5.29) and assuming that $E\left(Y|X\right) = \sum_y y f\left(y|x\right) = g\left(x, y\right)$ in equation (5.5.32). All we need to show is that $E\left[E\left(Y|X\right)\right] = E\left[g\left(X|Y\right)\right] = E\left(Y\right)$. This result validates $E\left[E\left(p_{t+1}\left|I_t\right.\right)\right] = E\left(p_{t+1}\right)$ in condition (5.5.24).

[11]Note that the expectation operator, $E\left(\cdot\right)$, is in linear form. The ideas of recursive expectations and the law of iterated expectations are demonstrated in the discussin of recursive projections in Chapter 4, Appendix, Section 4.5.2.

[12]See Wooldridge (2008), Appendix B for an introductory discussion of conditional expectations.

The second implication of RE is that the expectational errors are uncorrelated with any information available at time t: any information available to agents to form expectations at time t *does not systematically* generate forecast errors. To demonstrate this result under RE, consider any information, w_t, where $w_t \in I_t$:

$$\begin{aligned} E\left[\left(p_{t+1}-p_{t+1}^{e}\right)w_t\right] &= E\left[\left(p_{t+1}-E\left(p_{t+1}\left|I_t\right.\right)\right)w_t\right] \\ &= E\left(p_{t+1}w_t\right)-E\left[E\left(p_{t+1}\left|I_t\right.\right)w_t\right] \\ &= E\left(p_{t+1}w_t\right)-E\left(p_{t+1}w_t\right)=0, \end{aligned}$$

where $E\left[E\left(p_{t+1}\left|I_t\right.\right)w_t\right]=E\left[E\left(w_t p_{t+1}\left|I_t\right.\right)\right]=E\left(p_{t+1}w_t\right)$ and can be shown using the law of iterated expectations.

Solving Rational Expectations Models

The solution procedures for RE models require a different approach.[13] RE models do not rely merely on a mathematical expectation, which is a summary measure (expected value). Rather, RE models are based on conditional expectations, which is a mathematical expectation with a modified probability distribution ("information set"). Solution procedures involve "closing the model" where unknown variables (i.e., expectations) are expressed in terms of other "known" variables. The method of undetermined coefficients is a particular solution process that closes a model, and the minimum state variable (MSV) solution is the simplest solution when using the method of undetermined coefficients.

Application 1: A Simple Cobweb Model

The solution(s) for RE models include a REE. The REE imposes a consistency condition that an agent's choice is a best response to the choices made by others (Evans and Honkapohja 2001: 11). A simple way to demonstrate an REE is to use the cobweb model presented in equation (5.5.10). This particular equation shows that the movements of the price level at time t, (i.e., p_t) depend on the RE of the price level form at $t-1$, $\left(p_t^e = E\left(p_t|I_{t-1}\right)\right)$, and a composite stochastic error term, e_t:

$$p_t = a + bE\left(p_t|I_{t-1}\right) + e_t, \tag{5.5.34}$$

where $a=\frac{\alpha-\gamma}{\beta}$, $b=-\frac{\lambda}{\beta}<0$, and $e_t=\frac{\epsilon_t^d-\epsilon_t^s}{\beta}$. Agents "know" the model when they form their conditional expectations so their expectations can be written as:

$$\begin{aligned} E\left(p_t|I_{t-1}\right) &= E\left\{\left[a+bE\left(p_t|I_{t-1}\right)+e_t\right]|I_{t-1}\right\} \\ &= E\left(a|I_{t-1}\right)+E\left\{\left[bE\left(p_t|I_{t-1}\right)\right]|I_{t-1}\right\}+E\left(e_t|I_{t-1}\right) \\ &= a+bE\left[E\left(p_t|I_{t-1}\right)|I_{t-1}\right] \\ &= a+bE\left(p_t|I_{t-1}\right), \end{aligned} \tag{5.5.35}$$

where $E\left(a|I_{t-1}\right)=a$, $E\left[E\left(p_t|I_{t-1}\right)|I_{t-1}\right]=E\left(p_t|I_{t-1}\right)$, and $E\left(e_t|I_{t-1}\right)=0$.[14]

Following equation (5.5.35), the right-hand-side expression of $E\left(p_t|I_{t-1}\right)$ is moved to the left hand side of the equation:

$$(1-b)\,E\left(p_t|I_{t-1}\right)=a,$$

and $E\left(p_t|I_{t-1}\right)$ is equal to:

$$E\left(p_t|I_{t-1}\right)=\frac{a}{1-b}. \tag{5.5.36}$$

Equation (5.5.36) shows that agents form their conditional expectations of p_t using the structural parameters a and b. Inserting equation (5.5.36) into equation (5.5.34) yields:

$$\begin{aligned} p_t &= a+bE\left(p_t|I_{t-1}\right)+e_t \\ &= a+b\left[\frac{a}{1-b}\right]+e_t \\ &= \frac{a\,(1-b)+ab}{1-b}+e_t \\ p_t^{RE} &= \frac{a}{1-b}+e_t. \end{aligned} \tag{5.5.37}$$

[13] See Enders (2009), Chapter 1 for an introduction to these tools.

[14] These identities are based on the following. Since agents know the structure of the model, that is, the parameters of a and b, the existing information set would not affect the parameter values. Therefore, $E\left(a|I_{t-1}\right)=a$. We can show, using the law of iterated expectations, that $E\left[E\left(p_t|I_{t-1}\right)|I_{t-1}\right]=E\left(p_t|I_{t-1}\right)$. Intuitively, if an agent forms an expectation of a conditional expectation (based on the same information set), the conditional expectation does not change since there is no added information. Lastly, the conditional expectational of a stochastic error term is zero, $\left(E\left(e_t|I_{t-1}\right)=0\right)$, since an agent is unable to "forecast" white noise, e_t, given past information I_{t-1}.

Equation (5.5.37) is the REE and shows the movements of the price level over time given the RE in equation (5.5.36).

Furthermore, equations (5.5.36) and (5.5.37) also suggest the agents have made an optimal forecast in the model since the expectational error is simply stochastic noise:

$$p_t - E\left(p_t|I_{t-1}\right) = \left(\frac{a}{1-b} + e_t\right) - \frac{a}{1-b} = e_t. \tag{5.5.38}$$

The average "expected value" of the expectational effects is zero:

$$E\left[p_t - E\left(p_t|I_{t-1}\right)\right] = E\left(e_t\right) = 0.$$

Application 2: A Cobweb Model with Observable Variables

In the previous section, it was demonstrated that the variable of interest — the price level at time t — depends on its conditional expectations and a composite stochastic error term in equation (5.5.34). Assuming there are other observable variable(s), w_{t-1}, influencing the quantity supplied in equation (5.5.9):

$$q_t^s = \gamma + \lambda p_t^e + \delta w_{t-1} + \epsilon_t^s. \tag{5.5.39}$$

For convenience E_{t-1} is used as an expectation operator to represent the conditional expectations given information available at time $t-1$. The conditional expectations of price level at time t given the information available at time $t-1$ is written as:

$$E_{t-1}p_t = E\left(p_t|I_{t-1}\right). \tag{5.5.40}$$

In general, the conditional expectations of p_t given the information available at $t-j$ can be written as:

$$E_{t-j}p_t = E\left(p_t|I_{t-j}\right),$$

for all j.

To solve for the reduced form of the price level, both equations (5.5.39) and (5.5.8) are set equal to each other:

$$p_t = \left[\frac{\alpha-\gamma}{\beta}\right] - \left(\frac{\lambda}{\beta}\right)p_t^e - \left(\frac{\delta}{\beta}\right)w_{t-1} + \left[\frac{\epsilon_t^d - \epsilon_t^s}{\beta},\right]$$

or:

$$p_t = a + bp_t^e + dw_{t-1} + e_t',$$

where $a = \frac{\alpha-\gamma}{\beta}$, $b = -\frac{\lambda}{\beta}$, $d = -\frac{\delta}{\beta}$, and $e_t' = \frac{\epsilon_t^d - \epsilon_t^s}{\beta}$. The RE price at time t is written as:

$$p_t^e = E\left(p_t|I_{t-1}\right) = E_{t-1}p_t.$$

Therefore, the revised model of the price level is:

$$p_t = a + bE_{t-1}p_t + dw_{t-1} + e_t'. \tag{5.5.41}$$

Equation (5.5.41) is very similar to equation (5.5.34). But, equation (5.5.41) shows the price level, p_t, now depends on an extra observable variable, w_{t-1}. To solve for the REE, conditional expectations of both sides in equation (5.5.41) are taken:

$$\begin{aligned} E_{t-1}p_t &= E_{t-1}\left(a + bE_{t-1}p_t + dw_{t-1} + e_t'\right) \\ &= E_{t-1}a + E_{t-1}\left(bE_{t-1}p_t\right) + E_{t-1}\left(dw_{t-1}\right) + E_{t-1}e_t' \\ &= E_{t-1}a + bE_{t-1}\left(E_{t-1}p_t\right) + dE_{t-1}w_{t-1} + E_{t-1}e_t'. \end{aligned} \tag{5.5.42}$$

Note $E_{t-1}a = a$, $E_{t-1}\left(E_{t-1}p_t\right) = E_{t-1}p_t$, $E_{t-1}w_{t-1} = w_{t-1}$, and $E_{t-1}e_t' = 0$, thus we have:

$$\begin{aligned} E_{t-1}p_t &= a + bE_{t-1}p_t + dw_{t-1} \\ E_{t-1}p_t &= \frac{a}{1-b} + \frac{d}{1-b}w_{t-1}. \end{aligned} \tag{5.5.43}$$

Now substituting equation (5.5.43) into equation (5.5.41) and solving for the REE:

$$\begin{aligned} p_t &= a + b\left(\frac{a}{1-b} + \frac{d}{1-b}w_{t-1}\right) + dw_{t-1} + e_t' \\ p_t^{RE} &= \frac{a}{1-b} + \frac{d}{1-b}w_{t-1} + e_t'. \end{aligned} \tag{5.5.44}$$

Equation (5.5.44) is the REE where the price level depends on a constant term, an observable variable, w_{t-1}, and a composite stochastic error term, e_t'.

Application 3: The Cagan Hyperinflation Model

A well-known model, with implications for RE, is the Cagan Hyperinflation model (Cagan 1956) that describes the fundamental relation between the aggregate price level and the money supply. To begin, assume the quantity of real money demanded $(m_t - p_t)^d$ depends on the expected change in the price level:

$$(m_t - p_t)^d = \alpha - \beta\left(E_t p_{t+1} - p_t\right) + \epsilon_t,$$

where $\alpha, \beta > 0$. Assume also the quantity of real money supplied $(m_t - p_t)^s$ is determined by policymakers:

$$(m_t - p_t)^s = m_t - p_t,$$

where m_t and p_t are the log levels of money stock and price, respectively, $E_t p_{t+1}$ is the conditional expectations of p_{t+1} formed at t, and ϵ_t is a stochastic money demand shock. The quantity of money demanded is set with the quantity of money supplied to determine price level dynamics. The reduced form is:

$$p_t = a + bE_t p_{t+1} + dm_t + e_t, \tag{5.5.45}$$

where $a = -\frac{\alpha}{1+\beta}$, $b = \frac{\beta}{1+\beta}$, $d = \frac{1}{1+\beta}$, and $e_t = -\frac{\epsilon_t}{1+\beta}$.

The Cagan Model with a Constant Policy or Treatment

To make the model as simple as possible the "treatment" is assumed to be constant. In the Cagan model, the "treatment" or "policy" are monetary policy rules. For example, assume the treatment or policy, in this case assume the money stock m_t, does not change over time (i.e., $m_t = \bar{m}$). This implies policymakers decide to fix the money stock level in the economy. Equation (5.5.45) is rewritten as:

$$p_t = a' + bE_t p_{t+1} + e_t, \tag{5.5.46}$$

where $a' = a + d\bar{m}$, and $e_t = -\frac{\epsilon_t}{1+\beta}$.

The method of undetermined coefficients is used to solve for the model (5.5.46). From equation (5.5.46), the price level depends only on a constant term, its expectations, and a stochastic error term. We conjecture the RE solution is in the following form:

$$p_t = \Pi + e_t, \tag{5.5.47}$$

where Π is an unknown coefficient. Equation (5.5.47) is extended *one period forward* and conditional expectations for time t are taken:

$$\begin{aligned} E_t p_{t+1} &= E_t\left(\Pi + e_{t+1}\right) \\ &= E_t \Pi + E_t e_{t+1} \\ &= \Pi, \end{aligned} \tag{5.5.48}$$

where $E_t e'_{t+1} = 0$. Substituting equation (5.5.48) into equation (5.5.46):

$$\begin{aligned} p_t &= a' + bE_t p_{t+1} + e_t \\ &= a' + b\Pi + e_t. \end{aligned} \tag{5.5.49}$$

In equation (5.5.49), we see the *actual law of motion* (ALM) of p_t depends only on a constant term, $a' + b\Pi$, and a stochastic term, e_t, when RE is formed. By comparing equations (5.5.47) and (5.5.49), the result is:

$$\text{[From Equation (5.5.47)] } \Pi = a' + b\Pi \text{ [From Equation (5.5.49)]}. \tag{5.5.50}$$

It is straight-forward to solve the unknown parameter Π from (5.5.50):

$$\Pi = \frac{a'}{1-b}. \tag{5.5.51}$$

Equation (5.5.51) is put back in equation (5.5.48):

$$E_t p_{t+1} = \frac{a'}{1-b}. \tag{5.5.52}$$

Equation (5.5.52) is the RE agents form. Inserting (5.5.52) into (5.5.46), to get the dynamics of the price level:

$$\begin{aligned}
p_t &= a' + bE_t p_{t+1} + e_t \\
&= a' + b\left[\frac{a'}{1-b}\right] + e_t \\
p_t^{RE} &= \frac{a'}{1-b} + e_t. \qquad (5.5.53)
\end{aligned}$$

The Cagan Model with an Autoregressive Policy or Treatment

We can also have alternative treatment regimes. Assume the movement of the money supply follows a first-order autoregressive (AR(1)) policy rule:

$$m_t = \lambda + \gamma m_{t-1} + \xi_t, \qquad (5.5.54)$$

where ξ_t is a stochastic factor. Substituting equation (5.5.54) into equation (5.5.45) to get the reduced form for the price level:

$$p_t = a'' + bE_t p_{t+1} + hm_{t-1} + u_t + e_t, \qquad (5.5.55)$$

and: $a'' = \frac{\lambda-\alpha}{1+\beta}$, $b = \frac{\beta}{1+\beta}$, $h = \frac{\gamma}{1+\beta}$, $u_t = \frac{\xi_t}{1+\beta}$, and $e_t = -\frac{\epsilon_t}{1+\beta}$.

Applying the method of undetermined coefficient coefficients, based on equation (5.5.55), our conjecture for the RE solution is:

$$p_t = \Pi_0 + \Pi_1 m_{t-1} + \Pi_2 u_t + \Pi_3 e_t. \qquad (5.5.56)$$

Using equation (5.5.56), the equation is moved one period forward and then expectations for t are taken:

$$\begin{aligned}
E_t p_{t+1} &= E_t\left(\Pi_0 + \Pi_1 m_t + \Pi_2 u_{t+1} + \Pi_3 e_{t+1}\right) \\
&= E_t\Pi_0 + \Pi_1 E_t m_t + \Pi_2 E_t u_{t+1} + \Pi_3 E_t e_{t+1} \\
&= \Pi_0 + \Pi_1 E_t m_t \\
&= \Pi_0 + \Pi_1 m_t, \qquad (5.5.57)
\end{aligned}$$

where $E_t m_t = m_t$, and $E_t u_{t+1} = E_t e_{t+1} = 0$. Substituting equation (5.5.54) into equation (5.5.57):

$$\begin{aligned}
E_t p_{t+1} &= \Pi_0 + \Pi_1\left(\lambda + \gamma m_{t-1} + \xi_t\right) \\
&= \Pi_0 + \Pi_1\lambda + \Pi_1\gamma m_{t-1} + \Pi_1\xi_t. \qquad (5.5.58)
\end{aligned}$$

Inserting equation (5.5.58) into equation (5.5.55):

$$\begin{aligned}
p_t &= a'' + b\left(\Pi_0 + \Pi_1\lambda + \Pi_1\gamma m_{t-1} + \Pi_1\xi_t\right) + hm_{t-1} + u_t + e_t \\
&= \left(a'' + b\Pi_0 + b\lambda\Pi_1\right) + \left(b\gamma\Pi_1 + h\right)m_{t-1} + b\Pi_1\xi_t + u_t + e_t \\
&= \left(a'' + b\Pi_0 + b\lambda\Pi_1\right) + \left(b\gamma\Pi_1 + h\right)m_{t-1} + b\Pi_1\left[\left(1+\beta\right)u_t\right] + u_t + e_t \\
&= \left(a'' + b\Pi_0 + b\lambda\Pi_1\right) + \left(b\gamma\Pi_1 + h\right)m_{t-1} + \left[b\left(1+\beta\right)\Pi_1 + 1\right]u_t + e_t, \qquad (5.5.59)
\end{aligned}$$

where $u_t = \frac{\xi_t}{1+\beta}$, and now $\xi_t = (1+\beta)\,u_t$. According to equations (5.5.56) and (5.5.59), these two equations are identical when:

$$\begin{aligned}
\Pi_0 &= a'' + b\Pi_0 + b\lambda\Pi_1, \qquad (5.5.60) \\
\Pi_1 &= b\gamma\Pi_1 + h, \qquad (5.5.61) \\
\Pi_2 &= b\left(1+\beta\right)\Pi_1 + 1, \qquad (5.5.62) \\
\Pi_3 &= 1. \qquad (5.5.63)
\end{aligned}$$

From conditions (5.5.61)-(5.5.63), the unknown coefficients can be solved:

$$\begin{aligned}
\Pi_0 &= \frac{a'' + b\lambda\Pi_1}{1-b} = \frac{a''\left(1-b\gamma\right) + bh\lambda}{\left(1-b\gamma\right)\left(1-b\right)}, \qquad (5.5.64) \\
\Pi_1 &= \frac{h}{1-b\gamma}, \qquad (5.5.65) \\
\Pi_2 &= b\left(1+\beta\right)\Pi_1 + 1 = \frac{bh\left(1+\beta\right)}{1-b\gamma} + 1, \qquad (5.5.66) \\
\Pi_3 &= 1. \qquad (5.5.67)
\end{aligned}$$

Substituting solutions (5.5.64)-(5.5.67) into equation (5.5.56), the RE solution is obtained:

$$p_t^{RE} = \frac{a'' (1 - b\gamma) + bh\lambda}{(1 - b\gamma)(1 - b)} + \frac{h}{1 - b\gamma} m_{t-1} + \left[\frac{bh(1+\beta)}{1 - b\gamma} + 1\right] u_t + e_t. \tag{5.5.68}$$

Application 4: Models with Multiple Expectations

In this application a RE model is introduced with two rational expectations formulations. An example is Sargent and Wallace's (1975) "ad hoc" model consisting of an aggregate supply equation, an IS equation and an LM equation. A general reduced-form model is:

$$y_t = a + bE_{t-1}y_t + dE_{t-1}y_{t+1} + e_t. \tag{5.5.69}$$

Equation (5.5.69) implies agents' expectations of y_t and y_{t+1} are formed at time $t-1$.

Using the simplest REE:

$$y_t = \Pi_0 + \Pi_1 e_t. \tag{5.5.70}$$

The expression of equation (5.5.70) one period forward is:

$$y_{t+1} = \Pi_0 + \Pi_1 e_{t+1}. \tag{5.5.71}$$

Taking expectations of equations (5.5.70) and (5.5.71) at time $t-1$, respectively:

$$E_{t-1}y_t = \Pi_0, \tag{5.5.72}$$

and:

$$E_{t-1}y_{t+1} = \Pi_0. \tag{5.5.73}$$

Substituting equations (5.5.72) and (5.5.73) into equation (5.5.69):

$$y_t = a + b\Pi_0 + d\Pi_0 + e_t. \tag{5.5.74}$$

Solving for Π_0:

$$\Pi_0 = \frac{a}{1 - b - d}.$$

From equation (5.5.74), we see that:

$$\Pi_1 = 1.$$

Therefore, the REE is:

$$y_t^{RE} = \frac{a}{1 - b - d} + e_t. \tag{5.5.75}$$

Equation (5.5.75) is also called the *minimum state variable* (MSV) solution or "fundamental" solution (McCallum 1983). This is a linear solution that depends on a minimal set of variables. In this example, the REE of y_t depends only on an intercept, $\frac{a}{1-b-d}$, and a stochastic error term (e_t). Note, a variation of this procedure is applied in this chapter.

Another possible solution for model (5.5.69) is an AR(1) solution. We conjecture the AR(1) solution:

$$y_t = \Pi_0 + \Pi_1 y_{t-1} + \Pi_2 e_t. \tag{5.5.76}$$

The expectations of y_t and y_{t+1} formed at $t-1$ are, respectively:

$$E_{t-1}y_t = \Pi_0 + \Pi_1 y_{t-1}, \tag{5.5.77}$$

and:

$$\begin{aligned} E_{t-1}y_{t+1} &= \Pi_0 + \Pi_1 E_{t-1}y_t \\ &= \Pi_0 + \Pi_1 (\Pi_0 + \Pi_1 y_{t-1}) \\ &= \Pi_0 + \Pi_0\Pi_1 + \Pi_1^2 y_{t-1}. \end{aligned} \tag{5.5.78}$$

Substituting equations (5.5.77) and (5.5.78) into equation (5.5.69):

$$\begin{aligned} y_t &= a + b(\Pi_0 + \Pi_1 y_{t-1}) + d\left(\Pi_0 + \Pi_0\Pi_1 + \Pi_1^2 y_{t-1}\right) + e_t \\ &= (a + b\Pi_0 + d\Pi_0 + d\Pi_0\Pi_1) + \left(b\Pi_1 + d\Pi_1^2\right) y_{t-1} + e_t. \end{aligned} \tag{5.5.79}$$

Using equations (5.5.76) and (5.5.79), Π_0, Π_1, and Π_2 can be solved:

$$\Pi_0 = -\frac{a}{d},$$

$$\Pi_1 = \frac{1-b}{d},$$

and:

$$\Pi_2 = 1.$$

Therefore, the AR(1) REE is:

$$y_t^{RE} = -\frac{a}{d} + \frac{1-b}{d} y_{t-1} + e_t. \tag{5.5.80}$$

McCallum (1983) also terms this AR(1) REE a *"bubble"* solution since it involves the concept of a "self-fulfilling prophecy." The reason is equation (5.5.75) can fundamentally determine the dynamic behavior of y_t, but if agents "believe" and use y_{t-1} to form expectations, then the RE solution becomes equation (5.5.80) and a self-fulfilling prophecy. McCallum (1983) argues the MSV solution — not necessarily the AR(1) REE — should be the solution of interest unless an alternative assumption is made to focus on the bubble solution in the model.[15]

[15] A more general REE for this model can be derived. The general solution is:

$$y_t^{RE} = -\frac{a}{d} + \frac{1-b}{d} y_{t-1} + e_t + he_{t-1} + ku_{t-1}, \tag{5.5.81}$$

where h, k are arbitrary values of coefficients, and u_t is an extra stochastic term (i.e., a *sunspot* variable) where $E_{t-1}u_{t-1} = 0$. This general solution in equation (5.5.81) is also called the ARMA(1,1) sunspot solution for model (5.5.69).