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1 ARCH and generalizations.

For further or alternative readings, the very up-to-date survey by Bollerslev, Chou, and Kroner (1992) is highly recommended.

Many financial time series exhibit *volatility clustering*, which means that the series have periods where volatility is low and other periods where volatility is high. As econometricians we will understand "volatility" to mean "conditional variance". The conditioning set will always be the behavior of relevant variables up to "time t" - the time when we observe the series. There have recently been developed many different models for data that shows volatility clustering, and it is still a *very* active research area. The main differences between competing models will often be the choice of which variables to condition on, and (as usual) the choice of functional forms. The problems of modeling series with varying volatility is of course well known in econometrics under the heading of heteroskedasticity; but most of the interest in time-varying volatility models comes from finance.
One of the main ideas of asset-pricing is that the variability of an asset should be reflected in its price. One would expect an asset with high variance ("risk") to give a higher return (for investors to want to hold it). With some polishing and generalizations this is the main point of the CAPM-, and APT-models that are common in finance. But standard economic reasoning also says that risk should not be measured as the unconditional variance but rather as the conditional variance, and therefore finance characters are interested in modeling conditional heteroskedasticity in its own right (e.g. for the purpose of pricing options).

1.1 Engle's ARCH model

Consider the following simple scalar model

$$\begin{aligned} y_t &= \mu + e_t \; , \; t = 1, ..., T \\ e_t &= z_t \sigma_t \\ \sigma_t^2 &= \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 = \omega + a(L)\epsilon_t^2 \end{aligned} \tag{1}$$

where $\alpha_1, .., \alpha_q$, μ and ω are scalar parameters to be estimated. z_t is supposed to have mean zero and variance one, and will often (but not always) be assumed to be normally distributed. One has to assume that ω and α_i are all positive in order to obtain positive values for the estimate of the condition variance. In practice you have to assume this by either penalize the likelihood by setting it to a large negative number when negative values are met or by parameterizing it for example as the square of the parameter. I tend to prefer the later method since the former method potentially will create problems because the likelihood then will not be differentiable.

1.2 The GARCH model

Bollerslev (1986) suggested the following natural generalization of the ARCH model. Let the $e_t = \sigma_t$ as before, but now let

$$\sigma_t^2 \ = \ \omega \ + \ \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \ + \ \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2$$

which is a natural generalization corresponding to an ARMA model for the variance. This model is called a GARCH(p,q) model. Also in the GARCH model one will restrict the parameters to be positive, which will ensure a positive estimate of the conditional variance (even though I am not sure whether this is also a necessary condition in higher order models). More compactly we write

$$\sigma_t^2 \ = \ \omega \ + \ b(L)\sigma_t^2 \ + \ a(L)\epsilon_t^2 \ .$$

Bollerslev and Engle (1986) looks at the case where the variance process follows the equivalent of an ARIMA model, allowing for unit roots in the lag polynomials. In the case where

$$\alpha_1 \ + \ ... \ + \ \alpha_q \ + \ \beta_1 \ + \ ... \ + \ \beta_p \ = \ 1$$

they refer to the model as an IGARCH(p,q) model.

1.3 The E-GARCH model

One limitation of the GARCH models is that it a priori restricts the shocks to the model to have the same effect on the conditional variance whether the shocks are negative or positive. This may or may not be a reasonable assumption but one would like to be able to test this. The positivity constraints on the parameters can also be viewed as restrictive, since it rules out cyclical behavior in the conditional variance. For those reasons (among others) Nelson (1991) suggests the EGARCH(p,q) model:

$$log(\sigma_t^2) \ = \ \omega \ + \ \sum_{i=1}^{p} \beta_i log(\sigma_{t-i}^2) \sum_{i=1}^{q} \alpha_i (\phi z_{t-i} + \gamma[|z_{t-i}| - E|z_{t-i}|]) \ .$$

In the EGARCH the parameters are not restricted to be positive. Note that the term $|z_{t-i}| - E|z_{t-i}|$ is positive if the error term is larger than its expected value and negative otherwise. One can use other models for $log(\sigma_t^2)$, but for the particular model suggested by Nelson, he shows that the model seems to behave well asymptotically. We will look a little bit at the issue of stationarity of *ARCH models.

1.4 Stationarity of ARCH models

The GARCH model is covariance stationary if $A(1) + B(1) < 1$. It turns out that if $A(1) + B(1) = 1$ then the process is still stationary; but not covariance stationary since the variance is infinite. Notice that this is very different from the ARMA models where strict stationarity and covariance stationarity coincides (if the intial conditions are chosen properly). One can also show for the standard GARCH model that if ω is equal to zero then the conditional variance of the process will converge to zero almost surely. I will not go into details of the stochastic properties of *ARCH processes, but you should be aware that they can be quite tricky. It is obvious that the form of the ARCH models is chosen to give convenient estimations, and not to give convenient theoretical properties.

1.5 ARCH-M models

As mentioned in the introduction, one of the major motivations for looking at conditional variances is that financial theory says that the expected return on an asset should be correlated with its conditional variance. (The expected return will actually depend on the covariance with other assets as well as on the variance, but we will not go into finance theory here). Therefore one may want to combine the models for σ_t^2, whether one prefers ARCH, GARCH, or whatever, with a model for the mean (f.eks. a mean return). So now one would model

$$y_t \;=\; g(\sigma_t, b) \;+\; e_t \;,\; t \;= 1, ..., T \;,$$

where b is a parameter. (Of course one will usually also want to include regressors but that is supressed here). The ARCH-M model was first suggested by Engle, Lillien and Roberts (1987). It is usually more complicated to estimate ARCH-M models, because of the fact that the model for the conditional mean now depends on the conditional variance, making the model a lot more non-linear.

1.6 Estimation of ARCH models

The most commonly used estimation strategy is Maximum Likelihood, with an assumption of normality of the error terms. (One may also use the normal likelihood function without wanting to claim that the error terms are normally distributed, in which case one speaks of Quasi Maximum Likelihood estimation). For financial data this is often not a reasonable assumption and there has been articles in the literature that performs Maximum Likelihood using distributions like the t-distribution, that has heavier tails than the normal distribution. There are also articles in the literature that estimates ARCH models using GMM.

1.7 Other ARCH models

A lot of research is still being devoted to ARCH models. Some other of the newer research concerns factor-ARCH models, non-parametric ARCH models (you can have a nonparametric representation of the conditional variance of the probability density), multivariate ARCH, and all possible combinations. There is also STARCH (structural ARCH), and threshold ARCH and probably a lot of others. How about a non-parametric multivariate threshold factor-GARCH-M model? (I don't know if anybody has done that one yet). In the paper Ho, Perraudin and Sørensen (1992) we suggest an alternative to ARCH, modeling conditional heteroskedasticity in continuous time, which has some major advantages, but this becomes a bit technical.