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INTEREST RATE THEORY

The one year **spot rate** is defined as the yield on a pure discount bond of one year maturity. So for a discount bond paying X$ in one year with a present price of Q_0, you find the spot rate from solving

$$Q_0 = \frac{X}{1+S_1} \Rightarrow S_1 = \frac{X}{Q_0} - 1 \ .$$

For example, for the 3-year spot rate with price Q_0 and pay-off X:

$$Q_0 = \frac{X}{(1+S_3)^3} \ \Rightarrow \ S_3 = (\frac{X}{Q_0})^{(1/3)} - 1 \ .$$

In general, the T-year spot rate for a bond with price P_0 and pay-off X is

$$Q_0 = \frac{X}{(1+S_T)^T} \ \Rightarrow \ S_T = (\frac{X}{Q_0})^{(1/T)} - 1 \ .$$

The 1-year spot rate is usually different from the 2-year spot rate which again is different from the 3-year spot rate and so on. In other words, at any given date there is no single interest rate – even in the absence of default risk, call-ability, etc. The phrase **term structure of interest rates** refers to the patterns of interest rates for safe zero-coupon bonds at any given data.

The **yield curve** is a plot of the spot rate against time to maturity – S_T on the $Y-$ axis and T on the $X-$ axis (NOTE that "T" here is NOT the date, but the time to maturity at some given time). The yield curve can take various shapes, although an upward sloping yield curve is the most common. You can find pictures of today's yield curve in publications such as the Wall Street Journal. The "term structure of interest rates" is really just another word for the "yield curve." We will study some theories about the term structure of interest rates.

We will use the notation m_t for the pricing kernel, which we will usually interpret as $\beta \frac{U'(C_t)}{U'(C_{t-1})}$. In terms of modern theory, the price of discount bonds paying X_{t+s}, s periods from now, at time t is $E_t\{m_{t+1}X_{t+1}\}$ for $s = 1$, while the price of a two-period discount bond is $E_t\{m_{t+1}m_{t+2}X_{t+2}\}$, etc.

We usually consider an inflation protected bond with a safe payout of unity ($X = 1$). For such bonds, the one-year pay-out to a two-period discount bond is equivalent to selling the discount bond after a year at price $E_{t+1}\{m_{t+2}\}$ and the price of this one-year investment is $E_t\{m_{t+1}E_{t+1}(m_{t+2})\} = E_t E_{t+1}\{m_{t+1}(m_{t+2})\} = E_t\{m_{t+1}m_{t+2}\}$ by the law of iterated expectation. The point here is that the price of an asset (that does not expire after one period) cannot depend on how long agents wants to keep it. In general, the price of a T period discount bond is

$$Q(T)_t = E_t\{m_{t+1}...m_{t+T}\}$$

For the case, where the pricing kernel is the marginal utility, it is pretty obvious

$$Q(T)_t = E_t\{\beta\frac{U'(C_{t+1})}{U'(C_t)}...\beta\frac{U'(C_{t+T})}{U'(C_{t+T-1})}\} = E_t\{\beta^T\frac{U'(C_{t+T})}{U'(C_t)}\}.$$

Of course, this is just the Euler equation for an investment that pays of T periods into the future. (Recall, that there are Euler equations for any trade-off an agent can do without facing constraints. In this case, do not forget that more periods into the future get discounted more.)

We will not have time to go into modern models of the term structure but, for example, a general equilibrium model of the term structure involves building a dynamic general equilibrium model and then generating the pricing kernels from the Euler equation and then pricing the discount bonds. Of course, unless you are willing to assume something simple like a linear production technology this quickly become impossible to do analytically. However, you would usually build a model around the Euler equations shown here.

For a nominal bond, the real return is $\frac{1}{P_{t+s}}$, where P_{t+s} is the price level at period s relative to the current price level. To highlight the role of inflation, assume $C_{t+s} = C_t$, and inflation is certain, then

$$Q(T)_t = E_t\{\beta^s\frac{U'(C_{t+s})}{U'(C_t)}\frac{1}{P_{t+s}}\} = \beta^s\frac{1}{P_{t+s}}.$$

If the discount rate is θ and the inflation rate from period t to period $t+s$ is π_s, the bond price is $\frac{1}{(1+\theta)^s(1+\pi_s)}$. That is, a bond investor wants to be compensated for postponing consumption and for money losing real value, That is, the gross interest rate on the bond is the product of (1+s-period discount rate) and (1+ inflation rate). So approximately, the return for a one-year nominal bond will be the sum of the discount rate and the annual (expected) inflation rate. As bond markets are still mainly nominal, this is super important for financial markets, but for simplicity, I return to the inflation protected bond. Typically, the expected inflation rate is simply added, although in cases of high inflation, the correlation of inflation with the pricing kernel (which is probably some measure of the business cycle) may add a risk premium.

An important concept when you consider the term structure of interest rates is the **forward interest rate**. The forward interest rate f_{st} is the interest rate on a loan maturing t periods from now, but with the loan taken out s periods in the future ($s < t$ of course).
If you want to lend money (buy a bond) with a two year maturity you can do this in 2 ways:

a) Buy a 2-year discount bond with 2-year return $(1+S_2)^2$, where S_2 is the spot-rate.
or
b) Buy a 1-year discount bond with 1-year return $1+S_1$, and write a contract that states that you

are going to spend $(1+S_1)\$$ on a bond with return f_{12}, where f_{12} is the forward interest rate for a loan signed today, to be delivered at the end of year 1 and repaid at the end of year 2.
The contracts a) and b) should have equal value, since both contracts involved delivering 1$ (or $X\$$) today with a fixed return 2 years later. Therefore, the yield should be same, which means that:

$$(1+S_2)^2 \;=\; (1+S_1)*(1+f_{12}) \quad ,$$

In general, any t year loan is equivalent to an s year loan and a $t-s$ period forward loan, taken out at time s, so we have

$$(1+S_t)^t \;=\; (1+S_s)^s*(1+f_{st})^{(t-s)} \quad ,$$

which implies that

$$(1+f_{st})^{(t-s)} \;=\; \frac{(1+S_t)^t}{(1+S_s)^s}$$

or

$$f_{st} \;=\; \left(\frac{(1+S_t)^t}{(1+S_s)^s}\right)^{1/(t-s)} - 1$$

For simplicity we will just denote the one period forward rate at time s by f_s, so that $f_s = f_{s,s+1}$ by definition. We will in particular use the one period forward rates. Using the above equation we find

$$f_1 \;=\; \frac{(1+S_2)^2}{(1+S_1)} \;-\; 1 \ ,$$

$$f_2 \;=\; \frac{(1+S_3)^3}{(1+S_2)^2} \;-\; 1 \ ,$$

and in general

$$f_t \;=\; \frac{(1+S_{t+1})^{(t+1)}}{(1+S_t)^t} \;-\; 1 \ .$$

EXAMPLE: Assume $S_2 = 6\%$ and $S_1 = 5\%$, so the yield curve (for maturities 1 and 2 periods) is upward sloping. What is the forward rate f_1? Using the equations above we find:

$$(1+.06)^2 \;=\; (1+0.05)^1*(1+f_1)^1 \ .$$

or

$$1+f_1 \;=\; 1.06^2/1.05 = 1.0701 \;\Rightarrow\; f_1 \;=\; .0701 \ .$$

Note that the 2-period spot rate is an "average" of the 1-period spot rate and the forward rate. The spot rate is close to, but not exactly equal to the simple arithmetic average; here $(.0701+.05)/2 = .06005$. Therefore, when the 2-period spot rate is 1 point above the 1-period spot rate, it implies that the forward rate is approximately 2 percentage points higher than the 1-period spot rate. EXAMPLE: Assume $S_4 = 8\%$ and $S_2 = 7\%$. We can then find the forward rate f_{24} as

$$f_{24} \;=\; \sqrt{\frac{1.08^4}{1.07^2}} - 1 \;=\; .09 \;.$$

If you observe an upward sloping yield curve, this means that the forward rates are all higher than the spot rates, in the sense that $f_t > S_{t+1}$ for all t (remember that f_t is the interest rate on a loan delivered at t maturing in $t+1$ just as S_{t+1} is the current spot rate on a loan maturing in $t+1$). This follows because

$$1 + f_t \;=\; \frac{(1+S_{t+1})^{(t+1)}}{(1+S_t)^t} \;=\; \frac{(1+S_{t+1})^t}{(1+S_t)^t} * (1+S_{t+1}) \;>\; 1 + S_{t+1} \;.$$

The **expectations hypothesis** states that forward interest rates are unbiased predictors of corresponding future interest rates. For example, if the two-year spot rate is 6% and the one-year spot rate is 5%, then the forward rate for a 1-year loan taken out next year is (approximately) 7%. The pure expectations hypothesis states that the expected 1-year spot rate in a year is 7%. Define the one year spot rate in year t as $S_{t,1}$. The expectations hypothesis then can be stated as,

$$f_1 \;=\; ES_{1,1} \;,$$

or more generally as

$$f_t \;=\; ES_{t,1} \;.$$

How to test the expectations hypothesis? You run the regression

$$f_t \;=\; b_0 + b_1 S_{t,1} + u_t \;,$$

and perform an F-test for $b_0 = 0,\ b_1 = 1$. The expectations hypothesis is an example of a rational expectations model. It implies that investors on average can forecast the future interest rate, i.e. $f_t = S_{t,1} + u_t$, where u_t is a mean zero error term. (In the literature you will often see that authors performing the "reverse regression" $S_{t,1}$ on f_t.)

Forward rates and marginal utility

If we let the payout to a discount bond be 1, then the price of a k-period discount bond, in terms of marginal utility is

$$Q(k)_t = \beta^k \frac{E_t U'(C_{t+k})}{U'(C_t)} \;,$$

so the k-period spot-rate is

$$1 + S_k = \left(\frac{U'(C_t)}{\beta^k E_t U'(C_{t+k})}\right)^{(1/k)} \;.$$

You can see that the rate is high if current consumption is low or if expected future consumption is high relative to current consumption. Higher (today) uncertainty about future consumption will tend to drive down interest rates via a precautionary savings motive. (Note: usually, when people say that something is caused by "a motive" it is bad writing, but here interest rates may be affected by a desire to save, not by actual saving, which may or may not be possible depending, for example, on whether goods are storable in the model.)

For forward rates, focussing on two periods, we have

$$1 + f_1 = \frac{(1 + S_2)^2}{1 + S_1} \,,$$

so

$$1 + f_1 = \frac{U(C_t)}{\beta^2 E_t U'(C_{t+2})} \Big/ \frac{U(C_t)}{\beta E_t U'(C_{t+1})}$$

implying

$$1 + f_1 = \frac{E_t U'(C_{t+1})}{\beta E_t U'(C_{t+2})} \,.$$

Observe: a) If consumption is constant, the forward rate (and all spot rates) equals the discount rate. b) Higher expected consumption in period $t+1$ relative to period $t+2$ lowers the forward rate. c) Higher uncertainty about period $t+2$ ($t+1$) lowers (increase more consumption uncertainty (they go up via Jensen's inequality) and to uncertainty in real ras) the forward rate. This is as expected in terms of precautionary savings motives and relative scarcity. But the formula also clearly shows how it is the expectations at t and the relative uncertainty at t, that determines the forward rate.

It should be obvious how interest rates react totes and uncertainty in inflation; namely, uncertainty in returns matters if it correlates with the pricing kernel, as is the logic of the CAPM, CCAPM, etc. The yield curve is sometimes taken as a predictor of future upturns or downturns (an inverted yield-curve may predict a downturn), but the current (April 2023) inverted yield curve, if by most observers taken to indicate that the Federal Reserve is going to bring down inflation soon.