ECONOMICS 7344, Spring 2022
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Complete Markets

Consider assets $i = 1, ..., N$ with returns $r_i(s)$ if state s occurs. The vector of returns is $R(s)$. Assume there are S states of the world (to be revealed after trading) and that $N = S$. Consider dummies Dum_s that take the value 1 if state $s = 1, ..., S$ occurs. Any asset i has return of the form

$$r_i = a_{i1}Dum_1 + ... + a_{iS}Dum_S \,,$$

where a_{is} is the return if state s occurs. We can define the matrix $R = \{r_1, ..., r_S\}'$ such that a given column lists the returns to the S assets in state s, vectors $D_i = (0, .., 1, .., 0)'$, with a 1 in the i'th component, and the matrix $D = \{D_1, ...D_S\}'$, and we then have

$$R = A\,D\,.$$

If $X(s)$ is a random vector that with probability π_s has 1 in the $s'th$ row, 0 otherwise, then $R(s) = A\,DX(s)$; i.e., this given the return to all assets in a given state. a_{ij} is the typical element of the $S \times S$ matrix A. We have

$$\begin{pmatrix} r_1(1) & ... & r_1(s) \\ \vdots & ... & \vdots \\ r_S(1) & .... & r_S(s) \end{pmatrix} = \begin{pmatrix} a_{11} & ... & a_{1s} \\ \vdots & \vdots & \vdots \\ a_{S1} & ... & a_{SS} \end{pmatrix} \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & 1 \end{pmatrix}.$$

We say that markets are complete (or we need to stress the Arrow-Debreu framework: that a full set of Arrow securities exist) if trading in the assets with returns r_i is equivalent to trading in a full set of Arrow securities. This holds if the matrix A can be inverted, so that

$$D = A^{-1}R \quad (**)\ .$$

In this case, if you pick a column randomly in R (equivalent to picking a state of the world s) and reshuffle (say, purchase) the amounts of the assets using the elements in A^{-1} as weights, you get the return to the Arrow security for state s. The formula (**) shows how all the Arrow securities can be constructed from the list of returns in R. We have shown that if A has full rank, i.e., the returns to the assets are linearly independent, then being able to trade freely in those assets will be equivalent to trade in Arrow securities, and the outcomes for consumption and asset prices could be found by solving for the prices for the Arrow securities and go from there.

- We need at least as many assets as states-of-the-world.
- The returns need to be linearly independent (i.e., none of the rows in A can be written as a linear combination of the others
- If $N > S$, you still have complete markets if you can choose a subset of assets so that the matrix A can be inverted.
- If PO is the vector of payouts, then the gross returns are (for case of $N = 3$, with subscripts denoting assets and superscripts denoting state-of-the-world) but this is easily generalized):

$$\begin{pmatrix} R_1^1 & R_1^2 & R_1^3 \\ R_2^1 & R_2^2 & R_2^3 \\ R_3^1 & R_3^2 & R_3^3 \end{pmatrix} = \begin{pmatrix} P_1^{-1} & 0 & 0 \\ 0 & P_2^{-1} & 0 \\ 0 & 0 & P_3^{-1} \end{pmatrix} \begin{pmatrix} PO_1^1 & PO_1^2 & PO_1^3 \\ PO_2^1 & PO_2^2 & PO_2^3 \\ PO_3^1 & PO_3^2 & PO_3^3 \end{pmatrix}$$

  So if the payouts are linearly independent, so are the returns and *vice versa*.

Example: Two states of the world, A and B. You have an Arrow security for state A and a safe asset with return r^f. We have

$$r^A = 1\,D_A + 0 \quad (1)$$
$$r^f = 1\,D_A + 1\,D_B \quad (2)$$
$$(3)$$

We can find the Arrow securities as

$$\begin{pmatrix} D_A \\ D_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} r^A \\ r^f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} r^A \\ r^f \end{pmatrix} = \begin{pmatrix} r^A \\ r^f - r^A \end{pmatrix}$$

Some times I have posed exam questions where one or several agents have access to a safe asset and an Arrow security for, say, state A. There is still another state of the world B. To solve this, you almost always *have* to proceed as follows: Assume that the agents trades in two arrow securities A and B. Solve the question. Not, if the answer is that the agent wants to buy B^B units of Arrow security B and B^A units of Arrow security A, then to answer in terms of the safe asset and Arrow security B, you observe that the agent only can get payout in state B via the safe asset. So the agent will purchase B^B units of the safe asset. The safe asset carries along payout in state A, so the agent will purchase $B^A - B^B$ units of Arrow security A. Now, you can easily see that if state A occurs, the agent will receive B^A units from his or her investments. And in state B, s/he receives B^B. (If you try to solve directly for the safe asset, you have to deal with the Euler equation for that asset having several non-zero terms, which makes it hard to deal with.)